# Probability distributions of directed polymers in (1 + 1)-dimensional random media

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In this paper we study directed polymers in $(1+1)$-dimensional random media with special emphasis on $P(x,t)$, the probability distribution to reach a transverse distance $x$ for a polymer of length $t$. $P(x,t)$ follows the scaling form $P(x,t)\sim\langle x^2(t)\rangle^{-1/2}f(\xi)$, where $\langle x^2(t)\rangle$ is the transverse mean-square displacement, which asymptotically obeys $\langle x^2(t)\rangle\sim t^{4/3}$, and $\xi$ is the scaling variable $\xi=x/\langle x^2(t)\rangle^{1/2}$. The numerical results indicate that $f(\xi)\simeq\exp(-c\xi^{\delta})$, where the exponent $\delta$ evolves with $\xi$ such that $\delta\simeq 2$ for $\xi\to 0$ and $\delta\gtrsim 2.5$ for $\xi\gg 1$, demonstrating an "enhanced" Gaussian behavior. We discuss these results in the context of enhanced diffusion.

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The scaling properties of directed polymers (DP's) in random media [1] have been a matter of intensive study and debate mainly due to possible mapping of DP's on various other problems [2–6]. This seemingly simple model of DP's is hampered by difficulties similar to those encountered in spin glasses [7,8].

The DP problem in $(1+1)$ dimensions is defined by a walk in two dimensions with coordinates $(x,t)$, where the walk is directed in the longitudinal $t$ direction but can fluctuate in the transverse $x$ direction [1]. With no randomness the mean-square displacement describing the transverse fluctuations is simply the expected Brownian behavior $\langle x^2\rangle\sim t$. In a random medium, characterized by uncorrelated Gaussian or white noise, it has been established that $\langle x^2\rangle\sim t^{4/3}$, namely an enhanced-diffusion-type behavior in the transverse direction. Energetic and geometrical modifications of the original DP model have been recently proposed by changing the nature of the noise term [3,9–11] and by extending the transverse motion to fractals [12]. An interesting aspect of DP's in random media is the possibility of observing a transition from enhanced to regular transverse fluctuations as the relative strength of the randomness decreases (strong-to-weak-coupling transition). Such a transition is expected in higher dimensions, $d>3$ [4].

The mean-square displacement $\langle x^2(t)\rangle$ is of course only the second moment of the more informative probability distribution $P(x,t)$. The behavior of $P(x,t)$ should reflect the details of the DP's in the different coupling regimes in ways similar to previous studies of propagators in systems characterized by anomalous diffusion [13,14].

In this paper we concentrate on DP's in $(1+1)$ dimensions, a case that belongs inherently to the strong-coupling regime, and therefore displays enhanced mean-square displacements. We study $P(x,t)$, the probability distribution to reach a transverse distance $x$ for a DP of length $t$, in $(1+1)$ dimensions. We will compare the results with other propagators which correspond to anomalous diffusion [13,14].

The formalism applied for the description of the DP in random media was introduced by Kardar and Zhang [1]. The weight function for a polymer in a random potential field is expressed by the path integral [1]

$$W(x,t)=\int_{(0,0)}^{(x,t)}\mathcal{D}x'(t')\exp\left\{-\int_0^t dt'\left[(1/4D)\left[\frac{dx'}{dt'}\right]^2+\eta(x',t')\right]\right\}, \tag{1}$$

where $W(x,t)$ denotes the weight of all directed polymers joining the points (0,0) and $(x,t)$ and $D$ is a diffusion coefficient. The randomness is given by uncorrelated Gaussian noise with $\delta$-function correlation

$$\langle\eta(x,t)\rangle=0\,,\qquad \langle\eta(x,t)\eta(x',t')\rangle=\lambda^2\delta(t-t')\delta(x-x')\,. \tag{2}$$

The path-integral expression has an equivalent partial-differential-equation representation:

$$\dot{W}=[D\nabla^2+\eta(x,t)]W\,, \tag{3}$$

subject to the initial condition $W(x,0)=\delta(x)$. Without the noise term we immediately recognize the ordinary diffusion equation with the solution $W(x,t)\sim t^{-1/2}\exp(-x^2/4Dt)$, which corresponds to $\langle x^2(t)\rangle\sim t$. In the presence of noise the weight function is more complicated and depends on the noise configuration.

Here we have to calculate the configurational average

$$P(x,t)=\left\langle W(x,t)\Big/\int dx\, W(x,t)\right\rangle . \tag{4}$$

It has been suggested that $P(x,t)$ follows the scaling form [1,15]

$$P(x,t)\sim t^{-2/3}f(x/t^{2/3}) , \tag{5}$$

which yields the enhanced transverse mean-square displacement $\langle x^2(t)\rangle \sim t^{4/3}$ derived through

$$\langle x^2(t)\rangle=\int dx\, x^2P(x,t). \tag{6}$$

We solved Eq. (1) using a discretized version and the transfer-matrix method discussed by Kardar [16] and used by Kardar and Zhang [1]. This method is described by the following recursion relation:

$$W(x,t)=e^{\eta(x,t)}\{W(x,t-1)+e^{-1/4}[W(x-1,t-1)+W(x+1,t-1)]\} , \tag{7}$$

where we set $D=1$. The values of $\eta$ were randomly chosen from a Gaussian distribution of width $\lambda$. In our calculations the averaging was taken over $10^4$ realizations.

Figure 1 shows the transverse mean-square displacement for $\lambda=1$. $\langle x^2(t)\rangle$ clearly shows enhanced behavior at long times that follow $t^{4/3}$, as expected [1]. At early times, however, it seems to fit a regular behavior $\langle x^2(t)\rangle\sim t$ and the crossover time depends on the randomness parameter $\lambda$ [12].

The probability distribution $P(x,t)$ was obtained by numerical calculations, using Eqs. (4) and (7) and averaging over $5\times10^5$ realizations. It is presented in Fig. 2 for the randomness parameter $\lambda=1$ and for times $t=10$, 30, 100, and 300 and is plotted vs $\xi$. $\xi$ is the scaling variable $\xi=x/\langle x^2(t)\rangle^{1/2}$. The scaled nature of $P(x,t)$ as a function of $\xi$ is clear as well as the deviation from a regular Gaussian shape. To emphasize the deviations we included in Fig. 2 the corresponding normal distribution as a dashed line.

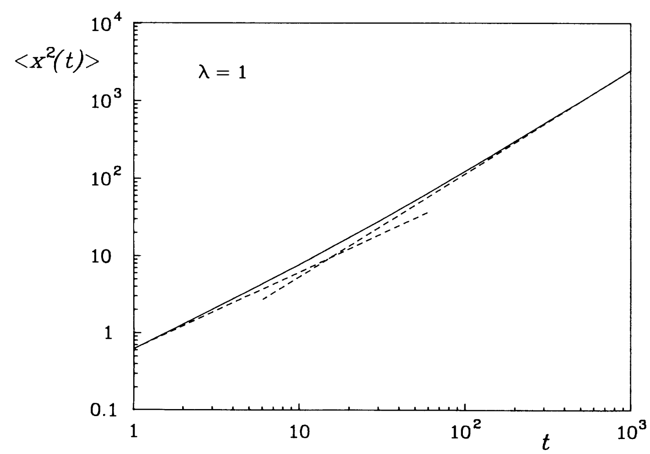


FIG. 1. The mean-squared transverse fluctuations $\langle x^2(t)\rangle$ as a function of time $t$ for dimension $(1+1)$. The solid line gives the simulation results, the dashed lines denote the slopes for the regular and enhanced behavior, 1 and $\frac{4}{3}$, respectively.

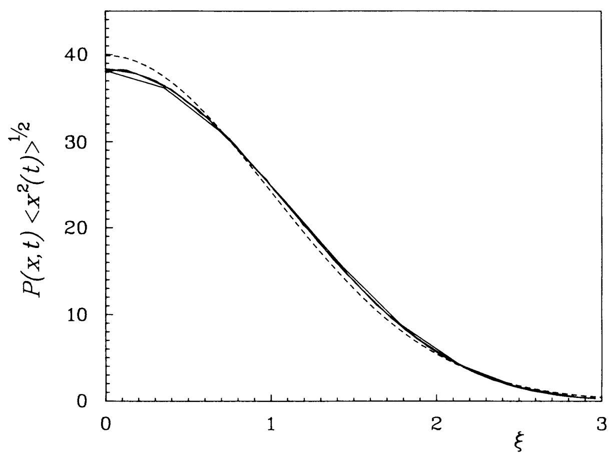


FIG. 2. The probability distribution $P(x,t)$ as a function of the scaling variable $\xi$. Plotted is $P(x,t)\langle x^2(t)\rangle^{1/2}$ for times $t=10$, 30, 100, and 300. The scaling variable is given by $\xi=x/\langle x^2(t)\rangle^{1/2}$. The dashed line denotes the normal distribution.

In order to check in more detail the scaling properties of the probability distribution we plotted in Fig. 3 $-\ln[P(x,t)/P(0,t)]$ vs $x^2/\langle x^2(t)\rangle$. Scaling is obeyed and deviations from Gaussian behavior are again noticeable. We therefore tried the more general case $f(\xi)=\exp(-\xi^\delta)$, typical to anomalous transport problems [13,14]. We found that a single exponent $\delta$ did not fit the whole $\xi$ range. Assuming this form of $f(\xi)$ the results indicate the following limits:

$$f(\xi)\sim\begin{cases}\exp(-c_1\xi^2) & \text{for } \xi\to0\\ \exp(-c_2\xi^\delta), \quad \delta\gtrsim\frac{5}{2} & \text{for } \xi\to\infty .\end{cases} \tag{8}$$

In Fig. 4 we display the value of $\delta$ as a function of $\xi^2$. $\delta$ was calculated from an analytical form fitted to the numerical data. We considered Padé approximants with different polynomial degrees in powers of $\xi^{1/2}$. A con-

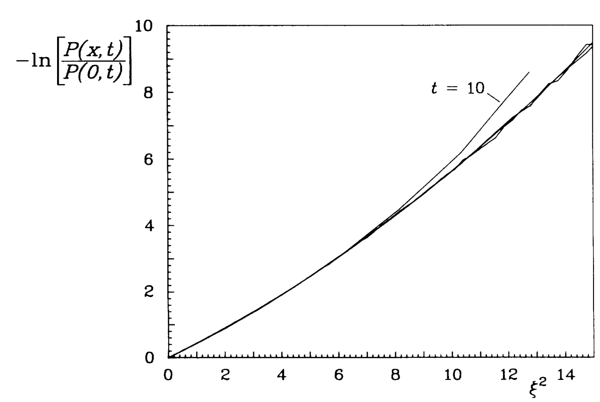


FIG. 3. The probability distribution $P(x,t)$ as a function of the scaling variable $\xi^2$. Plotted is $-\ln[P(x,t)/P(0,t)]$ for times $t=10$, 30, 100, and 300.

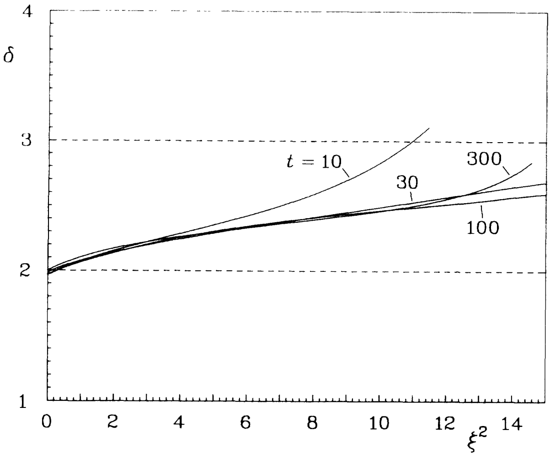


FIG. 4. The exponent δ as a function of the scaling variable $\xi^2$ for times $t=10$, 30, 100, and 300. The dashed lines denote the limiting values 2 and 3, respectively.

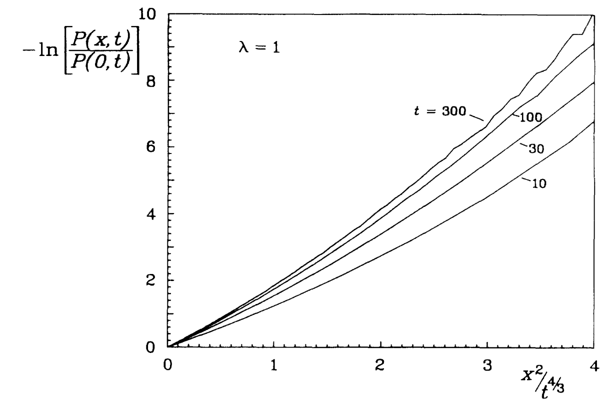


FIG. 5. The probability distribution $P(x,t)$ as a function of the scaling variable $x^2/t^{4/3}$. Same data as in Fig. 3.

tinuous change is observed in δ from a Gaussian ($\delta=2$) to an enhanced Gaussian behavior ($\delta \gtrsim 2.5$). A lower value of δ was obtained in Ref. [15]. Although the data show a slowly increasing trend in the values of δ, they do not attain the predicted asymptotic value of $\delta=3$ [17], which would interestingly fulfill the Fisher shape and size relationship [18]. Nevertheless, it is possible that $\delta=3$ holds asymptotically and that an extension of the scaling function by a power-law prefactor $f(\xi)\sim\xi^{\beta}\exp(-c_2\xi^3)$ fits the numerical results in the intermediate $\xi$ regime. Such prefactors turned out to be crucial in the scaling analyses of $P(x,t)$ for the motion in random velocity fields [14] and for random walks on fractal substrates [19,20]. In fact, considering the power-law prefactor and fitting the exponent $\beta$, we found that the scaling form concurs excellently with the simulation results for $\xi>1$. This tends to support the conjecture that $\delta=3$ [17].

Our results, as displayed in Fig. 3, should be confronted with the scaling properties of $P(x,t)$ when using $x^2/t^{4/3}$ as the scaling variable which relies on the asymptotic behavior of $\langle x^2(t)\rangle$ [15]. We have performed this scaling and show it in Fig. 5. The figure demonstrates that the data collapse is less satisfactory than what follows from the presentation in Fig. 3 for the same range of variables. This is consistent with the slow approach of $\langle x^2(t)\rangle$ to its asymptotic limit as shown in Fig. 1.

Comparing the probability distribution in Eq. (8) to other enhanced-diffusion models [13,14] we find strong differences. In the one-dimensional random-velocity model a particle moves within a layered medium where there is a constant velocity in the longitudinal direction of the layer. Transverse to the layer the particles are assumed to follow diffusional motion. For this model the mean-square displacement follows an enhanced diffusional behavior with $\langle x^2(t)\rangle\sim t^{3/2}$. The propagator $P(x,t)$ has the following limiting behaviors:

$$P(x,t)\sim\begin{cases} t^{-3/4}\exp(-c_1\xi^2) & \text{for } \xi\to 0\\ t^{-3/4}\xi^{5/3}\exp(-c_2\xi^{4/3}) & \text{for } \xi\to\infty\end{cases} \tag{9}$$

with the scaling variable being $\xi=x/t^{3/4}$. We thus observe an ordinary Gaussian behavior in the small-$\xi$ regime and a stretched Gaussian behavior (slower than Gaussian) in the large-$\xi$ regime. As aforementioned, the power-law prefactor of the $\xi\to\infty$ form was found to be important for intermediate $\xi$ regimes.

Finally we mention the behavior found in continuous-time random-walk models with space-time-coupled memories [13]. Here, enhanced diffusion is observed with the exponents depending on the parameters given in the particular models. However, no unique behavior is obeyed for the propagator $P(x,t)$; rather one finds several regimes for the different scaling variables $\xi$.

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