Statistical mechanics of social impact

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We discuss the mean-field theory for a class of probabilistic cellular automata that can describe the dynamics of social impact. The models exhibit complex intermittent behavior.

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I. INTRODUCTION

Statistical mechanics has proven to be a powerful tool for interdisciplinary research, most notably with regard to the study of neural networks [1]. A network approach to the social sciences comes from the theory of cellular automata [2,3], as applied to the so-called voter problem, or to different kinds of majority rule, in which individuals adjust their opinions (i.e., their states) to match the majority of their close neighbors [4–7]. Although their rules of opinion change following social interaction processes seem intuitively plausible, one may ask how well the assumptions of these models relate to knowledge of social influence processes developed by social scientists.

The theory of social impact was formulated by Latané [8], who claimed with considerable empirical support that the impact of a group of individuals on a given person can be seen as proportional to three classes of factors: the "strength" of the members of the group (how credible or persuasive they are), their "immediacy" (a decreasing function of their social "distance" from the individual), and their number N (impact scaling as N^x, with $x \simeq \frac{1}{2}$). It has been shown that this general function can describe a wide variety of situations in which social impact is exerted, regardless of the form in which influence takes place. This function can be fitted to such diverse phenomena as bystander intervention in emergencies [9], tipping in restaurants [10], social loafing [11], interest in news events [8], stage fright [12], conformity [13], and, of course, attitude change [13], where an individual is affected by his or her social environment.

The formal analysis of opinion formation in groups was initiated by Abelson [14], who has shown that a wide class of linear models of individual attitude change lead to complete uniformity of opinions as a generic stationary state. Recently, Nowak, Szamrej, and Latané [15,16] have proposed a new class of computer models of opinion formation based on Latané's theory of social impact [8]. These models exhibit a richer variety of stationary states, including the emergence of well-localized and dynamically stable clusters (domains) of individuals who share minority opinions, similar to real-world phenomena. The emergent self-organization of minority and majority members can be related to similar effects discussed by Axelrod [17] involving the clustering of optimal strategy choices in the so-called "prisoners' dilemma."

It is the aim of this paper to represent a mathematical framework for the class of models introduced by Nowak, Szamrej, and Latané. Although these models are related to known models of cellular automata [4–7], they are in many respects novel. Contrary to the models studied in the literature, those in question are characterized by long- or moderate-range interactions and by an intrinsic disorder. As we shall see below, these two features of the models are essential for describing a dynamical theory of social impact (influence).

The class of models, as well as theory that we present here, should be of general interest for statistical physicists who study discrete statistical models, such as neural networks, cellular automata, kinetic Ising models, etc., for the following reasons.

(1) The models provide a class of exactly solvable dynamical models of statistical mechanics. They are especially interesting, since they describe disordered systems. The disorder has a special character, since each of the elements of the system is characterized by random "strength" parameters. The interactions between the elements are also random and proportional to these strength parameters.

(2) The method of solving the models in the framework of mean-field theory is also novel and consists of introducing dynamical "order" parameters that even in the simplest cases have the form of a function. The complexity of order parameters here does not, however, have much in common with the complexity of the order parameters of spin glasses and replica symmetry breaking (cf. Ref. [18]). It reflects rather the amount of ordering among the elements of the system that share a given level of "strength."

(3) The dynamics of the models generically exhibits an interesting intermittent behavior, which, we think, is of a general nature and characterizes various other decaying systems. In particular, although a small amount of noise in the system tends to destroy minority clusters, the approach to uniformity is very complex and has the form of "staircase" dynamics. It consists typically of several rapid steps, each of them followed by a long period of intermittent quasistationary behavior.

(4) The models and methods presented here may be used to describe physical systems. In particular, we have in mind applications to the physics of evaporation processes (for a review, see Ref. [19]), to the theory of random ferromagnets (cf. Ref. [20]) that are composed of various species of magnetically active atoms, and to the

physics of neural networks [3].

The present paper is organized as follows. In Sec. II we describe the models of Nowak, and Szamrej, and Latané in detail, including a rather careful discussion of the assumptions used to construct the model and their relation to empirical data. Although these assumptions and their supporting data are grounded in social science rather than physics, they show how firm are the empirical fundaments on which we stand.

Next, we present exact solutions of specific versions of the models for several underlying geometries.. We start in Sec. III with the simplest case of a fully connected model, in which every element interacts with every other. Such a model may be solved exactly within the framework of mean-field theory, using an appropriately defined function as a dynamical "order" parameter. We present here heuristic arguments to show that the decay of minority groups in fully connected models will generically exhibit "staircase" dynamics. This is proven in Sec. IV, where we discuss the influence of noise on the dynamics. In Sec. V we briefly present the results for the case of hierarchical geometries, in which the interaction of elements decays with ultrametric distance (cf. Refs. [18,21]). In Sec. VI we discuss the case of strongly diluted and randomly connected models. Here we apply directly the theory of Derrida *et al.* that has been formulated in the context of Kaufmann's model of cellular automata [22–24] and asymmetric neural networks [25,26].

In Sec. VII we turn to the discussion of social impact models realized in two-dimensional (2D) Euclidian space. The interaction of elements in such cases decreases with increasing distance between the elements. For such models mean-field theory is an approximation only. Nevertheless, it allows for a satisfactory comparison of qualitative and some quantitative results of the analytic theory with numerical simulations.

The paper ends with Sec. VIII, which contains our conclusions. We have tried to keep our presentation on an elementary level in order to make it accessible to scientists from other specialties with interdisciplinary interests.

II. DESCRIPTION OF THE MODELS

The models of Nowak and Latané are based on several assumptions, which we have listed and discussed below.

(1) *Two-state elements.* The models in question belong to the category of cellular automata consisting of N individuals, each holding one of two opposite opinions. The states of every individual are therefore binary, $\sigma_i = \pm 1$, as in the standard Ising model of a ferromagnet.

One might expect that the opinions of individuals on specific subjects should vary gradually and be described by some continuous variable. As empirical data show, very frequently this is not the case. In fact, the distribution of opinions on "important" issues measured on some multivalued scale is typically bimodal and peaked at extreme values [27].

(2) *Disorder and random "strength" parameters.* Each individual is characterized by two random strength parameters, which we call persuasiveness p_i and supportiveness s_i. These parameters determine how effectively a given individual may interact with and influence other individuals either to change or to confirm their opinion.

Although assumption (2) seems intuitively reasonable to most observers, it is remarkably controversial with regard to personality differences [28]. Here we focus on the fact that differences of age, intelligence, socio-economic status, etc., may affect the degree to which individuals are influential.

(3) *"Social" space.* Each individual is characterized by a location in social space, so that each pair (i,j) of individuals is characterized by a distance d_{ij}. Interactions between individuals tend to decrease with this distance. The determination of the nature of this metric is itself a fascinating problem of sociometry, and in principle d_{ij} should be empirically determined from sociometric data and could correspond to some peculiar geometry.

In this paper we study some simple geometries that may capture some aspects of the actual geometry of social space. We consider (a) the trivial geometry of a fully connected model, where all the distances between individuals are equal; (b) a hierarchical geometry with ultrametric distance, where the individuals are divided into a hierarchy of groups and the distance between two individuals is determined by the hierarchy level of the group to which both belong; (c) a strongly diluted model, in which individual is linked to a random set of other individuals; in this model there is no metric structure, since the "distance" d_{ij} has to be considered as an asymmetric function of i and j, which is infinite for most of the ordered pairs (i,j); and (d) two-dimensional Euclidian geometry.

The first three geometries (a)–(c) obviously cover some aspects of the social space that we want to describe. Additionally, as will be discussed below, they allow for exact solutions of the model. The two-dimensional Euclidian metric was used for the computer simulations reported in Refs. [15] and [16]. In fact, recent empirical data show very clearly that the frequency of social interactions that influence individual attitudes is indeed a decreasing function of the Euclidian distance between the homes of the interacting persons [29].

(4) *Social impact theory.* Individuals are assumed to exchange, compare, adjust, and influence each other's attitudes. The total impact I_i that the ith individual experiences from his or her social environment is a function of the persuasive impact of those individuals who hold the opposite opinion to σ_i, relative to the supportive impact of those individuals who share the same opinion,

$$I_i = I_p\left[\sum_j \frac{t(p_j)}{g(d_{ij})}(1-\sigma_i\sigma_j)\right] - I_s\left[\sum_j \frac{t(s_j)}{g(d_{ij})}(1+\sigma_i\sigma_j)\right], \tag{2.1}$$

where $g(\ )$ is some decreasing function of the distance d_{ij} and $t(\ )$ is the strength scaling function. In Eq. (2.1) both impacts are functions of the sums of the influences of all individuals j who hold, respectively, the opposite or the same opinion as i. Note that the strength scaling function may be taken to be $t(x)=x$, provided we redefine appropriately the distribution $p(p_i,s_i)$. In the present paper we adopt this approach. In the course of dynamics, individuals adjust their opinions according to the value of the total impact they experience, so that, without noise,

$$\sigma_i' = -\mathrm{sgn}(\sigma_i I_i) , \tag{2.2}$$

where σ_i and σ_i' denote the opinion of the ith individual at consecutive time steps. We assume a synchronous version of the dynamics (2.2), in which all individual attitudes are updated in parallel. It typically takes somewhat longer to achieve stable equilibria using standard serial Monte Carlo methods, because of the likehood of selecting some individuals too rarely and some too frequently. Serial methods, on the other hand, avoid periodic asymptotic states that might occur with the parallel method. Such states can occur when persuasiveness exceeds supportiveness, such that two opposing individuals may each persuade the other and both oscillate between the two opinions. Nevertheless, we have noted no essential differences among numerical results from using parallel, standard Monte Carlo or serial "Knight's tour" methods. Numerical simulations [16] indicate, in fact, that the models in question are robust with respect to the effects of asynchrony.

(5) *Presence of noise.* Obviously, in reality, opinion change is not so deterministic as Eq. (2.2) suggests. There are various random elements in this process, and they may be phenomenologically modeled by introducing noise into the dynamics (2.2).

A probabilistic version of Eq. (2.2) may be realized by allowing for violations of rule (2.2) with a given probability. In the theory that we present in this paper, as well as in numerical simulations, however, two kinds of noise models have been used: a uniform white noise (for a concise review see, for instance, [30]), i.e.,

$$\sigma_i' = -\mathrm{sgn}(\sigma_i I_i + h) , \tag{2.3}$$

where h are random variables that are statistically independent for different time instants. Alternatively, we have used a site-dependent white noise, i.e.,

$$\sigma_i' = -\mathrm{sgn}(\sigma_i I_i + h_i) , \tag{2.4}$$

where h_i are random variables that are statistically independent for different individuals and different time instants. For this kind of noise, we have assumed that the noise variables have uniform statistical properties; i.e., the probability distribution $p(h_i)$ is both site and time independent.

The first kind of noise simulates global events affecting the group as a whole, such as, for instance, the availability of new information, the actions of public figures, etc. Site-dependent noise is a rough description of all the events, other than social impact, that are experienced by individuals and that influence their attitudes (cf. individual experiences, thought processes, etc. [31]).

Summarizing, we would like to stress that three major properties differentiate these models from those studied in the literature: First, for any geometry considered, we studied models that have *moderate- or long-range interactions.* For instance, in the case of fully connected models, infinite-range interactions were assumed, whereas in numerical simulations in two dimensions, the interactions were often characterized by an algebraic decrease of couplings with increasing distance. Second, the models incorporate *random strength parameters* or individual differences in persuasiveness and supportiveness. In the present paper we use a model in which p_i and s_i are randomly and independently chosen for each individual, with a probability density $p(p_i,s_i)$. Finally, the models include an element in the *form of the impact functions I_p and I_s*. Here, however, we will use the choice $I(x)=x$ and leave the discussion of alternatives to future publications [32].

Given the dynamics defined by Eq. (2.2), how do typical initial opinion distributions evolve in time? Such initial distributions may be distinguished by the numbers of opposing individuals, $m=\sum_j \sigma_j/N$, into three classes: (i) States with $|m|\simeq 1$ are close to uniformity of opinion; (ii) states with $0 \ll |m| < 1$ contain moderate numbers of minority members, which may be distributed uniformly in a statistical sense or concentrated in clusters (domains); and (iii) finally, when $|m|\simeq 0$, we have balanced opinion distributions, which, again, may be either statistically uniform or clustered.

Nowak, Szamrej, and Latané in Ref. [15] and Latané and Nowak in Ref. [16] performed extensive numerical studies of the dynamics described by Eq. (2.2). In their work, as we mentioned, they used Euclidian geometry with $g(d_{ij})=(d_{ij})^\alpha$, where the exponent α was varied between $2<\alpha\le 8$. Strength parameters s_i and p_i were distributed uniformly on the interval $[0,2s]$, and $[0,2p]$, respectively. Various forms of impact functions, such as, for instance, $I(x)=\sqrt{x}$ or $I(x)=x$, have been used.

Computer simulations [15,16] in the absence of noise indicate quite generally for class (i) states that uniformity is the most frequent final state. For class (ii) states, two scenarios are possible. With statistically uniform initial distributions, there is a *polarization* (i.e., decrease of minority numbers) and *clustering* of opinions, such that stable domains of minority opinion appear. When minorities are initially clustered, their decrease is inhibited, leading such states to be more dynamically stable. Similar conclusions hold for class (iii) data, except, of course, that the determination as to which of the opinions would form the minority is random. It is worth stressing that the formation of local coherent clusters of attitudes from initially uniform distribution of attitudes has been observed in empirical data. For example, the classic study of Festinger, Schachter, and Back [33] of social pressures in informal groups documented the development of "group standards" geographically clustered among residents of specific courtyards of a student housing project at MIT.

The influence of small amounts of noise in the dynamics, according to Eq. (2.3), did not seem to affect these results strongly. This is rather strange, since the simple intuition from random ferromagnet theory would suggest that the only thermodynamically stable state in the presence of small noise is a uniform state. Although minority clusters showed a tendency to decay in the presence of noise, the time scale of this decay seemed to exceed the computer simulation time for small noise levels. In separate simulations [32] it was observed that minority clusters perturbed by a strong noise at one time instant and by relatively weak noise in the following time steps change their shape and decrease in area. Typically, however, after some time such clusters reach a new, apparent equilibrium and remain stable for periods longer than the simulation.

These findings have implications for a variety of applications in the social sciences, such as the use of public opinion polls to predict elections. Currently, the prediction of election results is usually based solely on assessing the proportion of people holding different attitudes or, at most, on plotting temporal trends in the distribution of attitudes. In the light of the theoretical and numerical results described above, predictions should also take into account the degree to which attitudes are clustered. For instance, a well-clustered 30% minority may remain in a stable equilibrium, while a 30% minority scattered among majority members may be subject to rapid declines in number. Thus public opinion polls should assess the degree to which individuals that are "close" to one another share the same opinion, as well as determining the proportion of who agree with the majority.

In the sections below, we look for analytic results that could give more insight into the structure of the dynamical models of social impact and that could, at least qualitatively, explain the results of numerical simulations.

III. FULLY CONNECTED MODELS

We shall begin our analysis with a fully connected model that has infinite-range interactions. Such a model allows for an exact solution in the framework of mean-field theory.

For such a model, one chooses $g(d_{ij})=N$ for all $i\neq j$. The scaling with N assures the existence of $N\rightarrow\infty$ limit. In order to describe a nontrivial competition between social impact and self-supportiveness, we additionally set $g(0)=1/\beta$.

In the noiseless limit, the evolution equation (2.2) may be reduced for large N to the form

$$\sigma_i'=\text{sgn}(m)\Theta(|m|-|a_i|)+\sigma_i\text{sgn}(a_i)\Theta(|a_i|-|m|)\ , \quad (3.1)$$

where m denotes a weighted majority-minority difference

$$m=\sum_j (s_j+p_j)\sigma_j/N(s+p)\ , \quad (3.2)$$

s and p are the means of s_j and p_j, respectively, and $\Theta(\cdot)$ denotes a unit-step function.

In the infinite-N limit, we expect that the mean-field theory is exact, and so the relative fluctuations of m are of the order of $1/\sqrt{N}$. In the limit of large N, m is practically no longer a random variable and may be replaced by its mean.

The random parameters a_i introduced in Eq. (3.1) describe the effective self-supportiveness and are defined for each of the individuals as

$$a_i=\frac{s-p}{s+p}+\frac{\beta}{s+p}s_i\ . \quad (3.3)$$

We restrict our attention to the case in which all $a_i\geq 0$ for any realization of the random variable s_i, reserving discussion of the case where a_i's may take negative values for Ref. [32].

It is interesting to realize that the "order parameter" appropriate for the dynamics defined by Eq. (3.1) is a function of a variable $\xi\in[0,\infty]$,

$$n(\xi)=\left\langle\frac{1}{N}\sum_j\frac{s_j+p}{s+p}\sigma_j\Theta(a_i-\xi)\right\rangle\ , \quad (3.4)$$

where brackets $\langle\ \rangle$ denote the average with respect to random variables s_i and p_i. $n(\xi)$ measures the weigthed majority-minority difference calculated only for those individuals who have effective self-supportiveness greater than ξ. As we shall see below, $n(\xi)$, which gives the description of the dynamics (3.4) averaged with respect to the quenched disorder, is indeed a proper order parameter since (i) in the absence of noise it defines different, stationary states of the dynamics (i.e., nonergodic phases); in fact our model has infinitely many stationary states; (ii) it determines uniquely an approach toward stationary states in the noiseless case; and (iii) in the presence of noise the only stationary state, or stable phase in the thermodynamical sense, is a nearly uniform state with $m\simeq\pm 1$. The function $n(\xi)$, however, turns out still to be very useful for description of an approach and a departure from metastable states.

The complexity of the order parameter $n(\xi)$ is a direct result of the disorder, but is only weakly reminiscent of the order parameter of spin glasses [18]. The function $n(\xi)$ does not describe any replica symmetry breaking, although it does describe infinitely many stationary states. Its physical relevance is that, to specify the state of the system completely in the presence of disorder, it is necessary to describe the ordering among individuals of a given strength. In fact, the derivative of $n(\xi)$ with respect to ξ, when it exists, is directly related to the weighted majority-minority difference of those σ_j's whose strength is given according to Eqs. (3.1) and (3.3) by the expression

$$s_i=\frac{1}{\beta}[\xi(s+p)-(s-p)]\ . \quad (3.5)$$

The order parameter (3.4) fulfills the equation

$$n'(\xi)=[g(m,\xi)+n(|m|)]\Theta(|m|-\xi)+n(\xi)\Theta(\xi-|m|)\ , \quad (3.6)$$

where $n(0)=m$ and

$$g(m,\xi)=\left\langle\frac{1}{N}\text{sgn}(m)\sum_j\frac{s_j+p_j}{s+p}\Theta(|m|-a_j)\Theta(a_j-\xi)\right\rangle\ . \quad (3.7)$$

In fact, Eq. (3.6) can be derived without the mean-field assumption. In such a case, $n(\xi)$, $g(m,\xi)$, and m would correspond to unaveraged quantities. The fact that Eq. (3.6) does not depend explicitly on random variables s_i,

p_i, and fluctuating σ_i provides yet another argument for the correctness of the mean-field approach. Note also that Eq. (3.6) implies that $n(\xi)$ is continuous for all ξ, provided that was initially. It may, however, be nondifferentiable at $\xi=m_k$, where m_k denote successive values of the weighted majority-minority difference.

Note that according to Eq. (3.6), the function $n(\xi)$ does not change its shape for $\xi \geq |m| = |n(0)|$. The "magnetization" m fulfills therefore the recurrence that follows from Eq. (3.6) by putting $\xi=0$:

$$m' = g(m,0) + n(|m|) \, . \tag{3.8}$$

Direct inspection of expressions (3.4) proves that

$$n(\xi) = m - \left\langle \frac{1}{N} \sum_j \frac{s_j + p_j}{s+p} \sigma_j \Theta(\xi - a_i) \right\rangle \, . \tag{3.9}$$

Letting $\xi = |m|$ in the above equation and introducing the result to Eq. (3.8), we obtain

$$m' = m + \left\langle \frac{1}{N} \sum_j \frac{s_j + p_j}{s+p} [\mathrm{sgn}(m) - \sigma_i] \Theta(|m| - a_j) \right\rangle \, . \tag{3.10}$$

Note that the change of m is always nonnegative, when $\mathrm{sgn}(m)=1$, since $1-\sigma_i \geq 0$. Similarly, the change of m is nonpositive, for $\mathrm{sgn}(m) = -1$, so that

$$|m'| = |g(m,0) + n(|m|)| \geq |m| \, . \tag{3.11}$$

From this inequality it follows that m is a monotonic function of a time step. The actual value of $|m|$ is therefore the maximal among all of the values of $|m_k|$. In such a case, according to Eq. (3.6), the initial function $n_0(\xi)$ determines the value of the function $n(|m|)$. That, in turn, implies that $n_0(\xi)$ determines fully the evolution of the parameter m. For positive initial $m_0 = n_0(0)$, we obtain, for instance,

$$m' = g(m,0) + n_0(m) \, . \tag{3.12}$$

The map (3.12) provides an analytic solution for the dynamics (3.1), since it reduces it to the solution of a single algebraic equation. Knowing the explicit form of $n_0(m)$, we may, using Eqs. (3.12) and (3.6), find explicitly the dynamics of m and then that of $n(\xi)$.

The map (3.12) has the following properties: Denoting $g(m,0)+n_0(m)=f(m)$, it is easy to show that $f(0)=m_0$ and $f(m) \geq m_0$ for all positive m's. The latter inequality follows from Eq. (3.10) when we let $m'=f(m)$ and σ_i equal its initial value $\sigma_i(0)$:

$$f(m) = m_0 + \left\langle \frac{1}{N} \sum_j \frac{s_j + p_j}{s+p} [1 - \sigma_i(0)] \Theta(m - a_j) \right\rangle \, . \tag{3.13}$$

Moreover, from Eq. (3.4) it follows that $-1 \leq n(\xi) \leq 1$, whereas from Eq. (3.7) it follows also that $0 \leq g(m,0) \leq 1$. As a result, $f(\ )$ is bounded. Writing

$$f(m) = \left\langle \frac{1}{N} \sum_j \frac{s_j + p_j}{s+p} [\Theta(m - a_j) - \sigma_i(0)] \Theta(a_j - m) \right\rangle \, , \tag{3.14}$$

and replacing $\Theta(m-a_j)$ by $1-\Theta(a_j-m)$, we get

$$f(m) = 1 - \left\langle \frac{1}{N} \sum_j \frac{s_j + p_j}{s+p} [1 + \sigma_i(0)] \Theta(a_j - m) \right\rangle \, , \tag{3.15}$$

so that $f(m) \leq 1$. From Eq. (3.13), on the other hand, we infer that it is an increasing function of m.

All these facts imply that m is an increasing and bounded function of the time step; i.e., map (3.12) has at least one stable fixed point. It may also have several stable fixed points separated by unstable ones. The fixed points values fulfill the equation

$$\frac{1}{N} \sum_j \frac{s_j + p_j}{s+p} [1 - \sigma_j(0)] \Theta(|m| - a_j) = 0 \, . \tag{3.16}$$

m will tend to the smallest of the stable fixed points as time step goes to infinity.

It is important to stress that the situation in which $f(\ )$ has many fixed points may be generic. Here we discuss two examples. In both of them we assume $s_i = p_i$ and vary the form of the distribution $p(s)$ or the initial correlations of the strength of the individuals and their opinion. Let us assume that the random variable that describes the initial state of each of the individuals, $\sigma_i(0)$, is a function of s_i distributed with the probability $p(\sigma, s_i)$, which is identical for all i. Let us also denote $\sigma(s) = \sum_{\sigma=\pm 1} \sigma p(\sigma, s)$.

Example A. Single-peaked strength distribution $p(s)$, no initial strength correlations. For this case we may choose for instance $\sigma(s) = \sigma$,

$$p(s) = \frac{1}{2\tilde{s}} \, , \tag{3.17}$$

for $\bar{s} - \tilde{s} \leq s \leq \bar{s} + \tilde{s}$, and zero otherwise. $\bar{s}$ denotes here the mean value of s and $\tilde{s}^2/3$ its variance. Elementary calculations show that, in this case,

$$m_0 = \sigma \tag{3.18}$$

and

$$f(m) = \sigma \, , \tag{3.19a}$$

for $m \leq \beta(\bar{s} - \tilde{s})$,

$$f(m) = \sigma + \frac{1-\sigma}{4\bar{s}\tilde{s}} \left[\left[\frac{m}{\beta} \right]^2 - (\bar{s} - \tilde{s})^2 \right] \, , \tag{3.19b}$$

for $\beta(\bar{s} - \tilde{s}) \leq m \leq \beta(\bar{s} + \tilde{s})$, and

$$f(m) = 1 \, , \tag{3.19c}$$

for $\beta(\bar{s} - \tilde{s}) \leq m$. This function has two stable and one unstable fixed points, provided $\beta(\bar{s} - \tilde{s}) \geq \sigma$ and $\beta(\bar{s} - \tilde{s}) \leq 1$. It is easy to generalize this example and to show that $f(\ )$ will typically have $K+1$ stable and K unstable fixed points if the distribution $p(p_j, s_j)$ is multipeaked [32] and has K peaks.

Example B. Uniform strength distribution $p(s)$ and initial opinion-strength correlations. For this case we may choose

$$p(s)=\frac{1}{2\bar{s}} \tag{3.20}$$

for $s \le 2\bar{s}$,

$$\sigma(s)=-1 \tag{3.21a}$$

for $s \le s_1$, and

$$\sigma(s)=1 \tag{3.21b}$$

for $s > s_1$, with s_1 being a real parameter. After performing some algebra, we obtain

$$m_0=\frac{2\bar{s}^2-s_1^2}{2\bar{s}^2} . \tag{3.22}$$

Note that $m_0>0$ for $s_1<\bar{s}/\sqrt{2}$. The function $f(m)$ is then given by

$$f(m)=\frac{2\bar{s}^2-s_1^2}{2\bar{s}^2}+\frac{1}{2\bar{s}^2}\left[\frac{m}{\beta}\right]^2 \tag{3.23a}$$

for $m<\beta s_1$, and

$$f(m)=1 \tag{3.23b}$$

for $m \ge \beta s_1$. Evidently, $f(m)$ has two stable and one unstable fixed points, provided $\frac{1}{2}<\beta s_1 \le 1$, and

$$\frac{\beta\bar{s}}{2} > \left[1-\frac{1}{2}\left[\frac{s_1}{\bar{s}}\right]^2\right]^{1/2} . \tag{3.24}$$

It is possible to generalize this result for more complicated initial correlations $\sigma(s)$. For instance, choosing

$$\sigma(s)=-1$$

for $s \le s_1$ and $s \ge s_2$, while

$$\sigma(s)=1$$

otherwise, we obtain $f(m)$ that grows quadratically for $s \le s_1$, is constant for $s_1<s<s_2$, and grows quadratically again for $s>s_2$. Such a function may have three stable and two unstable fixed points. We stress that some correlations between the opinions of individuals and their strength parameters are quite common in sociological data.

The noiseless model results in a whole continuum of asymptotic states corresponding to different minority numbers and determined uniquely by the initial function $n_0(\xi)$. All of these states are marginally stable in the linear sense, since small changes of initial data inevitably change asymptotic behavior.

In order to understand the effects of noise, it is very useful to evoke a physical analogy. The function

$$m-f(m)=\frac{\partial V(m)}{\partial m} ,$$

where $V(m)$ may be regarded as a potential. The dynamics of m from this perspective is a dissipative overdamped motion in the potential $V(m)$. When the function $f(m)$ possesses $2K+1$ fixed points m_1^*, $m_2^*,\dots,m_{2K+1}^*$, the stable fixed points $m_1^*,m_3^*,\dots$ correspond to the local minima of the potential. The unstable fixed points $m_2^*,m_4^*,\dots$ describe locations of the maxima of the potential. Typically, the values of the minima and maxima are descending:

$$V(m_1^*)>V(m_3^*)> \cdots ,$$

$$V(m_2^*)>V(m_4^*)> \cdots .$$

The starting point of the dynamics, $m(0)$, is smaller that m_1^* for $m_0>0$. The noiseless motion corresponds to the approach to the closest minimum of the potential, i.e., m_1^*. This minimum should remain metastable even in the presence of noise.

For uniformly bounded noise h, the minimum of the potential at m_1^* will remain globally stable, provided the noise h is small enough, and cannot carry the system over the energy barrier $V(m_2^*)-V(m_1^*)$. For unbounded noise, such as Gaussian noise, the minimum of the potential should become metastable, but its lifetime should be extremely long. In particular, for Gaussian noise with zero mean and dispersion $\bar{h}$, the lifetime should behave as $\exp\{[V(m_2^*)-V(m_1^*)]/\bar{h}^2\}$. In principle, the system should always end up in the uniform or a nearly uniform minimum that corresponds to the point m_{2K+1}^*. The dynamics of the system will consist of several intermittent steps. We call this kind of dynamics "staircase" dynamics.

A simple explanation of this form of the dynamics can be formulated as follows. In most discussed examples that lead to multiple minima of the potential $V(m)$, one may distinguish several groups of minorities that are characterized by different values of the strength parameters. The staircase dynamics corresponds to successive decays of such groups, starting from the weakest one to the strongest. The first step of the dynamics with $m_1^* \simeq m_0$ may sometimes result from a small value of the initial weighted majority-minority difference.

In the next section we shall show more rigorously that this simple physical explanation of the results is quite appropriate.

IV. STAIRCASE DYNAMICS IN THE PRESENCE OF NOISE

In order to understand the effects of noise on the dynamics (3.1), we shall first analyze the case of vanishing disorder, when $p(p_j,s_j)$ becomes a product of Dirac's δ functions centered at p and s. In such a situation all individuals are characterized by a single self-supportiveness that we denote by a. In this case there is no point in introducing the function $n(\xi)$. m itself is then an appropriate order parameter and fulfills an equation analogous to Eq. (3.12), in which the right-hand side (RHS) is a function of m only and does not depend on $n_0(\xi)$. The dynamics of m is uniquely determined by its initial value m_0. In the noiseless case there are two kinds of asymptotic states: marginally stable states with $|m|<a$ and stable states with $|m|=1$.

We shall consider below two kinds of white noise: uniform noise that acts in the same way on every individual and site-independent noise.

In the presence of uniform white noise denoted by h,

$$m' = \mathrm{sgn}(m+h)\Theta(|m+h|-a) + m\Theta(a-|m+h|) . \tag{4.1}$$

m does not self-average over the different realizations of h, and Eq. (4.1) must be considered as a stochastic map. Direct inspection of this map shows that when we start with $0<m<a$, m will remain constant, provided the noise h_k will never exceed the value $a-m$ at any time steps $k=1,2,3,\ldots$. For uniformly bounded noise such that $h_k \le \delta$, this will happen for sufficiently small δ. For unbounded noise, such as Gaussian noise, the noise will necessarily exceed the value of $a-m$ in the course of time. If we take Gaussian noise with mean zero and variance δ, however, in the limit of small δ the probability that it remains smaller than $a-m$ is close to 1. The lifetime of the state characterized by $m<a$ is proportional to the inverse of the probability of noise exceeding $a-m$, which, in turn, is given by

$$p(h>a-m) \propto \exp[-(|m|-a)^2/2\delta^2] , \tag{4.2}$$

i.e., exponentially small. We may therefore expect that m will remain constant for such a long number of steps, and then finally it will jump and reach the value 1. Of course, the noise may also take negative values and become $h<-m-a$, so that m in such a case will attain the value -1. The probability of such an event, however, is much smaller than the one given by Eq. (4.2). Also, in principle, m may jump back from the value 1 to some value $m'<a$, provided the noise becomes negative and fulfills $h<a-1$. Again, the probability of such an event will be exponentially small in the small-δ limit:

$$p(h<a-1) \propto \exp[-(1-a)^2/2\delta^2] . \tag{4.3}$$

In the asymptotic limit we shall observe a balance of jumps between the values of $m=1$ and some $m<a$. Each of the actual states will have an exponentially long lifetime. The reason for such behavior is clear: The noise must compete with or cooperate with self-supportiveness a in order to induce a change of state. Sufficient numbers of individuals must change their opinion under the actual influence of noise to lead to a global change of the state of the system.

Similar conclusions may be drawn in the case of site-independent noise. We expect that the global effects of noise in this case will be even weaker, since each of the individuals is affected by a different noise value and the global effects of the noise may average to zero.

In the presence of site-dependent white noise h_i, the weighted majority-minority difference is a self-averaging quantity, and the random map is replaced by its averaged version,

$$m' = \langle \mathrm{sgn}(m+h)\Theta(|m+h|-a) + m\Theta(a-|m+h|) \rangle , \tag{4.4}$$

where the brackets $\langle\ \rangle$ denote that we now average over h. Since m is self-averaged, this average is performed with respect to the explicit h dependence of the RHS of Eq. (4.4) only. In the case of uniformly bounded noise, such that for every realization of the random variable h we have $h<\delta$, some of the marginally stable states with $m<a-\delta$ remain stable. For larger values of m, the function on the RHS of Eq. (4.4) will grow. Eventually, it will saturate and exhibit a stable fixed point for $m \simeq 1$.

In the case of Gaussian noise, all marginally stable states become unstable. The function on the right-hand side has a characteristic sigmoid shape. It is worth noting that the rates $\lambda(m)$ that describe the departure from the originally marginal fixed points with some $m<a$ are extremely small for low noise levels. For Gaussian white noise, we may calculate them by rewriting Eq. (4.4) in the form

$$m' = m[1+\lambda(m)] , \tag{4.5}$$

so that we easily estimate

$$\lambda(m) \simeq \frac{1-m}{m(2\pi\delta^2)^{1/2}} \exp[-(|m|-a)^2/2\delta^2] . \tag{4.6}$$

The averaged dynamics will therefore consist of very slow, but monotonic growth of m for $|m|<a$. Such intermittent behavior [34] will be followed by a fast decay toward $|m| \simeq 1$, as soon as $|m|$ reaches a. For obvious reasons we call such behavior "staircase" dynamics with a single step. It should be stressed, however, that the introduction of disorder and individual differences will add additional complexity to the dynamics. For instance, the dynamics in the presence of disorder will typically start with some period of fast decay and then become interrupted by a series of very long periods of apparent stationarity.

Let us now consider the disordered case and start again with a discussion of uniform white noise h. In this case the order parameter $n(\xi)$ does not self-average over the realizations of the noise h. On the other hand, mean-field theory in the sense of self-averaging over individuals is still valid, and $n(\xi)$ is self-averaged with respect to the random strength parameters.

In the presence of noise, Eq. (3.6) becomes a stochastic functional map and takes the form

$$n'(\xi) = [g(m+h,\xi) + n(|m+h|)]\Theta(|m+h|-\xi) + n(\xi)\Theta(\xi-|m+h|) . \tag{4.7}$$

We shall consider below a generic situation, in which in the absence of noise the function $f(m)$ of Eq. (3.13) has three positive fixed points: two stable ones m_1^*, m_3^* and an unstable one m_2^*, with $m_1^* < m_2^* < m_3^*$.

We shall look at the behavior of $n(\xi)$ in the vicinity of the first fixed point. As we shall see below, without losing generality, we may assume that at the time instant $t=0$ the system approached some state $n(\xi, t=0)$ and that the noise was turned on in the following time steps.

At the first time instant $t=0$, the function $n(\xi)$ can therefore be represented by the expression

$$n(\xi,t=0)=[g(\tilde{m},\xi)+n_0(\tilde{m})]\Theta(\tilde{m}-\xi)+n_0(\xi)\Theta(\xi-\tilde{m}) , \tag{4.8}$$

where $n_0(\xi)$ describes the state of the system in the remote past, whereas $\tilde{m}$ is a parameter that fulfills

$$n_0(0)=m_0\leq\tilde{m}\leq m_1^* . \tag{4.9}$$

The weighted majority-minority difference in this state is given by $m(t=0)=f(\tilde{m})=g(\tilde{m},0)+n_0(\tilde{m})$. It is elementary to check the following.

(a) For small h such that $m+h\leq\tilde{m}$, the function $n(\xi)$ at $t=1$ will remain unchanged,

$$n(\xi)=[g(\tilde{m},0)+n_0(\tilde{m})]\Theta(\tilde{m}-\xi)+n_0(\xi)\Theta(\xi-\tilde{m}) , \tag{4.10a}$$

and, consequently, the value of the weighted majority-minority difference will be

$$m=n(0)=f(\tilde{m}) . \tag{4.10b}$$

(b) For small h that $m+h\geq\tilde{m}$, the function $n(\xi)$ at $t=1$ will become

$$n(\xi)=[g(m+h,0)+n_0(m+h)]\Theta(m+h-\xi)+n_0(\xi)\Theta(\xi-m-h) , \tag{4.10c}$$

and, consequently, the value of the weighted majority-minority difference will be

$$m=n(0)=m(t=0)+h . \tag{4.10d}$$

As we see, the form of the function remains unchanged and only the parameter $\tilde{m}$ has been replaced by $m(t=0)+h$. For small uniform noise, we can exclude the possibility that m will change sign as a result of the action of large negative noise, similarly, the probability of such change will ne negligible for Gaussian noise, provided $m(t=0)$ is sufficiently larger than zero. That means that the change of $\tilde{m}$ in Eqs. (4.10) determines the whole evolution of $n(\xi)$. The stochastic equation for $\tilde{m}$ reads

$$\tilde{m}'=[f(\tilde{m})+h]\Theta(f(\tilde{m})+h-\tilde{m})+\tilde{m}\Theta(\tilde{m}-f(\tilde{m})-h) . \tag{4.11}$$

Note that, for positive $\tilde{m}$, positive noise is the most destabilizing. It is easy to see that $\tilde{m}$ may only grow. This growth may be, however, limited for uniformly bounded noise $|h|\leq\delta$. The growth of $\tilde{m}$ is limited to those regions of $\tilde{m}$ where

$$\tilde{m}-f(\tilde{m})\leq\delta . \tag{4.12}$$

This condition divides the set of $\tilde{m}$ into two separate subsets, provided $\max_{m_1^*\leq\tilde{m}\leq m_2^*}[\tilde{m}-f(\tilde{m})]=E\geq\delta$. Let us denote the boundaries of these sets by $\tilde{m}_1$ and $\tilde{m}_2$. When $\tilde{m}_2\geq\tilde{m}_1+\delta$, $\tilde{m}$ cannot jump from a value $\tilde{m}\leq\tilde{m}_1$ to some value $\tilde{m}\geq\tilde{m}_2$, since

$$\tilde{m}'\leq f(\tilde{m})+h\leq\tilde{m}_1+\delta<\tilde{m}_2 . \tag{4.13}$$

This means that for small uniformly bounded noise, $\tilde{m}$ will remain in the vicinity of m_1^* forever. Similar considerations apply to the case of Gaussian noise with variance δ. The probability of jumping from one region to another will decrease exponentially with the size of the noise. In this case $\tilde{m}$ will remain fluctuating around m_1^* only a finite time T and jump to the next stable point m_3^*. The time T in the limit $\delta\to 0$ may for any practical purposes be considered infinitely long.

The case of site-dependent noise requires more complicated calculations. The function $n(\xi)$ self-averages over the noise, and the functional map (4.7) has to be replaced by

$$n'(\xi)=\langle[g(m+h,\xi)+n(|m+h|)]\Theta(|m+h|-\xi)+n(\xi)\Theta(\xi-|m+h|)\rangle , \tag{4.14}$$

where the average over the noise $\langle\ \rangle$ applies to the explicit dependence of h on the RHS of Eq. (4.14). We assume again that the noise was turned on at $t=1$ and that at $t=0$ the function $n(\xi)$ had the same form as in Eq. (4.8).

Since, for any particular realization with sufficiently small noise, the form of the function $n(\xi)$ remains unchanged, it is reasonable to look for the solution of Eq. (4.14) in the form

$$n(\xi,t=0)=\int d\tilde{m}\,\rho(\tilde{m})\{[g(\tilde{m},\xi)+n_0(\tilde{m})]\Theta(\tilde{m}-\xi)+n_0(\xi)\Theta(\xi-\tilde{m})\} , \tag{4.15}$$

where the function $\rho(\tilde{m})$ is the normalized density,

$$\int_0^\infty d\tilde{m}\,\rho(\tilde{m})=1 . \tag{4.16}$$

The weighted majority-minority difference is for $n(\xi)$ given by Eq. (4.15) equal to

$$m_\rho=\int_0^\infty d\tilde{m}\,\rho(\tilde{m})f(\tilde{m})=\int_0^\infty d\tilde{m}\,\rho(\tilde{m})[g(\tilde{m},0)+n_0(\tilde{m})] , \tag{4.17}$$

and depends functionally on the density ρ.

Substituting the ansatz (4.15) into Eq. (4.14), we obtain after some algebra the exact evolution equation for the density in the form

$$\rho'(\tilde{m})=p(\tilde{m}-m_\rho)\int_0^{\tilde{m}}d\tilde{m}'\,\rho(\tilde{m}')+\int^{\tilde{m}-m_\rho}dh\,p(h)\rho(\tilde{m}) . \tag{4.18}$$

This equation is valid, provided the noise, as well as the support of the function $\rho(\tilde{m})$, has appropriate lower bounds, so that the evolution cannot lead to any sign changes of the m_ρ, i.e., $\mathrm{sgn}(m_\rho+h)=\mathrm{sgn}(\tilde{m})=1$ for all h and $\tilde{m}$. Denoting $\Delta(\tilde{m})=\int^{\tilde{m}}dm\,\rho(m)$, $E(h)=\int^h dh'\,p(h)$, we can rewrite Eq. (4.18) in the compact form

$$\Delta'(\tilde{m})=E(m-m_\rho)\Delta(\tilde{m}) . \tag{4.19}$$

It is easy to see that the only stable asymptotic solution of this equation corresponds to the situation when $\Delta(\tilde{m})=0$, for all $\tilde{m}$ such that $E(\tilde{m}-m_\rho)<1$. For noise that is unbounded from above, this implies the solution $\Delta(\tilde{m})=\Theta(\tilde{m}-\tilde{m}^*)$, with $\tilde{m}^*\to\infty$, or

$$\rho(\tilde{m})=\lim_{\tilde{m}^*\to\infty}\delta(\tilde{m}-\tilde{m}^*)\ . \tag{4.20}$$

Note that the asymptotic solution corresponds to $m_\rho=1$ since $\lim_{\tilde{m}^*\to\infty}f(\tilde{m}^*)=1$. For the case when the support of $p(h)$ is compact, the support of $\rho(\tilde{m})$ must contain $\tilde{m}$'s that are greater than any of the elements of the support of $p(\tilde{m}-m_\rho)$. An explicit solution of Eq. (4.19) will depend on the form $n_0(\xi)$ that enters the definition of m_ρ.

For the case of Gaussian noise, Eq. (4.19) is an approximation only, since it excludes the possibility of a change in the sign of $\tilde{m}$ for any of the realizations of $\tilde{m}$ and h. The contribution of such sign changes, however, will typically be exponentially small, as we discussed in the first part of this section.

Equation (4.19) does describes the staircase dynamics. If we start the evolution with a well-localized distribution $\rho(\tilde{m})$ peaked at some $m<m_1^*$, it will shift to the position of $m_\rho\simeq f(m)$ and slightly diffuse in the next step. On the other hand, a wide distribution $\rho(\tilde{m})$ will initially reduce its width to the size of the order of the variance of the noise and move toward m_1^*. After this happens, a slow buildup of the distribution for $\tilde{m}>m_1^*$ begins. This happens on a time scale that increases exponentially with the decrease of the noise variance. As soon as the sufficient part $\rho(\tilde{m})$ is concentrated for $\tilde{m}>m_2^*$ so that m_ρ becomes greater than m_2^*, the distribution will rapidly shift toward m_3^* and shrink to the size of the noise variance.

Despite the simplicity of the final state in the presence of noise, the discussion in this section leads to the conclusion that all of the considered models exhibit a complex staircase dynamics with multiple steps. It is a general type of dynamics, which characterizes decay in a wide class of complex systems.

V. MODELS WITH HIERARCHICAL GEOMETRY

The theory presented in previous sections may easily be generalized for the case of hierarchical models or geometries, in which individuals are divided into groups, groups into subgroups, and so forth. Within each level of the hierarchy, the interactions among all pairs of members of the same group are uniform, but different in magnitude from those among members of different groups. In other words, the interactions decrease with distance, which in hierarchical models is ultrametric. The distance between individuals i and j may be, for instance, defined as a decreasing function of the lowest hierarchy level in which both i and j belong to the same group.

In this section we briefly consider the simplest version of such a hierarchy, in which N individuals are divided into p groups of equal, extensive size. Since the methods used are direct analogs of those presented in detail in Secs. III and IV, we sketch only the results.

We consider again models of the class described in Sec. II. We assume once more that the function that characterizes self-interactions and self-supportiveness is $g(0)=1/\beta$ and does not depend on the group to which a given individual belongs. On the other hand, we set $g(d_{ij})=N/p$, if i and j belong to the same group, while $g(d_{ij})=N/\alpha p$, if i and j belong to different groups. In this way we allow for nontrivial competition between self-supportiveness and social impact.

In the following we shall enumerate groups with lowercase Greek letters ν, μ, etc. Individuals within groups will be enumerated with lowercase Roman letters j, i, etc. We shall denote the state of the ith individual in the νth group as σ_i^ν and his strength parameters as s_i^ν and p_i^ν. We assume that the statistical properties of the system are uniform.

For each of the νth groups, $\nu=1,\dots,p$, we introduce the weighted majority-minority differences

$$m_\nu=\frac{p}{N}\sum_{j\in\nu}\frac{s_j^\nu+p_j^\nu}{s+p}\sigma_j^\nu\ . \tag{5.1}$$

These parameters characterize each of the groups separately. It turns out that it is useful to introduce also additional parameters

$$M_\nu=m_\nu+\alpha\sum_{\nu'\neq\nu}m_{\nu'}\ . \tag{5.2}$$

These parameters, in turn, provide a weighted average of the weighted majority-minority difference.

The noiseless dynamics may be fully described by introducing a set of functions $n_\nu(\xi)$, $\nu=1,\dots,p$, as an order parameter

$$n_\nu(\xi)=\left\langle\frac{p}{N}\sum_{j\in\nu}\frac{s_j^\nu+p_j^\nu}{s+p}\sigma_j^\nu\Theta(a_j^\nu-\xi)\right\rangle\ , \tag{5.3}$$

where $a_j^\nu=[\beta s_j+(s-p)(1+\alpha p+\alpha)]/(s+p)$. The order parameters fulfill the equations

$$n'_\nu(\xi)=[g(M_\nu,\xi)+n_\nu(|M_\nu|)]\Theta(|M_\nu|-\xi)+n_\nu(\xi)\Theta(\xi-|M_\nu|)\ , \tag{5.4}$$

with $n_\nu(0)=m_\nu$.

One can proceed as before in order to obtain an analytic solution of the form of Eq. (3.12) and to show that the dynamics (5.4) will also exhibit a staircase character. This can be done only for a limited set of initial conditions, namely, those that do not lead to any change of sign of any of M_ν. In the asymptotic limit, however, we expect that this will always be the case, so that the method will allow us to identify all of the stationary states of the dynamics (5.4).

Let us assume that at $t=0$ the state of the system is described by the functions $n_{0\nu}(\xi)$. At $t=1$ it takes then the form

$$n_\nu(\xi)=[g(\tilde{M}_\nu,\xi)+n_{0\nu}(|\tilde{M}_\nu|)]\Theta(|\tilde{M}_\nu|-\xi)+n_{0\nu}(\xi)\Theta(\xi-|\tilde{M}_\nu|)\ , \tag{5.5}$$

where $\tilde{M}_\nu=m_{0\nu}+\alpha\sum_{\nu'\neq\nu}m_{0\nu'}$ and $m_{0\nu}=n_{0\nu}(0)$. Using the same argumentation as in Secs. III and IV, it is easy to show that the form of the function $n_\nu(\xi)$ remains unchanged; i.e., at $t=2$,

$$n_\nu(\xi)=[g(\tilde{M}'_\nu,\xi)+n_{0\nu}(|\tilde{M}'_\nu|)]\Theta(|\tilde{M}'_\nu|-\xi) +n_{0\nu}(\xi)\Theta(\xi-|\tilde{M}'_\nu|) , \quad (5.6)$$

where

$$\tilde{M}'_\nu=\left[f_\nu+\alpha\sum_{\nu'\neq\nu} f_{\nu'}(\tilde{M}'_{\nu'})\right] \times\Theta\left[\left|f_\nu+\alpha\sum_{\nu'\neq\nu} f_{\nu'}(\tilde{M}'_{\nu'})\right|-|M_\nu|\right] +M_\nu\Theta\left[|M_\nu|-\left|f_\nu+\alpha\sum_{\nu'\neq\nu} f_{\nu'}(\tilde{M}'_{\nu'})\right|\right] \quad (5.7)$$

and

$$f_\nu(\tilde{M}_\nu)=g(\tilde{M}_\nu,0)+n_{0\nu}(\tilde{M}_\nu) . \quad (5.8)$$

The expression (5.7) is valid provided the parameters $\tilde{M}_\nu$ do not change their signs in the course of dynamics. The actual value of the weighted majority-minority differences are given by

$$m_\nu=f_\nu(\tilde{M}_\nu) , \quad (5.9)$$

and M_ν's are expressed as in formula (5.2).

The dynamics (5.7) is very similar in character to that discussed in the previous sections. Also, one can easily incorporate noise in the theory. There are, however, two additional features.

(a) The dynamics introduces a class of stable stationary states, such that $m_\nu\simeq-1$, for $\nu=1,\ldots,k$, and $m_\nu\simeq1$, for $k<\nu\leq p$. These states are linearly stable in the absence of disorder. For instance, if k fulfills the inequality $(p+1)/2-1/2\alpha<k<(p-1)/2+1/2\alpha$, then such states exist and are stable in the absence of noise, because when m_ν's are close to their asymptotic values, $\mathrm{sgn}(M_\nu)=\mathrm{sgn}(m_\nu)=\mathrm{const}$. From Eq. (5.7) we easily infer as in Sec. IV that $|\tilde{M}'_\nu|\geq|\tilde{M}_\nu|$ so that $|m'_\nu|\geq|m_\nu|$. For the generic form of the function $f_\nu(\ )$, this implies that the asymptotic state $|m_\nu|\simeq1$ must exist. Obviously, this state is stable in the noiseless limit. In the presence of finite but small noise, the corresponding state with $|m_\nu|\simeq1$ can be proven to be linearly stable, using the analog of Eq. (4.19).

(b) There are types of critical phenomena induced by changes of the control parameter α that determines the distance between individuals that belong to different groups.

We illustrate these features by discussion of a simple example where only two groups are present and $p=2$.

Example. The evolution equation for the parameters $\tilde{M}_\nu$ may take for $p=2$ even a simpler form than that of Eq. (5.7). Namely, such simplification happens when we consider two kinds of initial conditions: symmetric ones, with $n_{01}(\xi)=n_{02}(\xi)=n_0(\xi)$, or asymmetric ones, with $n_{01}(\xi)=-n_{02}(\xi)=n_0(\xi)$. Denoting $f(\tilde{M})=g(\tilde{M},0)+n_0(|\tilde{M}|)$, we obtain for symmetric initial conditions $\tilde{M}_1=\tilde{M}_2=\tilde{M}$, where $\tilde{M}$ evolves according to

$$\tilde{M}'=(1+\alpha)f(\tilde{M}) . \quad (5.10)$$

For the generic case, $f(\ )$ has two stable fixed points m_1^* and m_3^*, such that $m_3^*\simeq1$, and one unstable fixed point m_2^*. For small α the evolution equation (5.10) will generically have also three fixed points $m_i^*(\alpha), i=1,2,3$. It will always have at least one stable fixed point $m_3^*(\alpha)\simeq1+\alpha$. The increase of α may, however, destroy the stability of $m_1^*(\alpha)$, leading to intermittency [34] for $\alpha\geq\alpha_c$, where the critical value of α_c is determined from $m_1^*(\alpha_c)=m_2^*(\alpha_c)$. For an asymmetric initial condition, $\tilde{M}_1=-\tilde{M}_2=\tilde{M}$ and

$$\tilde{M}'=(1-\alpha)f(\tilde{M}) . \quad (5.11)$$

In this case, for small α, the dynamics will also exhibit three fixed points $m_1^*(\alpha)$, $m_2^*(\alpha)$, and $m_3^*(\alpha)\simeq1-\alpha$. With increasing α the fixed point $m_1^*(\alpha)$ remains stable, whereas the other two will vanish.

Summarizing models with hierarchical geometry may be solved exactly using the same methods as in the case of fully connected models. Even the simple geometry introduces features, such as the appearance of linearly stable states and geometry-induced critical phenomena.

VI. STRONGLY DILUTED MODELS

In this section we consider yet another type of geometry, or rather social network architecture, namely, strongly diluted networks. The underlying assumption is that at each instant every individual is affected by the impact of K randomly chosen individuals that we call ancestors. The set of direct ancestors of a given individual i changes at each time step. When the set of ancestors is not too large, such a model may be exactly solved with the help of the method of Derrida *et al.* [22–24]. The reason is that K input sites are chosen at random among the total number of N sites. The number of sites that belong to the tree of ancestors of the ith individual at the time step t is $(K+1)^t$ (note that typically a given individual is also his and/or her ancestor). As long as $t\ll\ln N$, all the sites that belong to the tree of ancestors in the limit $N\to\infty$ are different; i.e., there are no feedback loops.

The architecture of strongly diluted networks corresponds to the fact that individual opinions are affected by different individuals at different points in time. The drawback of such models is that they are fully asymmetric; i.e., when the ith person experiences the impact from jth person, the reverse situation has a probability close to zero.

The dynamics of such a model takes a similar form to that described in Eq. (3.1):

$$\sigma'_i=\mathrm{sgn}(m_i)\Theta(|m_i|-|a_i|)+\sigma_i\mathrm{sgn}(a_i)\Theta(|a_i|-|m_i|) . \quad (6.1)$$

The weighted majority-minority difference is now defined locally,

$$m_i=\frac{1}{K}\sum_{j(i)}\frac{s_j\sigma_j}{\bar{s}} , \quad (6.2)$$

where $j(i)=j_1,\ldots,j_K$ denote direct ancestors of the ith individual at the time $t+1$. We use here the normalization $g(d_{ij})=K$ for $i\neq j$, $g(0)=1/\beta$, to allow for the competition of self-supportiveness and social impact. We have also assumed $s_j = p_j$ so that the self-supportiveness parameter is simply

$$a_i = \beta s_i \ . \tag{6.3}$$

The state of the ith individual at time t is a function of the states and strengths of all of its ancestors:

$$\sigma_i = \tilde{\tau}(t, s_i, \{s\}_{\text{anc}}) \ . \tag{6.4}$$

Denoting the right-hand side of Eq. (6.1) by $F(\sigma_{j_1}(t), s_{j_1}, \ldots, \sigma_{j_k}(t), s_{j_K}, \sigma_i(t), s_i)$, we may rewrite Eq. (6.1) in the form

$$\sigma_i'(t+1) = \sum_{\sigma_i = \pm 1} \left[\frac{1 + \sigma_i \tilde{\tau}_i(t)}{2} \right] \sum_{\sigma_{j(i)} = \pm 1} \prod_{j(i)} \left[\frac{1 + \sigma_{j(i)} \tilde{\tau}_{j(i)}(t)}{2} \right] F(\sigma_{j_1}, s_{j_1}, \ldots, \sigma_{j_k}, s_{j_K}, \sigma_i, s_i) \ . \tag{6.5}$$

In the course of evolution, the effects of ancestors and their random strengths will tend to self-average. The quantity of interest is therefore

$$\tau_i(s_i) = \langle \tilde{\tau}(t, s_i, \{s\}_{\text{anc}}) \rangle_{\text{anc}} \ , \tag{6.6}$$

where the average is over the random strength of ancestors and over statistics of initial conditions. Assuming that no initial correlations between the states of different individuals were present at $t=0$ and noting that the trees of ancestors of different individuals are different with probability 1, the average (6.6) can be done independently for each of the factors entering the RHS of Eq. (6.5). We obtain

$$\sigma_i'(t+1) = \left\langle \sum_{\sigma_i = \pm 1} \left[\frac{1 + \sigma_i \tau_i(s_i)}{2} \right] \sum_{\sigma_{j(i)} = \pm 1} \prod_{j(i)} \left[\frac{1 + \sigma_{j(i)} \tau_{j(i)}(s_{j(i)})}{2} \right] F(\sigma_{j_1}, s_{j_1}, \ldots, \sigma_{j_k}, s_{j_K}, \sigma_i, s_i) \right\rangle_{s_{j(i)}} , \tag{6.7}$$

where $\langle \ \rangle$ denotes now only the average over the strengths of the direct ancestors of the ith individual at $t+1$.

Equation (6.7) may be written in the compact form

$$\tau_i'(s_i) = \langle \operatorname{sgn}(m_i) \Theta(|m_i| - |a_i|) + \tau_i(s_i) \operatorname{sgn}(a_i) \Theta(|a_i| - |m_i|) \rangle_{m_i} \ , \tag{6.8}$$

where the brackets denote the average over the random variable

$$m_i = \frac{1}{K} \sum_{j(i)} \frac{s_{j(i)} \sigma_{j(i)}}{\bar{s}} \ , \tag{6.9a}$$

which is the sum of statistically independent variables that are distributed according to

$$p(\sigma_j, s_j) = \left[\frac{1 + \sigma_j(s_j)}{2} \right] p(s_j) \ . \tag{6.9b}$$

When initial conditions $\sigma_i(0)$ are statistically independent and uniform [i.e., the probability distributions of $\sigma_i(0)$ are independent, but equal for different sites], the solution of Eq. (6.8) remains uniform at any time, i.e., $\tau_i(s_i) = \tau(s_i)$. The distribution of m_i becomes then also site independent, so that if we denote $m_i = \tilde{m}$,

$$\tau'(s_i) = \langle \operatorname{sgn}(\tilde{m}) \Theta(|\tilde{m}| - |a_i|) + \tau(s_i) \operatorname{sgn}(a_i) \Theta(|a_i| - \tilde{m}|) \rangle_{\tilde{m}} \ . \tag{6.10}$$

Introducing the order parameter

$$n(\xi) = \left\langle \frac{1}{K} \sum_{j=1}^{K} \frac{s_j \tau_j(s_j)}{\bar{s}} \Theta(a_i - \xi) \right\rangle_s \ , \tag{6.11}$$

we obtain easily the analog of Eq. (3.6) or (4.14),

$$n'(\xi) = \langle [g(\tilde{m}, \xi) + n(|\tilde{m}|)] \Theta(|\tilde{m}| - \xi) + n(\xi) \Theta(\xi - |\tilde{m}|) \rangle_{\tilde{m}} \ , \tag{6.12}$$

where the random variable $\tilde{m}$ is distributed in accordance with Eqs. (6.9), with $\tau_j(s_j) = \tau(s_j)$.

Denoting the mean value of $\tilde{m}$ as m and writing

$$\tilde{m} = m + h \ , \tag{6.13}$$

we immediately see that

$$n'(\xi) = \langle [g(m+h, \xi) + n(|m+h|)] \Theta(|m+h| - \xi) + n(\xi) \Theta(\xi - |m+h|) \rangle_h \ , \tag{6.14a}$$

and

$$n(0) = m = \left\langle \frac{1}{K} \sum_{j=1}^{K} \frac{s_j \tau(s_j)}{\bar{s}} \right\rangle_{s_j} \ . \tag{6.14b}$$

The random variable h is defined by

$$h = \frac{1}{K} \sum_{j=1}^{K} \left[\frac{s_j \sigma_j}{\bar{s}} - \left\langle \frac{s_j \tau(s_j)}{\bar{s}} \right\rangle_{s_j} \right] , \tag{6.15}$$

and its distribution is uniquely determined by the distributions $p(\sigma_j, s_j)$ given by expression (6.9).

Note that h is a sum of statistically independent random variables with mean zero. In the limit of large K, h may be well approximated by a Gaussian random variable with variance

$$\langle h^2 \rangle = \frac{1}{K} \int ds [(s^2/\bar{s}^2) - m^2] p(s) \ . \tag{6.16}$$

In this limit the dynamics (6.14) is the same as that obtained for the fully connected model with site-independent Gaussian noise [Eq. (4.14)]. The only difference is that the noise level (6.16) depends this time on the actual configuration of the system. This dependence, however, is relatively weak, since $0 \le |m| \le 1$ and

$$\left\langle \frac{1}{K}\frac{(s-\bar{s})^2}{\bar{s}^2}\right\rangle \le \langle h^2\rangle \le \left\langle \frac{1}{K}\frac{s^2}{\bar{s}^2}\right\rangle . \tag{6.17}$$

The theory of Sec. IV may be directly applied to solve Eq. (6.14). The dynamics described by (6.14) will again exhibit a staircase character. The time scale of jumps between the steps depends on the noise level and is governed by expression (6.16). This time scale increases to infinity when the number of direct ancestors $K \to \infty$.

Note that the central limit argument that we used in the case of large K does not apply when K is moderate or small. In this case one has to use Eq. (6.10) for $\tau(s)$ with the random variable $\tilde{m}$ defined by Eqs. (6.9). The statistical properties of $\tilde{m}$ depend then functionally on $\tau(s)$ in a self-consistent manner. Such a situation is well known in the physics of disordered media [26,35].

Summarizing, strongly diluted models are equivalent to fully connected models in the presence of site-independent noise h. The noise properties, however, depend self-consistently on the actual state of the system. Dilution will tend to destroy minority groups and to speed up the staircase dynamics.

VII. MODELS WITH EUCLIDIAN GEOMETRY

It is interesting to see whether the typical features of the infinite-range and hierarchical models can be also be observed for long- and moderate-range models [15,16,32]. For such models the dynamics are

$$\sigma_i' = \mathrm{sgn}(m_i)\Theta(|m_i|-|a_i|)+\sigma_i \mathrm{sgn}(a_i)\Theta(|a_i|-|m_i|) , \tag{7.1}$$

where both m_i and a_i are defined locally. For example,

$$m_i = \sum_{j\neq i}\frac{(s_j+p_j)\sigma_j}{(s+p)g(d_{ij})} . \tag{7.2}$$

The mean-field approximation corresponds now to the assumption that in Eq. (7.1) the actual value of m_i may be substituted by a mean of m_i with respect to the disorder. That is evidently an approximation. One should stress, however, that in the numerical simulations, algebraically decaying and moderate-range interactions were assumed [15,16]. For such slowly decaying interactions, one may expect that mean-field theory will provide a relatively good approximation of the dynamics.

It is convenient to introduce a field-theoretical description of the model, in which we substitute an integral over two-dimensional Euclidian space R^2 for the sum over individuals j, with an appropriate factor describing the density of individuals. Let $x \in R^2$ and $n(x,[\xi])$ be a functional

$$n(x,[\xi]) = \left\langle \int D_2 x' \frac{s(x')+p(x')}{(s+p)g(x-x')}\Theta(a(x')-\xi(x'))\right\rangle . \tag{7.3}$$

In the mean-field limit, it fulfills an equation analogous to Eq. (3.6),

$$n'(x,[\xi]) = g(x,[m,\xi]) + n(x,[\tfrac{1}{2}(\xi+|m|+|\xi-|m||)]) , \tag{7.4}$$

where $n(x,[0])=m(x)$, whereas $g(x,[m,\xi])$ is a local, functional analog of the function $g(m,\xi)$ defined by expression (3.7):

$$g(x,[m,\xi]) = \left\langle \int D_2 x' \frac{[s(x')+p(x')]\mathrm{sgn}[m(x)]}{(s+p)g(x-x')}\Theta(|m(x')|-a(x'))\Theta(a(x')-\xi(x'))\right\rangle . \tag{7.5}$$

Apparently, Eq. (7.5) has a very complicated form and does not seem to have any advantage over the direct description of the dynamics given by Eq. (7.1). However, just as before, we can prove that if $\mathrm{sgn}[m(x)]=\mathrm{const}$ for all x, then $|m'(x)| \ge |m(x)|$. The functional analog of map (3.12) may be then easily derived:

$$m'(x) = g(x,[m]) + n_0(x,[|m|]) . \tag{7.6}$$

Now it is evident that the advantage of mean-field theory lies in the fact that we reduced the full dynamics described by the functional equation with disorder (7.1) to the averaged functional equation (7.6). The functional equation for $m(x)$ holds, provided $m(x)$ does not change its sign. This may happen, for instance, if we are close to uniformity, and $\mathrm{sgn}[m(x)]=\mathrm{const}$. In this way we show that for such initial states the dynamics will have a staircase character in the presence of small amounts of noise. In general, when $\mathrm{sgn}[m(x)] \neq \mathrm{const}$, we can only conclude that the dynamics will have at least one intermittent step, if at some point $\mathrm{sgn}[m(x)]$ becomes constant, or it will remain trapped for very long times with nonconstant $\mathrm{sgn}[m(x)]$. In such situations correlations between the actual configuration of the system $\sigma(x)$ and the strength parameters arise. The results of computer simulations [15] indicate clearly that the latter possibility is, in fact, typical, and even in the presence of small noise such correlations build up in the course of evolution. We illustrate these findings by discussing two explicit examples.

Example A. We consider the case of finite-range interactions, when $g(x)=g$ for $|x|<R$ and $g(x)=\infty$ otherwise. Let $p(x)=s(x)$, and the probability distribution is $p(s(x))=1/2\tilde{s}$ for $s(x)\in[\bar{s}-\tilde{s},\bar{s}+\tilde{s}]$ and zero otherwise. Let also $\sigma(x)=-1$ for $|x|\le r$, $\sigma(x)=1$ for $|x|>r$, with $0\le r<R/\sqrt{2}$. It is easy to check then that

$$m(x)=m_0=\pi\frac{R^2-2r^2}{g}\geq 0 \qquad (7.7a)$$

for $|x|<R-r$. $m(x)$ grows for larger $|x|$ and becomes

$$m(x)=\pi\frac{R^2}{g} \qquad (7.7b)$$

for $|x|\geq R+r$. Since from Eq. (7.6) we infer that

$$m'(x)=m(x)+\left\langle\int D_2x'\frac{1-\sigma(x')}{g(x-x')}s(x')\Theta(m(x')-a(x'))\right\rangle , \qquad (7.8)$$

the configuration $\sigma(x)$ will be stationary, provided

$$\left\langle\int_{|x'|<r} D_2x'\frac{2s(x')}{g(x-x')}\Theta(m(x')-a(x'))\right\rangle=0 . \qquad (7.9)$$

The sufficient condition for Eq. (7.9) to hold is

$$m_0<\beta(\bar{s}-\tilde{s}) . \qquad (7.10)$$

Note that if condition (7.9) holds, a destabilization of the considered configuration would require a change of m_0 of the order of $\beta(\bar{s}-\tilde{s})-m_0$. Such change in the presence of small noise will have an exponentially small probability.

Example B. In general, sgn$[m(x)]\neq$const. As numerical examples show, $\sigma(x)$ usually remains trapped in the configuration with regions of positive and negative $m(x)$. Although for such stationary configurations the sign of $m(x)$ varies locally, Eq. (7.6) still allows them to be determined, since the sign of $m(x)$ remains at least constant in time. For instance, for the uniform strength distribution $p(x)=s(x)$, $p(s(x))=1/2\bar{s}$, for $s(x)\in[0,2\bar{s}]$, the necessary condition for stationarity is again

$$\left\langle\int D_2x'\frac{\text{sgn}[m(x')]-\sigma(x')}{g(x-x')}s(x')\times\Theta(m(x')-a(x'))\right\rangle=0 . \qquad (7.11)$$

For this condition to be fulfilled usually requires local correlations of $\sigma(x)$ with the strength parameters. Condition (7.11) is fulfilled if

$$\sigma(x)=\text{sgn}[m(x)] \qquad (7.12)$$

for $|m(x)|\geq a(x)$ and for arbitrary $\sigma(x)$ for $|m(x)|<a(x)$. That leads to the nonlinear integral equation, for the stationary $m(x)$,

$$m(x)=\left\langle\int D_2x'\left[\frac{\text{sgn}m(x')}{g(x-x')}s(x')\Theta(|m(x')|-a(x'))+\frac{\sigma(x')}{g(x-x')}s(x')\Theta(a(x')-|m(x')|)\right]\right\rangle. \qquad (7.13)$$

This equation has many solutions. In particular, we may assume cylindrical symmetry and let sgn$[m(x)]=-1$ for $|x|<r$, sgn$[m(x)]=-1$ otherwise, with some $r\gg R$. Let us assume the same $g(x)$ as in Example A and denote

$$r(x)=\int D_2x'\frac{\text{sgn}[m(x')]s(x')}{g(x-x')} . \qquad (7.14)$$

This function takes the value

$$r(x)=-\pi\frac{R^2}{g} \qquad (7.15a)$$

for $|x|<r-R$. $f(x)$ is cylindrically symmetric, and it grows for larger $|x|$. Eventually, it becomes

$$r(x)=\pi\frac{R^2}{g} \qquad (7.15b)$$

for $|x|\geq R+r$. If $\pi R^2/g>2\beta\bar{s}$, then there exist the two radii r_- and r_+ such that

$$r(x)=\pm 2\beta\bar{s} ,$$

for $|x|=r_\pm$. There exists also r_0 such that $f(x)=0$ for $|x|=r_0$. Obviously, $r_-<r_0<r_+$. Moreover, for convex cluster shapes such as the presently considered circular shape, $r_0\leq r$. From Eq. (7.13) we obtain then

$$m(x)=r(x)+\left\langle\int D_2x'\frac{[\sigma(x')-\text{sgn}m(x')]s(x')}{g(x-x')}\times\Theta(a(x')-|m(x')|)\right\rangle . \qquad (7.16)$$

It is easy to see that this equation is fulfilled for $m(x)=r(x)$, provided we choose $\sigma(x)=\text{sgn}[r(x)]$, i.e., $\sigma(x)=-1$ for $r_-<|x|<r_0$ and $\sigma(x)=1$ for $r_0<|x|<r_+$.

The question of stability and the selection of specific solutions of Eq. (7.13) are very complicated, but at least in simple situations, such as a cylindrically symmetric case, it may be studied with the help of our theory. We leave this problem, as well as the detailed comparison of our theory with numerical simulations, to planned future publications [32].

VIII. CONCLUSIONS

Summarizing, we have formulated the statistical mechanics of a class of models of cellular automata that describe the dynamics of social impact and set a paradigm for the social sciences [36]. The models incorporate intrinsic disorder, but nevertheless allow for analytic solutions. We have formulated a kind of mean-field theory that can be applied to systems in which interactions are separable functions of random strength parameters of the interacting elements. The dynamics is most conveniently described by order parameters that characterize correlations between ordering among the elements of the system and their random strength. Generically, in the presence of small noise, such dynamics exhibit "staircase" behavior, which consists of several intermittent steps, corresponding to increased ordering among the groups of elements that have weaker strength parameters. The time scale of approaching these successive steps increases rapidly with decreasing noise level. We presented analytic results for a variety of simple geometries and network architectures. Our results explain qualitatively the results obtained in numerical simulations.

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