Transition operators in electromagnetic-wave diffraction theory: General theory

G. E. Hahne
Computational Chemistry Branch, NASA Ames Research Center, Moffett Field, California 94035-1000
(Received 22 July 1991; revised manuscript received 8 January 1992)

The objective of this paper is the establishment of a formal theory of the scattering of time-harmonic electromagnetic waves from impenetrable, immobile obstacles, with given linear, homogeneous, and generally nonlocal, boundary conditions of Leontovich (i.e., impedance) type for the wave on the obstacle's surface $\partial\Omega$. As in an analogous treatment of acoustic-wave diffraction by the author [G.E. Hahne, Phys. Rev. A **43**, 976 (1991); **43**, 990 (1991)], the theory is modeled on the theory of the complete Green's function and the transition (T) operator in time-independent formal scattering theory of nonrelativistic quantum mechanics. For each nonzero free-space wave number k_0, the electromagnetic field is described kinematically in terms of a six-component entity comprising the direct sum of the electric and the magnetic three-vector field at each point of position space; an electromagnetic source is described correspondingly as a six-component entity comprising the direct sum of the time-harmonic electric and magnetic current distributions. Accordingly, the Green's function and the T operator are 6×6 matrices of two-point, complex-valued, k_0-dependent functions. A simplified expression is obtained for the T operator for a general case of nonlocal, homogeneous Leontovich boundary conditions for the electromagnetic wave on $\partial\Omega$. Analogous to the acoustic case, all the nonelementary operators that enter the expression for the T operator are formally simple, rational algebraic functions of a certain invertible, linear, k_0-dependent operator $\breve{Z}^{+}_{k_0}$, which is called the radiation impedance operator; $\breve{Z}^{+}_{k_0}$ is an operator of the class that maps the linear space of complex tangent-vector fields on $\partial\Omega$ onto itself. The nonlocal operator $\breve{Z}^{+}_{k_0}$ is defined only implicitly, in that, apart from a simple transformation made for technical reasons, it is the operator that maps the tangential magnetic field on $\partial\Omega$ of an outgoing-wave solution to the source-free Maxwell equations into the uniquely corresponding tangential electric field. The paper concludes with a derivation of an expression for the differential scattering cross section for plane electromagnetic waves in terms of certain matrix elements of the T operator for the obstacle, and a proposal for a class of Leontovich boundary conditions that, if realized, would yield exactly zero scattering amplitude at a given k_0. There are four appendixes: The first appendix recapitulates the theory of the free-space Green's function for the time-harmonic Maxwell field and the relationship of this Green's function to certain linear functional operators defined on the space of tangent-vector fields on the obstacle's boundary. The second appendix establishes mathematical conditions on the defining operators for the Leontovich boundary conditions, which conditions are sufficient to guarantee the uniqueness and existence of the complete Green's function, and reciprocity for the Green's functions of purely outgoing-wave type. The third appendix argues that any of a certain class of time-harmonic, linear electromagnetic scattering problems admits to a partial decoupling into a separate problem for the region interior and the region exterior to a dividing surface; a complete set of interior solutions determines exactly one equivalence class of Leontovich boundary conditions for the exterior electromagnetic field, so that the solution of a scattering problem then reduces to a problem in functional analysis in the space of tangent-vector fields on the dividing surface, as described in the main part of the paper. The fourth appendix describes a linear network analog to the formal scattering theory established herein.

PACS number(s): 42.25.Fx, 03.40.Kf, 03.80.+r, 02.30.+g

I. INTRODUCTION

We shall address the physical and mathematical problems of the scattering of time-harmonic classical electromagnetic waves by a fixed, impenetrable obstacle, such that the electromagnetic field is required to satisfy a given set of boundary conditions of impedance type on the surface of the obstacle. The principal objective is the determination of a formally simple expression for the so-called transition (T) operator associated with the given scattering problem [this entity is defined in Eq. (38)]; once this operator is known—whether analytically or numerically—to sufficient accuracy, computation of the complete Green's function, and in turn the scattering wave functions and scattering (S) matrix, is reduced to quadratures.

The motivation for establishing T-operator theory in this context is the hope that the degree of generality which is inherent in this construct is great enough that it can provide an overview and unification of many existing schemes for the treatment of electromagnetic-wave diffraction problems, while being sufficiently close to the physical and computational worlds that further development of approximation methods for predictive analysis of various types of diffraction problems is facilitated.

The concept of the transition operator in the theory of the complete Green's function for a scattering problem originated in quantum-mechanical collision theory, and is apparently due to Møller [1,2], as stated by Watson [3]. Some further important early papers in the development of what is now called formal scattering theory were those of Lippmann and Schwinger [4], Chew and Wick [5], Chew and Goldberger [6], Gell-Mann and Goldberger [7], and DeWitt [8,9]. Textbook treatments of time-independent formal scattering theory can be found in Ref. [10], Chap. 2.5; in Ref. [11], Chap. 5; in Ref. [12], Chap. 8; and in Ref. [13], Chap. 8. In two previous papers [14,15], the author developed an analogous theory for the scattering of time-harmonic acoustic scalar waves from fixed obstacles, with given surface boundary conditions of impedance (also called Robin) type, that is, given a nonlocal, homogeneous, linear relationship between the surface acoustic overpressure distribution and the surface normal fluid velocity distribution—see Ref. [14], Eq. (15).

It will be shown in what follows here that electromagnetic-wave scattering from obstacles admits an analogous theoretical construction. Note that impedance boundary conditions—called Leontovich boundary conditions in the electromagnetic case—are defined [see Eqs. (35) and (40)] as a nonlocal, homogeneous, linear relationship between the tangential magnetic-field distribution and the tangential electric-field distribution on the obstacle's surface. It is for our purposes both mathematically convenient and physically expedient to deal directly with electric and magnetic fields and currents, so that the duplex, or Heaviside, form of Maxwell's equations will be employed—see, for example, Ref. [16], p. 128 or Ref. [17], Eq. (6.150). Scalar potentials, vector potentials (Ref. [17], Sec. 6.4), and Hertz polarization potentials (Ref. [18], p. 431; Ref. [19], Chap. 14-5; Ref. [20], p. 195) will not be introduced explicitly except for the vector potentials that are employed in the theory of the radiation impedance operator—see Ref. [21], Chap. 2.6, and Sec. V, below.

We shall in this paper take advantage of the circumstance that the scattering obstacles are presumed fixed in spatial position, with time-independent physical properties, by considering only time-harmonic electromagnetic currents and fields. If c is the speed of light in vacuum, the time dependence $\exp(-ik_0ct)$ is to be understood and is dropped; the wave number k_0 can be any nonzero real number. The restriction $k_0 \neq 0$ has the disadvantage that the theory as presented here does not afford a completely developed framework for transforming diffraction problems for fixed obstacles into the time domain; a compensating advantage is that we do not need to deal with the atypical electric and magnetic fields (for example, exterior electric and magnetic monopole fields) associated with static charge and current distributions.

For the mathematical treatment of scattering from moving obstacles it is generally necessary to work in the space-time domain rather than in the space-frequency domain as is done here. A problem of this category entails predicting the results of directing a pulsed electromagnetic signal at an obstacle that moves in a prescribed manner, and that has a given local response (such as electric permittivity and conductivity, magnetic permeability, etc.) or surface response (in the form of a generalization of Leontovich boundary conditions to the three-dimensional manifold that represents the trajectory of the obstacle's surface in space-time) to an electromagnetic field. It appears that the kinematics applied herein of taking three-vector electric and magnetic current densities as the electromagnetic sources does not admit of an efficient generalization to the formal treatment of transient scattering phenomena; a different description of electromagnetic sources and fields, which makes use of polarization densities and Hertz polarization potentials, seems to lend itself more readily to the treatment of at least the formal theory, and possibly also to concrete problems, of describing electromagnetic-wave scattering phenomena when pulsed sources and moving obstacles are involved. This version of electromagnetic-wave scattering theory will be addressed in planned future publications.

The organization of the remainder of this paper roughly parallels that of Ref. [14], and is as follows. In Sec. II we shall specify the kinematical structure that we need to describe the geometry of a generic diffraction problem, time-harmonic electromagnetic fields and currents, and Green's functions. In Sec. III we shall first describe the elementary dynamics of the time-harmonic electromagnetic field, that is, Maxwell's differential field equations and the boundary conditions imposed on the field, and then define certain Green's functions and the T operator associated with the given diffraction problem. In Sec. IV we obtain a representation theorem for Green's functions, and apply the results to the study of the reciprocity property. In Sec. V we shall, based on some existing results that are recapitulated in Appendix A, develop the theory of what will be called the radiation impedance operator $\check{Z}^{+}_{k_0}$ for source-free, outgoing-wave solutions to Maxwell's equations. In Sec. VI we determine an expression for the T operator in terms of (i) elementary operators, (ii) the operators, presumed known, which determine the impedance boundary conditions, and (iii) the radiation impedance operator. Section VII concludes the paper with (i) the derivation of an expression in terms of T-operator matrix elements for the scattering amplitude and the differential cross section for scattering of linearly polarized plane waves into other such waves, and (ii) a demonstration that there exists formally a type of Leontovich boundary conditions such that the realization of these conditions would entail an identically zero scattering amplitude for the given obstacle and wave number. In Appendix A we obtain an explicit expression for the free-space Green's function, and define certain surface-limiting operators derived from this Green's function. The latter operators will be called "primitive" operators herein; these are defined in Ref. [21], p. 63, and are the electromagnetic analogs of the operators defined in Ref. [14], Sec. III A. In Appendix B we establish restrictions on the defining operators for Leontovich boundary conditions that are sufficient to guarantee the existence and uniqueness of a solution to the given electromagnetic-wave scattering problem. In Appendix C we discuss the question of treating the problem of electromagnetic-wave scattering by a geometrically and compositionally complex scattering obstacle by establishing a mathematically equivalent boundary value problem of Leontovich type on a geometrical surface that surrounds the obstacle. Finally, Appendix D shows that the formal theory of scattering from obstacles has an analog in the theory of a class of linear electrical network problems.

There is in preparation a second paper [22] which plans to develop certain elaborations and applications of the general formalism established here, in particular (i) the vector spherical-harmonic expansion for the radiation impedance operator for spherical $\partial\Omega$, (ii) the dominant singularity structure of the radiation impedance operator for smooth obstacle boundaries, and (iii) verification that the so-called physical-optics approximation [Ref. [23], Eq. (I.126)] and the geometrical (ray) optics limit for the complete Green's function for scattering from smooth-surfaced, convex, perfectly electrically conducting obstacles follow from an explicit "quasiplanar" approximation to the radiation impedance operator and applications of the method of stationary phase.

II. KINEMATICS OF ELECTROMAGNETIC FIELDS AND SOURCES

We consider scattering problems that take place in three-dimensional Euclidean space $\mathcal{E}^3$, which is divided into three disjoint parts: the first is the open set Ω, which is occupied by a medium presumed impenetrable to electromagnetic fields; the second is the connected, unbounded open set Ω^{ex}, which is taken to be empty space; and the third is the two-dimensional surface $\partial\Omega$, which is the boundary of Ω^{ex}, and which includes the boundary of Ω. The obstacle $\Omega\cup\partial\Omega$ can consist of one or more disjoint subsets. Permitted idealizations are that part of $\partial\Omega$ is an infinitely thin plate, or a thin screen with apertures, or a thin flange attached to entities with nonempty interiors; the idealization that $\partial\Omega$ is taken in part to be a one-dimensional object, as a length of infinitely thin wire, is not permitted.

We fix a Cartesian coordinate system in $\mathcal{E}^3$, and denote a generic point in $\mathcal{E}^3$ by a three vector $\mathbf{r}$, with components $(x,y,z)^\tau$—the "τ" means transpose, so that the entity is taken to be a column matrix. The volume measure on $\mathcal{E}^3$ is denoted by $d^3r=dxdydz$. Only if, and usually if, a point is in the surface $\partial\Omega$, we denote it with a subscripted three vector as $\mathbf{r}_\partial$, $\mathbf{r}_{\partial 1}$, $\mathbf{r}_{\partial a}$, or the like. The local unit outward (i.e., pointing outward Ω^{ex}) normal vector to $\partial\Omega$ at $\mathbf{r}_\partial$ is called $\hat{\mathbf{n}}(\mathbf{r}_\partial)$, and is defined everywhere on $\partial\Omega$ except on those isolated curves or points where $\partial\Omega$ is not smooth. The area measure on $\partial\Omega$ is that obtained from the restriction of the Euclidean metric in $\mathcal{E}^3$ to $\partial\Omega$ as in Ref. [21], Chap. 2.1, and is called dA or dA_1, etc., when an integral of the parametrized variable $\mathbf{r}_\partial$ or $\mathbf{r}_{\partial 1}$, respectively, over $\partial\Omega$ is to be carried out. If the obstacle is locally (that is, over a nonzero area) an infinitely thin sheet around $\mathbf{r}_\partial$, we can distinguish locally two faces of $\partial\Omega$, such that the two faces have oppositely directed outward normal vectors at $\mathbf{r}_\partial$; when taking limits as $\mathbf{r}\to\mathbf{r}_\partial$ of functions $F(\mathbf{r})$ defined on Ω^{ex}, defining boundary conditions on $\partial\Omega$, or doing integrals over $\partial\Omega$, one can, whether from necessity or convenience, treat these two faces as distinct parts of $\partial\Omega$.

Let $\mathcal{V}^3$ be the linear space of complex three-vector fields on $\mathcal{E}^3$. An element $\mathbf{E}\in\mathcal{V}^3$ can be construed as a column matrix of three complex-valued component functions on $\mathcal{E}^3$, say $(E_x(\mathbf{r}),E_y(\mathbf{r}),E_z(\mathbf{r}))^\tau$. Given two vector fields, say $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, the magnitude of both of which decreases sufficiently rapidly as $r=|\mathbf{r}|\to\infty$, we define the *bilinear* inner product $(\mathbf{E};\mathbf{B})_{\mathcal{V}^3}$ as

$$(\mathbf{E};\mathbf{B})_{\mathcal{V}^3}\equiv\int_{\mathcal{E}^3}\mathbf{E}(\mathbf{r})\cdot\mathbf{B}(\mathbf{r})d^3r=(\mathbf{B};\mathbf{E})_{\mathcal{V}^3}\ , \tag{1}$$

where the last equality indicates that the inner product is *symmetric* in the interchange of its vector-field arguments.

We consider linear operators that map $\mathcal{V}^3$ into itself; such operators are sometimes called *dyadics,* dyadic operators, or, when appropriate, dyadic Green's functions (Ref. [17], p. 265; Ref. [20], p. 17 and *passim*; Ref. [23], p. 13 and *passim*; Ref. [24]; Ref. [25], Chap. 13.1; and Refs. [26–30]). If Y is such an operator, then

$$\mathbf{B}\in\mathcal{V}^3 \text{ implies } Y\mathbf{B}\in\mathcal{V}^3\ , \tag{2}$$

and each ordered pair of vector fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ gives rise to the complex-number *matrix element* $(\mathbf{E},Y\mathbf{B})_{\mathcal{V}^3}$ of the operator Y. Note that the matrix element is separately linear in each of its constituent vector-field arguments and in its operator argument. We define the *transpose* Y^τ of an operator Y as the unique operator that causes the equality

$$(\mathbf{B};Y^\tau\mathbf{E})_{\mathcal{V}^3}=(\mathbf{E};Y\mathbf{B})_{\mathcal{V}^3} \tag{3}$$

to hold for all ordered pairs of vector fields $\mathbf{E}\in\mathcal{V}^3$ and $\mathbf{B}\in\mathcal{V}^3$. We call the operator Y *symmetric* if

$$Y^\tau=Y\ . \tag{4}$$

In applications, simple operators of the type in Eq. (2) are describable as 3×3 matrices of two-point integrable kernels on $\mathcal{E}^3$; the position-coordinate, Cartesian-component representatives of Y are of the form $Y_{jk}(\mathbf{r}_1;\mathbf{r}_2)$, and we have

$$(Y\mathbf{B})_j(\mathbf{r}_1)\equiv\int_{\mathcal{E}^3}\sum_{k=x}^{z}Y_{jk}(\mathbf{r}_1;\mathbf{r}_2)B_k(\mathbf{r}_2)d^3r_2\ . \tag{5}$$

More generally, finite-order derivatives of the operand can occur both before and after application of an integrable kernel in a linear operation. The identity operator $I_{\mathcal{V}_3}$ maps each vector field $\mathbf{B}\in\mathcal{V}^3$ into itself, and has representatives

$$I_{\mathcal{V}^3,jk}(\mathbf{r}_1;\mathbf{r}_2)=\delta_{jk}\delta^3(\mathbf{r}_1-\mathbf{r}_2)\ , \tag{6}$$

where δ^3 is the Dirac δ-function kernel on $\mathcal{E}^3$.

The electric-field intensity $\mathbf{E}(\mathbf{r})$ and magnetic-field intensity $c\mathbf{B}(\mathbf{r})$ are kinematically independent, but dynamically interdependent; also, the combined electric field and magnetic field at a point in space-time form an irreducible [31] six-component entity under the effects of homogeneous Lorentz transformations, including ordinary spatial rotations. We therefore sometimes consider these two vector fields jointly, and represent an electromagnetic field in $\mathcal{E}^3$ as an element Φ in the direct sum space $\mathcal{V}^3 \oplus \mathcal{V}^3 \equiv \mathcal{V}^{3\oplus 3}$ with six components $\Phi_{\alpha j}(\mathbf{r})$ at each point $\mathbf{r} \in \mathcal{E}^3$, where $\alpha = e,m$, $j = x,y,z$, $\Phi_{ej}(\mathbf{r}) = E_j(\mathbf{r})$, $\Phi_{mj}(\mathbf{r}) = cB_j(\mathbf{r})$, and Φ is a column matrix, as follows:

$$\Phi = (E_x(\mathbf{r}), E_y(\mathbf{r}), E_z(\mathbf{r}), cB_x(\mathbf{r}), cB_y(\mathbf{r}), cB_z(\mathbf{r}))^\tau . \tag{7}$$

We can similarly represent a general electromagnetic current density $\mathbf{J}_e(\mathbf{r})$ and magnetic current density $\mathbf{J}_m(\mathbf{r})$ jointly as a columnar six-component electromagnetic current density with components $\Upsilon_{\alpha j}(\mathbf{r})$, such that

$$\Upsilon = (J_{ex}(\mathbf{r}), J_{ey}(\mathbf{r}), J_{ez}(\mathbf{r}), J_{mx}(\mathbf{r}), J_{my}(\mathbf{r}), J_{mz}(\mathbf{r}))^\tau . \tag{8}$$

Given two electromagnetic fields $\Phi \in \mathcal{V}^{3\oplus 3}$ and $\Psi \in \mathcal{V}^{3\oplus 3}$, or a field Φ and a source $\Upsilon \in \mathcal{V}^{3\oplus 3}$, we define a symmetric, bilinear inner product $(\Phi,\Psi)_{\mathcal{V}^{3\oplus 3}}$ in the following manner:

$$\begin{aligned}(\Phi;\Upsilon)_{\mathcal{V}^{3\oplus 3}} &\equiv \int_{\mathcal{E}^3} \sum_{\alpha=e}^{m} \sum_{j=x}^{z} \Phi_{\alpha j}(\mathbf{r}) \Psi_{\alpha j}(\mathbf{r}) d^3 r \\ &= (\Psi;\Phi)_{\mathcal{V}^{3\oplus 3}} .\end{aligned} \tag{9}$$

Linear operators in the direct-sum space can be manufactured from, or decomposed into, 2×2 matrices of dyadic operators, as those appearing in Eq. (2). Matrix elements of such operators with respect to ordered pairs of vector fields in $\mathcal{V}^{3\oplus 3}$, the transpose of such an operator, and symmetry of an operator, are defined as in the three-component case. An electromagnetic Green's function Γ is such an operator,

$$\Gamma = \begin{bmatrix} \Gamma_{ee} & \Gamma_{em} \\ \Gamma_{me} & \Gamma_{mm} \end{bmatrix} , \tag{10}$$

where each of the four $\Gamma_{\alpha\beta}$ is a dyadic operator. The transposed operator Γ^τ proves to have the matrix

$$\Gamma^\tau \equiv \begin{bmatrix} (\Gamma^\tau)_{ee} & (\Gamma^\tau)_{em} \\ (\Gamma^\tau)_{me} & (\Gamma^\tau)_{mm} \end{bmatrix} = \begin{bmatrix} (\Gamma_e)^\tau & (\Gamma_{me})^\tau \\ (\Gamma_{em})^\tau & (\Gamma_{mm})^\tau \end{bmatrix} . \tag{11}$$

In Eq. (11), a notational ambiguity is resolved by parentheses: $(\Gamma^\tau)_{em}$ is by definition the e,m dyadic constituent of the transposed operator Γ^τ, while $(\Gamma_{em})^\tau$ is the transpose of the e,m constituent dyadic of the original operator Γ; Eq. (11) shows that these are generally not the same.

Next we consider vector fields on $\partial\Omega$. These will often arise as the limits of vector fields defined on $\Omega \cup \Omega^{\text{ex}}$, which may be discontinuous across $\partial\Omega$. The appending of a "+" (respectively, "−") to an argument, as $\mathbf{E}(\mathbf{r}_\partial +)$, indicates that the limit as $\mathbf{r} \to \mathbf{r}_\partial$, with $\mathbf{r} \in \Omega^{\text{ex}}$ (respectively, $\mathbf{r} \in \Omega$), is to be taken; if no explicit choice of limits is indicated, the + sign is to be understood. Let $\delta^2_{\partial\Omega}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})$ be the Dirac-δ-function kernel on $\delta\Omega$, such that for any continuous complex-valued function f on $\partial\Omega$ we have

$$f(\mathbf{r}_{\partial 1}) = \int_{\partial\Omega} \delta^2_{\partial\Omega}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) f(\mathbf{r}_{\partial 2}) dA_2 . \tag{12}$$

We will deal with fields of tangent vectors defined on $\partial\Omega$, and with "tangential" operators that map this space of tangent vector fields, which we call $\mathcal{V}^{\partial\Omega}$, into itself linearly. Operators of the latter type will be considered to annihilate any field of purely normal vectors on $\partial\Omega$, and will be denoted with a superimposed "breve" accent. One such operator is the identity operator $\breve{I}_\partial$ in the space of tangent-vector fields, which acts as a projection operator in the space of general vector fields on $\partial\Omega$, and has the Cartesian components

$$\breve{I}_{\partial,jk}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) \equiv \delta^2_{\partial\Omega}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2})[\delta_{jk} - \hat{\mathbf{n}}_j(\mathbf{r}_{\partial 2})\hat{\mathbf{n}}_k(\mathbf{r}_{\partial 2})] . \tag{13}$$

We further define an integral operator $\breve{X}_\partial$, which annihilates normal vector fields, and which has the action on tangential vector fields that the vector at each $\mathbf{r}_\partial$ is rotated by an angle $\pi/2$ in a right-handed sense about the local normal vector $\hat{\mathbf{n}}(\mathbf{r}_\partial)$,

$$\breve{X}_{\partial,jk}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) \equiv \delta^2_{\partial\Omega}(\mathbf{r}_{\partial 1};\mathbf{r}_{\partial 2}) \sum_{l=x}^{z} \epsilon_{jlk} \hat{\mathbf{n}}_l(\mathbf{r}_{\partial 2}), \tag{14}$$

where ϵ_{jlk} is the completely antisymmetric Levi-Civita symbol, with $\epsilon_{xyz} = +1$. We note the operator identity

$$\breve{X}^2_\partial = -\breve{I}_\partial . \tag{15}$$

It is convenient to define a complex-number "tangential" inner product $(\mathbf{E};\breve{I}_\partial \mathbf{B})_{\partial\Omega}$ between an ordered pair $\mathbf{E},\mathbf{B}$ of vector fields defined on $\partial\Omega$ as

$$(\mathbf{E};\breve{I}_\partial \mathbf{B})_{\partial\Omega} \equiv \int_{\partial\Omega} dA \sum_{j,k} E_i(\mathbf{r}_\partial)[\delta_{jk} - \hat{\mathbf{n}}_j(\mathbf{r}_\partial)\hat{\mathbf{n}}_k(\mathbf{r}_\partial)] B_k(\mathbf{r}_\partial). \tag{16}$$

If $\mathbf{E}$ and $\mathbf{B}$ are continuous vector fields everywhere on $\mathcal{E}^3$, we can think of Eq. (16) as a kind of matrix element in the sense of Eq. (3): each vector field in $\mathcal{V}^3$ gives rise to a vector field in $\mathcal{V}^{\partial\Omega}$ by restriction and tangential projection, and the inner product of a pair is then computed according to Eq. (16). It is possible, but we shall not attempt, to formulate the latter construction in terms of $\mathcal{V}^3$ matrix elements of an operator involving suitable Dirac δ functions.

The inner product of Eq. (16) is symmetric with respect to the interchange of its vector arguments. A general tangential operator, say $\breve{Y}$, gives rise to a complex number bilinear matrix element $(\mathbf{E};\breve{Y}\mathbf{B})_{\partial\Omega}$ for any pair $\mathbf{E},\mathbf{B}$ of vector fields on $\partial\Omega$. The transpose of the tangential operator $\breve{Y}$ is defined as the unique operator $\breve{Y}^\tau$ such that

$$(\mathbf{B};\breve{Y}^\tau \mathbf{E})_{\partial\Omega} = (\mathbf{E};\breve{Y}\mathbf{B})_{\partial\Omega} \tag{17}$$

for all ordered pairs of vector fields on $\partial\Omega$. The operator $\breve{Y}$ is called symmetric if

$$\breve{Y}^\tau = \breve{Y} , \tag{18}$$

and *skewsymmetric* if

$$\breve{Y}^\tau = -\breve{Y} . \tag{19}$$

We note that if $\breve{Y}$ is a tangential operator it has the property that left and right application of the tangential projection operator is redundant,

$$\check{Y}=\check{I}_\partial\check{Y}=\check{Y}\check{I}_\partial \ . \tag{20}$$

A tangential operator $\check{Y}$ has an *inverse* $\check{Y}^{-1}$ if

$$\check{Y}\check{Y}^{-1}=\check{I}_\partial=\check{Y}^{-1}\check{Y} \ . \tag{21}$$

The inverse is considered to be unique, so that one has, and defines,

$$(\check{Y}^{-1})^\tau=(\check{Y}^\tau)^{-1}\equiv\check{Y}^{-\tau} \ . \tag{22}$$

As noted in the text of Ref. [14] following Eq. (57), we can encounter entities more general than the operators of Eq. (2) and of the previous paragraph, but which nevertheless give rise to such bilinear matrix element forms; these entities do not necessarily map a vector field into a vector field, but combine finite-order differentiation of the field components standing on both sides of a matrix-element expression with kernel-type linear mappings, where the kernels can involve δ functions. We shall not attempt to give a careful definition of such entities, or (when the meaning is clear) distinguish them notationally from ordinary operators. We note that they are cases of the still more general entities called *distributions* [32], of the type that map the direct-product space of two vector fields, say $\mathcal{V}^{\partial\Omega}\otimes\mathcal{V}^{\partial\Omega}$ (that is, the linear space generated by taking finite complex linear combinations of ordered pairs of tangent vector fields on $\partial\Omega$), linearly into the complex numbers.

Any ordered pair of vector fields in $\mathcal{V}^{\partial\Omega}$ can be construed as a vector field (in a generalized sense) in the direct-sum space $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}\equiv\mathcal{V}^{\partial\Omega}\oplus\mathcal{V}^{\partial\Omega}$, and conversely. An ordered pair of vector fields in $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ gives rise to an inner product, which is defined in a manner analogous to Eq. (16). Linear operators that map $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ into itself will be denoted by upper-case roman letters, with a superimposed overcircle accent, as $\mathring{Y}$. Operators of this type comprise a 2×2 matrix of tangential operators, and the notions of transpose, symmetry, unit operator, and so on, are defined in an obvious manner. Analogous to the case of $\mathcal{V}^{\partial\Omega}$ discussed following Eq. (16), two continuous vector fields in $\mathcal{V}^{3\oplus3}$ give rise by restriction and tangential projection to a pair of fields and hence a $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ inner product or matrix element that depends only on the values of the original fields within $\partial\Omega$. This construction will be employed in the T operator derived in Sec. VI.

III. DYNAMICS OF ELECTROMAGNETIC-WAVE DIFFRACTION FROM OBSTACLES

Let the electric field, electric current density, and electric charge density be denoted by $\mathbf{E}(t,\mathbf{r}),\mathbf{J}_e(t,\mathbf{r}),\rho_e(t,\mathbf{r})$, and the corresponding magnetic quantities by $c\mathbf{B}(t,\mathbf{r}),\mathbf{J}_m(t,\mathbf{r}),\rho_m(t,\mathbf{r})$; it is convenient for the sake of dimensional simplicity to call $c\mathbf{B}$ "the" magnetic field. We use SI units, so that $\mathbf{E}$ and $c\mathbf{B}$ have units V/m, $\mathbf{J}_e$ and $\mathbf{J}_m$ have units A/m^2, while ρ_e and ρ_m have units C/m^3; Green's functions will be defined so as to have physical dimensions m^{-2}. We assume that the exterior region has the electromagnetic properties of empty space, that is, has electric permitivity ϵ_0, magnetic permeability μ_0, and zero electric and magnetic conductivities throughout Ω^{ex}. With these conventions and assumptions, Maxwell's differential equations for the electric and magnetic fields in Ω^{ex} or in obstacle-free space, with given charge-current sources, take the form

$$\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}-\nabla\times(c\mathbf{B})=-c\mu_0\mathbf{J}_e \ , \tag{23}$$

$$\nabla\cdot\mathbf{E}=\rho_e/\epsilon_0 \ , \tag{24}$$

$$\frac{1}{c}\frac{\partial(c\mathbf{B})}{\partial t}+\nabla\times\mathbf{E}=-c\mu_0\mathbf{J}_m \ , \tag{25}$$

$$\nabla\cdot(c\mathbf{B})=\rho_m/\epsilon_0 \ . \tag{26}$$

The above equations imply the differential conservation laws for electric and for magnetic charge,

$$\nabla\cdot\mathbf{J}_e+\frac{\partial\rho_e}{\partial t}=0 \ , \tag{27}$$

$$\nabla\cdot\mathbf{J}_m+\frac{\partial\rho_m}{\partial t}=0 \ . \tag{28}$$

We suppose now that all the field and source quantities have the time dependence $\exp(-ik_0ct)$, with $k_0\neq0$; we drop the exponential time dependence and denote the residual electromagnetic quantities with k_0 subscripts. The charge conservation equations now imply

$$\rho_{k_0,e}=(ik_0c)^{-1}\nabla\cdot\mathbf{J}_{k_0,e} \ , \tag{29}$$

$$\rho_{k_0,m}=(ik_0c)^{-1}\nabla\cdot\mathbf{J}_{k_0,m} \ . \tag{30}$$

That is, the current densities determine the charge densities, and only the time-harmonic forms of Eqs. (23) and (25) are needed to determine the electromagnetic fields in full, once the current density distributions are known. We have in matrix form, following multiplication by a factor i on both sides,

$$\begin{bmatrix} k_0 & -i\nabla\times \\ i\nabla\times & k_0 \end{bmatrix}\begin{bmatrix}\mathbf{E}_{k_0}\\ c\mathbf{B}_{k_0}\end{bmatrix}=\begin{bmatrix}-ic\mu_0\mathbf{J}_{k_0,e}\\ -ic\mu_0\mathbf{J}_{k_0,m}\end{bmatrix} . \tag{31}$$

The latter equations resemble the time-independent Schrödinger equation with a source term. (Although we shall make no use of this property here, Maxwell's equations can be decoupled into two three-component equations by the use of the vector combinations $\mathbf{E}\pm ic\mathbf{B}$ and $\mathbf{J}_e\pm i\mathbf{J}_m$ [33]. These and numerous other manipulations and properties of Maxwell's equations are discussed in Ref. [34].)

An electromagnetic Green's function is made up of dyadic operators, as in Eq. (11). We use subscripts to denote the type—whether electric (e) or magnetic (m)—of field and source, the wave number k_0, and the Cartesian components of the field and source vectors. A generic dyadic will have Cartesian components $\Gamma_{k_0,\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)$, where α and β can be e or m, and j and k range over x, y, and z. The first index (α or j) of a pair of indices and the first argument $\mathbf{r}_1$ are associated with the field point, and the second index of a pair (β or k) and second argument $\mathbf{r}_2$ are associated with the source. For example, $\Gamma_{k_0,em,xz}(\mathbf{r}_1;\mathbf{r}_2)$ is proportional to the x component of the electric field at $\mathbf{r}_1$, due to a unit magnetic source current that is located at $\mathbf{r}_2$, points in the z direction, and oscillates with frequency $k_0 c$.

We will have reduced the solution of Eq. (31) to quadratures once the electromagnetic field is obtained for arbitrarily positioned and directed unit sources of harmonic electric and magnetic currents (equivalently, for point oscillating electric and magnetic dipoles). These fields comprise a Green's function for the Maxwell field, and satisfy the differential equations

$$\sum_{k=x}^{z} \begin{bmatrix} k_0\delta_{jk} & -i(\nabla_1\times)_{jk} \\ i(\nabla_1\times)_{jk} & k_0\delta_{jk} \end{bmatrix} \times \begin{bmatrix} \Gamma_{k_0,ee,kl}(\mathbf{r}_1;\mathbf{r}_2) & \Gamma_{k_0,em,kl}(\mathbf{r}_1;\mathbf{r}_2) \\ \Gamma_{k_0,me,kl}(\mathbf{r}_1;\mathbf{r}_2) & \Gamma_{k_0,mm,kl}(\mathbf{r}_1;\mathbf{r}_2) \end{bmatrix} = \begin{bmatrix} \delta_{jl}\delta^3(\mathbf{r}_1-\mathbf{r}_2) & 0 \\ 0 & \delta_{jl}\delta^3(\mathbf{r}_1-\mathbf{r}_2) \end{bmatrix} . \quad (32)$$

Note that the components of Γ_{k_0} have physical dimensions *length*$^{-2}$, and are such that $(-i\mu_0 c)(\Gamma_{k_0})$ times current spectral density times volume equals an electromagnetic-field spectral density, where we consider k_0 and ct as Fourier conjugate variables. Various Green's functions will be defined, depending on whether or not an obstacle is present, on the type of radiation boundary condition at large distances from the source and possible obstacle, and on the surface boundary conditions (SBC's) on the obstacle, if present. In particular, the terms *free-space Green's function,* or *complete Green's function,* will be used if an obstacle is not present, or is present, respectively. The types of radiation and SBC's, and the corresponding notational conventions, will be described in several stages in the remainder of this section.

The *Silver-Müller radiation conditions* (cf. Ref. [21], Chap. 4.2) for electromagnetic fields are defined as follows. If the electric field $\mathbf{E}_{k_0}(\mathbf{r})$ and magnetic field $c\mathbf{B}_{k_0}(\mathbf{r})$ satisfy the time-harmonic Maxwell equations Eq. (31) with no sources in the region defined by $r \geq R$ for some fixed $R>0$, and if

$$\lim_{r\to\infty} \{ r[\hat{\mathbf{r}}\times\mathbf{E}_{k_0}(\mathbf{r}) \mp c\mathbf{B}_{k_0}(\mathbf{r})]\} = \mathbf{0} \, , \qquad \lim_{r\to\infty} \{ r[\hat{\mathbf{r}}\times c\mathbf{B}_{k_0}(\mathbf{r}) \pm \mathbf{E}_{k_0}(\mathbf{r})]\} = \mathbf{0} \quad (33)$$

for all $\hat{\mathbf{r}}$ (with suitable modifications for unbounded obstacles), the electromagnetic field is said to satisfy *outgoing-wave* (upper sign) or *ingoing-wave* (lower sign) conditions at infinity. These space-frequency domain criteria correspond to prescribing retarded and advanced signals, respectively, in the space-time domain .

We shall distinguish among the various types of Green's functions Γ_{k_0} and their dyadic constituents $\Gamma_{k_0,\alpha\beta}$, according to the boundary conditions that they satisfy, by superscripts: The first superscript can be 0, L, or E, where "0" means a Green's function for obstacle-free space, "L" means some particular homogeneous Leontovich boundary conditions are satisfied on $\partial\Omega$, and "E" means the special case of "L" that the obstacle is a perfect electrical conductor. Two further superscripts will be used, which will take the values "+" or "−", according to whether outgoing-, or ingoing-wave radiation conditions, respectively, are satisfied for the original sources of the total field. The details of the notation for complete Green's functions depend on physical assumptions about the source and the obstacle, and will be explained later in this section; first, we consider the case of free-space Green's functions.

We assemble two free-space Green's functions $\Gamma_{k_0}^{0++}$ and $\Gamma_{k_0}^{0+-}$ from dyadic constituents, depending whether the magnetic source terms are associated with retarded or advanced fields, respectively; in the following, the symbols $G_{k_0,\alpha\beta}^{0\pm}$ denote "conventional" Green's dyadics, as described in Appendix A,

$$\Gamma_{k_0}^{0+\pm} \equiv \begin{bmatrix} (\Gamma_{k_0}^{0+\pm})_{ee} & (\Gamma_{k_0}^{0+\pm})_{em} \\ (\Gamma_{k_0}^{0+\pm})_{me} & (\Gamma_{k_0}^{0+\pm})_{mm} \end{bmatrix} = \begin{bmatrix} G_{k_0,ee}^{0+} & G_{k_0,em}^{0\pm} \\ G_{k_0,me}^{0+} & G_{k_0,mm}^{0\pm} \end{bmatrix} . \quad (34)$$

In Eq. (34), the central expression is simply a definition of the names of the component dyadics of the operator, while the last equality specifies that the component dyadics are closely related to the usual dyadic Green's functions. Note that the second superscript on a Γ is always "+" herein, and means that electric currents are presumed to give rise exclusively to outgoing waves, while the third superscript can be "+" or "−," and entails the optional association of outgoing or ingoing waves with oscillating magnetic currents. The differential equations Eq. (32), along with radiation conditions as in Eq. (33), are sufficient to determine the free-space Green's functions; explicit expressions for the constituent Green's dyadics are given in closed "Cartesian" form in Appendix A.

We define a generalized, nonlocal version of Leontovich boundary conditions on $\partial\Omega$ (see Ref. [21], Chap. 4.1, Refs. [35–37], and Ref. [38], p. 70 for the local version). Let $\mathbf{E}_{k_0}(\mathbf{r}_\partial+)$ and $c\mathbf{B}_{k_0}(\mathbf{r}_\partial+)$ be the surface limits of the total electric and magnetic fields in Ω^{ex}, and let $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$ be given tangential operators with the same physical dimensions. Then we require that

$$\breve{A}_{k_0}\breve{I}_\partial\mathbf{E}_{k_0} - \breve{C}_{k_0}\breve{X}_\partial c\mathbf{B}_{k_0} = \mathbf{0}_\partial \, , \quad (35)$$

where $\mathbf{0}_\partial \in \mathcal{V}^{\partial\Omega}$ is the zero tangential vector field. The "E" case that the obstacle is a perfect electrical conductor is obtained by specializing to $\breve{A}_{k_0} = \breve{I}_\partial$ and $\breve{C}_{k_0}$ is the zero tangential operator $\breve{0}_\partial$, that is,

$$(\breve{I}_\partial\mathbf{E}_{k_0})(\mathbf{r}_\partial) = \mathbf{0}_\partial \, . \quad (36)$$

We can make a number of observations on the mathematical and physical content of Eq. (35). First, Eq. (35) is a representative of an equivalence class of Leontovich boundary conditions: the simultaneous replacements in Eq. (35)

$$\check{A}_{k_0} \to \check{Y}\check{A}_{k_0} \text{ and } \check{C}_{k_0} \to \check{Y}\check{C}_{k_0} , \tag{37}$$

where $\check{Y}$ is any invertible tangential operator, yield a physically equivalent boundary value problem. Second, the left-hand side (lhs) of Eq. (35) implicitly defines a linear mapping of $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ into $\mathcal{V}^{\partial\Omega}$. The kernel of this mapping [i.e., the set of fields in $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ that map into the zero field in $\mathcal{V}^{\partial\Omega}$, as in Eq. (35)] should comprise just the linear space of tangential electromagnetic fields in $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ that belong to a certain class of solutions to Maxwell's equations. What distinguishes these solutions from others is clarified physically by considering the obstacle to be not a black box, but a material substance through which electromagnetic waves can propagate and which responds to a signal by generating the electric (and possibly magnetic) currents that give rise to the scattered wave. The family of solutions of Maxwell's equations which are regular in the interior of the obstacle and have all source currents (as distinguished from response currents) in Ω^{ex} form a linear space; the tangential projections on $\partial\Omega$ of the exterior limiting values of these solutions also form a linear space, and it is this linear space that is to be the kernel of the mapping as defined by Eq. (35). Third, there is a complementary linear space in $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$, which can be construed as comprising the tangential projection of outgoing wave solutions to Maxwell's equations in Ω^{ex}; sufficient conditions on the operators $\check{A}_{k_0}$ and $\check{C}_{k_0}$ are established in Appendix B, such that no nontrivial electromagnetic field of this second class in $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ is mapped into the zero field in $\mathcal{V}^{\partial\Omega}$ by the mapping defined by Eq. (35). As described in connection with Eq. (79) and in Appendix C, when the equivalence class of operators $\check{A}_{k_0}$ and $\check{C}_{k_0}$ is chosen correctly the space of all fields in $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ is in effect the direct sum of these two subspaces, which, speaking informally, have the same (infinite) dimensionality: one subspace corresponds to just the regular, source-free solutions to Maxwell's equations in the interior Ω of the obstacle, and the other corresponds to the source-free outgoing-wave solutions for a given exterior environment in Ω^{ex}. Moreover (cf. Appendix C), it is possible to construct a projection operator that annihilates fields belonging to one subspace and acts as the unit operator on fields belonging to the complementary subspace.

The Leontovich boundary conditions Eq. (35) are intended as a means of simulating the electromagnetic response of a complex physical obstacle, analogous to the replacement of an N-terminal electrical circuit by a "black box," with frequency-dependent complex impedance "boundary conditions" determining the relationship between voltages at, and current flow through, each of the N terminals. (This analogy is given a more detailed realization in Appendix D.) It is plausible, but I am at present unable to prove, that any physically reasonable object whatever that generates currents linearly in response to electromagnetic fields can be replaced in the matter of its scattering properties by an equivalence class of fictitious Leontovich boundary conditions on a conveniently chosen, and possible artificial, circumscribing surface $\partial\Omega$. A constructive proof of this claim is given in Appendix C for the class of special cases that the obstacle is made up of a homogeneous, isotropic medium, the obstacle is fixed, and the source is time harmonic. It is shown in Appendix C that the determination of a representative pair of operators $\check{A}_{k_0}$ and $\check{C}_{k_0}$, which are to simulate the electromagnetic response of the interior region Ω, can be effected in a number of ways, and the equivalence class of such operator pairs is independent of the electromagnetic properties of the exterior region. An analogous result holds for the independence of the response of the exterior region Ω^{ex} to the electromagnetic properties of the obstacle. (The exterior region is normally taken to be empty space—cf. the theory of the radiation impedance operator in Sec. V.) Solution of the scattering problem reduces to matching appropriate interior and exterior fields across the boundary, which in mathematical terms entails functional analysis in the linear space of complex tangent-vector fields on the dividing surface $\partial\Omega$. The formal solution of the exterior problem, and extraction of a transition operator associated with the Leontovich boundary conditions that simulate the obstacle, is carried out in Sec. VI.

As stated, the presumed form Eq. (35) of the Leontovich conditions is believed to permit the simulation of an arbitrary scattering obstacle with a linear response. For example, the analogs of a short circuit (zero impedance) and of an open circuit (infinite impedance) can be interwoven for complementary linear subspaces of $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ by a suitable choice of the operators $\check{A}_{k_0}$ and $\check{C}_{k_0}$. Another example, which is discussed in the concluding remarks to Sec. VI, is the simulation of empty space. The simulation of an obstacle made up of a uniform, isotropic medium is discussed in Appendix C. Other possibilities, which have not been investigated, are the simulation of a grating or a rough surface by a suitable choice of $\check{A}_{k_0}$ and $\check{C}_{k_0}$ on a circumscribing smooth surface.

We suppose that the complete scattered wave generated by the obstacle reacting to the original, free-space electromagnetic wave can be represented by a linear transition operator $\mathcal{T}^{X+}_{k_0}$ (with $X=L$ or $X=E$); the operator comprises a 2×2 matrix of dyadic operators with components $\mathcal{T}^{X+}_{k_0,\gamma\zeta,pq}(\mathbf{r}_3;\mathbf{r}_4)$, where $\gamma,\zeta=e,m$ and $p,q=x,y,z$, such that, in operator shorthand,

$$\Gamma^{X+\pm}_{k_0}=\Gamma^{0+\pm}_{k_0}+\Gamma^{0++}_{k_0}\mathcal{T}^{X+}_{k_0}\Gamma^{0+\pm}_{k_0} . \tag{38}$$

The second summand on the right-hand side (rhs) of Eq. (38) is an operator product, that is, entails matrix multiplication and integration; written out componentwise, Eq. (38) means

$$\Gamma^{X+\pm}_{k_0,\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)=\Gamma^{0+\pm}_{k_0,\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)+\int_{\mathcal{E}^3}d^3r_3\int_{\mathcal{E}^3}d^3r_4\sum_{\gamma,p}\sum_{\zeta,q}\Gamma^{0++}_{k_0,\alpha\gamma,jp}(\mathbf{r}_1;\mathbf{r}_3)\mathcal{T}^{X+}_{k_0,\gamma\zeta,pq}(\mathbf{r}_3;\mathbf{r}_4)\Gamma^{0+\pm}_{k_0,\zeta\beta,qk}(\mathbf{r}_4;\mathbf{r}_2). \tag{39}$$

For the dyadic constituents of complete Green's functions, Leontovich boundary conditions take the form

$$(\check{A}_{k_0}\check{I}_\partial\Gamma^{L++}_{k_0,e\beta}-\check{C}_{k_0}\check{X}_\partial\Gamma^{L+\pm}_{k_0,m\beta})_{jk}(\mathbf{r}_{\partial 1};\mathbf{r}_2)=0 \; . \tag{40}$$

Equation (40) must hold for both $\beta=e$ and $\beta=m$, for all $\mathbf{r}_{\partial 1}\in\partial\Omega$ and $\mathbf{r}_2\in\Omega\cup\Omega^{\mathrm{ex}}$, and for either choice (emitting or absorbing) of magnetic source. For the E case we require

$$(\check{I}_\partial\Gamma^{E+\pm}_{k_0,e\beta})_{jk}(\mathbf{r}_{\partial 1};\mathbf{r}_2)=0 \; . \tag{41}$$

We shall in Appendix B establish a set of sufficiency conditions on $\check{A}_{k_0}$ and $\check{C}_{k_0}$, which guarantee the uniqueness and existence of the complete Green's functions.

We have stated the assumption that in our simulation the obstacle is considered impenetrable to electromagnetic fields. Accordingly, in addition to the surface boundary conditions (SBC's) of Eq. (40), and the radiation conditions at infinity, we shall require the complete Green's functions to satisfy the *extinction property,* that is, be zero whenever the field point is in the interior Ω: componentwise, we want

$$\Gamma^{L+\pm}_{k_0,\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)=0 \quad \text{whenever} \quad \mathbf{r}_1\in\Omega \; . \tag{42}$$

The property analogous to Eq. (42) was termed "the extended boundary conditions" following Eq. (16) in Ref. [14], but the present name seems preferable [39–41]. Note that, unlike Ref. [14], we do not necessarily demand or state that Eq. (42) must hold if the source point $\mathbf{r}_2\in\Omega$ while the field point $\mathbf{r}_1\in\Omega^{\mathrm{ex}}$; this property will hold if exclusively outgoing waves are generated by the original source currents, but not if ingoing waves are also present—see the remarks following Eq. (96) and Ref. [22] for results and discussion.

The uniqueness and existence of $\Gamma^{L+\pm}_{k_0}$ are presumed to be guaranteed, given that Eq. (32), a suitable adaptation of Eqs. (33), (40), and (42), are satisfied. Hence the T operator is determined in turn by Eq. (38), albeit only implicitly. The T operator will be analyzed in Sec. VI into the simplest form that seems attainable for general geometries and boundary conditions. The following observations may serve to give insight on the structure and physical significance of the T operator. As in the acoustic case (Ref. [14], Eq. (58)), the T-operator components in Eq. (39) will be zero if either $\mathbf{r}_3\in\Omega^{\mathrm{ex}}$ or $\mathbf{r}_4\in\Omega^{\mathrm{ex}}$, or both, will have an elementary structure if $\mathbf{r}_3\in\Omega$ or $\mathbf{r}_4\in\Omega$, or both, and will have a nonelementary structure only if both $\mathbf{r}_3\in\Omega$ and $\mathbf{r}_4\in\partial\Omega$. The $\mathcal{T}^{X+}_{k_0,\gamma\zeta,pq}(\mathbf{r}_3;\mathbf{r}_4)$ components have the following physical significance: The obstacle responds, linearly but nonlocally, to an incoming field of type ζ, in Cartesian direction q, at position $\mathbf{r}_4\in\Omega\cup\partial\Omega$, by creating a current distribution on $\Omega\cup\partial\Omega$; the T-operator components give the density distribution of that current at any position $\mathbf{r}_3\in\Omega\cup\partial\Omega$, for both types $\gamma=e,m$, and for all three directions $p=x,y,z$. One then superimposes all the fields emitted by all the currents generated by the given "incoming" field distribution; the emitted field then comprises the complete scattered field.

We always assume herein that the obstacle consists exclusively of ordinary electrical matter, and, correspondingly, that the obstacle generates only retarded fields and outgoing waves—hence the "+" superscript on $\mathcal{T}^{X+}_{k_0}$, and the choice of $\Gamma^{0++}_{k_0}$ as the left-hand (and in causal succession, last) operator in the second summand on the rhs of Eqs. (38) and (39). The initial free-space signal $\Gamma^{0+\pm}_{k_0}$ is assembled by using outgoing waves from electrical sources, and using either outgoing waves from, or ingoing waves to, magnetic sources, as in Eq. (34); hence the first summand, and the right-hand operator in the second summand, on the rhs of Eqs. (38) and (39) can be either $\Gamma^{0++}_{k_0}$ or $\Gamma^{0+-}_{k_0}$, depending on the presumed type of coupling of the initial source magnetic current to the electromagnetic field. A remark on consistency: we will be able to infer, at the end of the principal computation in Sec. VI, that the T operator for general Leontovich boundary conditions is such that there will be a discontinuity in the tangential electric field across $\partial\Omega$, that is, magnetic surface currents are generated in $\partial\Omega$. The Leontovich conditions are only a simulation of processes taking place inside a physical obstacle, however, and it is plausible on physical grounds that for a penetrable scattering object of a conventional type that can generate only electrical currents and charges in response to a field, the components $\mathcal{T}^{X+}_{k_0,\gamma\zeta,pq}(\mathbf{r}_3;\mathbf{r}_4)$ will be effectively zero whenever $\gamma=m$.

IV. REPRESENTATION THEOREMS AND RECIPROCITY FOR GREEN'S FUNCTIONS

We shall derive a convenient representation theorem for Green's functions of purely outgoing-wave type. We then apply cases of the theorem straightforwardly to the derivation of the reciprocity property for free-space and for E-type complete Green's functions. Another case of the theorem will be subjected to apparently *ad hoc* manipulations to establish reciprocity for Green's functions satisfying suitably constrained Leontovich SBC's; these manipulations and constraints can be justified in part by the work of Secs. V and VI and Appendixes B and C.

In order to define reciprocity, we first define the operator Π in $\mathcal{V}^{3\oplus 3}$ as follows:

$$\Pi\equiv\begin{bmatrix} I_{\mathcal{V}^3} & 0 \\ 0 & -I_{\mathcal{V}^3}\end{bmatrix} . \tag{43}$$

Now let $\mathcal{L}$ be a linear operator mapping the space $\mathcal{V}^{3\oplus 3}$ into itself; if the transpose $\mathcal{L}^\tau$ is related to $\mathcal{L}$ by

$$\mathcal{L}^\tau=\Pi\mathcal{L}\Pi \; , \tag{44}$$

we say that $\mathcal{L}$ satisfies the reciprocity condition, the term "self-reciprocal" being ambiguous.

We define the Heaviside differential operator $\mathcal{H}_{k_0}$ as

$$\mathcal{H}_{k_0}\equiv\begin{bmatrix} k_0 I_{\mathcal{V}^3} & -iI_{\mathcal{V}^3}\nabla\times \\ iI_{\mathcal{V}^3}\nabla\times & k_0 I_{\mathcal{V}_3}\end{bmatrix} . \tag{45}$$

An application of the divergence theorem shows that the operator $\mathcal{H}_{k_0}$ satisfies the reciprocity condition with respect to $\mathcal{V}^{3\oplus 3}$. Inasmuch as the Green's functions are in a limited domain right operator inverses of $\mathcal{H}_{k_0}$—see Eq. (32)—it is plausible that, given the imposition of suitable boundary conditions, they also can be made to be satisfy the reciprocity condition.

We consider two Green's functions $\Gamma_{k_0}^{X++}$ and $\Gamma_{k_0}^{Y++}$, both of purely outgoing-wave type. In what follows, we invoke a limited version of the summation convention: when a Cartesian index, such as j, k, l, p, or q, appear twice in a multiplicative expression, a sum of that index over x, y, and z is implied. The differential equations satisfied by the dyadic blocks of the former are, according to Eq. (32),

$$k_0\Gamma_{k_0,e\beta,pk}^{X++}(\mathbf{r}_3;\mathbf{r}_2)-i\epsilon_{plq}\left[\frac{\partial}{\partial r_{3l}}\right]\Gamma_{k_0,m\beta,qk}^{X++}(\mathbf{r}_3;\mathbf{r}_2)=\delta_{e\beta}\delta_{pk}\delta^3(\mathbf{r}_3-\mathbf{r}_2), \tag{46}$$

$$i\epsilon_{plq}\left[\frac{\partial}{\partial r_{3l}}\right]\Gamma_{k_0,e\beta,qk}^{X++}(\mathbf{r}_3;\mathbf{r}_2)+k_0\Gamma_{k_0,m\beta,pk}^{X++}(\mathbf{r}_3;\mathbf{r}_2)=\delta_{m\beta}\delta_{pk}\delta^3(\mathbf{r}_3-\mathbf{r}_2), \tag{47}$$

where $\beta=e$ or m. We now multiply both sides of Eq. (46) by $-\Gamma_{k_0,e\alpha,pj}^{Y++}(\mathbf{r}_3;\mathbf{r}_1)$ and sum over p, multiply both sides of Eq. (47) by $+\Gamma_{k_0,m\alpha,pj}^{Y++}(\mathbf{r}_3;\mathbf{r}_1)$ and sum over p, and add corresponding sides of the resulting equations. From the last equation we subtract corresponding sides of another, similar equation which differs from the first by the simultaneous index exchanges $X\leftrightarrow Y$, $j\leftrightarrow k$, $\alpha\leftrightarrow\beta$, and $\mathbf{r}_1\leftrightarrow\mathbf{r}_2$. Note that on the lhs of the resulting equation all the terms with k_0 as a coefficient sum to zero, while the remaining terms comprise an exact divergence in the variables $\mathbf{r}_3$. Both sides of the resulting equation are now integrated in the variable $\mathbf{r}_3$ over the domain $\Omega^{\text{ex}}(R)$ contained between $\partial\Omega$ and a very large two-sphere, centered at the origin, of radius R, called $S^2(R)$. An application of the divergence theorem reduces one side of the equation to two surface integrals, one over $\partial\Omega$ and the other over $S^2(R)$; the latter integral vanishes as $R\to\infty$ as a result of the radiation conditions Eq. (33). Making use of the definition of a transpose and of the operators $\check{X}_\partial$ and Π—see Eqs. (11), (15), and (43)—we find that

$$\begin{aligned}\Theta_{\Omega^{\text{ex}}}(\mathbf{r}_1)&(\Pi\Gamma_{k_0}^{X++})_{\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)-(\Gamma_{k_0}^{Y++\tau}\Pi)_{\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)\Theta_{\Omega^{\text{ex}}}(\mathbf{r}_2)\\&=-i\int_{\partial\Omega}dA_3[(\Gamma_{k_0}^{Y++\tau})_{\alpha e,jp}(\mathbf{r}_1;\mathbf{r}_{\partial 3})\check{X}_{\partial,pq}(\mathbf{r}_{\partial 3})\Gamma_{m\beta,qk}^{X++}(\mathbf{r}_{\partial 3};\mathbf{r}_2)+(\Gamma_{k_0}^{Y++\tau})_{\alpha m,jp}(\mathbf{r}_1;\mathbf{r}_{\partial 3})\check{X}_{\partial,pq}(\mathbf{r}_{\partial 3})\Gamma_{e\beta,qk}^{X++}(\mathbf{r}_{\partial 3};\mathbf{r}_2)]\ .\end{aligned} \tag{48}$$

In Eq. (48), we have used the unit step function $\Theta_{\Omega^{\text{ex}}}$, which is defined as follows for any open domain $\Delta\subseteq\mathcal{E}^3$:

$$\Theta_\Delta(\mathbf{r})\equiv\begin{cases}+1 & \text{if } \mathbf{r}\in\Delta\\ 0 & \text{otherwise}\ .\end{cases} \tag{49}$$

For the case that in Eq. (48) both $X=0$ and $Y=0$, we can shrink $\partial\Omega$ to the null set; the surface integral therefore vanishes, and hence $\Gamma_{k_0}^{0++}$ satisfies the reciprocity condition. For the case that both $X=E$ and $Y=E$, the integral over $\partial\Omega$ in Eq. (48) vanishes, since the tangential electric fields derived from $\Gamma_{k_0}^{E++}$ vanish everywhere on $\partial\Omega$; moreover, this Green's function satisfies the extinction property Eq. (42). Therefore, we have

$$\Gamma_{k_0,\alpha\beta,jk}^{E++}(\mathbf{r}_1;\mathbf{r}_2)=0\quad\text{if } \mathbf{r}_2\in\Omega\ , \tag{50}$$

and hence

$$(\Gamma_{k_0}^{X++\tau})_{\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)]=(\Pi\Gamma_{k_0}^{X++}\Pi)_{\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)\qquad\text{for } X=0 \text{ or } X=E\ , \tag{51}$$

for all indices, and for almost all arguments $\mathbf{r}_1$ and $\mathbf{r}_2$. Equation (51) restates in the language of Green's functions the familiar reciprocity theorem for time-harmonic electromagnetic fields—see Ref. [20], p. 205; Ref. [23], p. 13; Ref. [42], p. 64; and Refs. [43] and [44]. We note that Eq. (48) with $X=L$ and $Y=0$ will prove useful in Sec. V, in the reduction of $\Gamma_{k_0}^{L++}$ to the form of Eq. (38).

In order to investigate the validity of the reciprocity condition for the case of Leontovich SBC's, we put both $X=L$ and $Y=L$ in Eq. (48), that is, both Green's functions satisfy the same Leontovich boundary conditions. The rhs can be written in the following 2×2 dyadic matrix form, where a factor i is omitted:

$$\begin{bmatrix}(\Gamma_{k_0}^{L++\tau})_{ee} & (\Gamma_{k_0}^{L++\tau})_{em}\\ (\Gamma_{k_0}^{L++\tau})_{me} & (\Gamma_{k_0}^{L++\tau})_{mm}\end{bmatrix}\begin{bmatrix}\check{0}_\partial & -\check{X}_\partial\\ -\check{X}_\partial & \check{0}_\partial\end{bmatrix}\begin{bmatrix}L_{k_0,ee}^{L++} & \Gamma_{k_0,em}^{L++}\\ \Gamma_{k_0,me}^{L++} & \Gamma_{k_0,mm}^{L++}\end{bmatrix}, \tag{52}$$

where the integral over $\partial\Omega$ is understood. We want the rhs of Eq. (48) and hence Eq. (52) to be zero; this objective will be achieved, in view of Eq. (40) and its transpose, if the central matrix in Eq. (52) can be "factorized" as follows:

$$\begin{bmatrix} \breve{A}^{\tau}_{k_0}\breve{Q}_{k_0} & -\breve{T}_{k_0}\breve{X}_\partial \\ \breve{X}_\partial\breve{C}_{k_0}\breve{Q}_{k_0} & -\breve{X}_\partial\breve{U}_{k_0}\breve{X}_\partial \end{bmatrix} \begin{bmatrix} \breve{0}_\partial & -\breve{X}_\partial \\ -\breve{X}_\partial & \breve{0}_\partial \end{bmatrix} \begin{bmatrix} \breve{A}_{k_0} & -\breve{C}_{k_0}\breve{X}_\partial \\ \breve{X}_\partial\breve{Q}^{-1}_{k_0}\breve{R}^{\tau}_{k_0} & -\breve{X}_\partial\breve{Q}^{-1}_{k_0}\breve{S}^{\tau}_{k_0}\breve{X}_\partial \end{bmatrix} = \begin{bmatrix} \breve{0}_\partial & -\breve{X}_\partial \\ -\breve{X}_\partial & \breve{0}_\partial \end{bmatrix}, \tag{53}$$

where $\breve{Q}_{k_0}$, $\breve{R}^{\tau}_{k_0}$, $\breve{S}^{\tau}_{k_0}$, $\breve{T}_{k_0}$, and $\breve{U}_{k_0}$ are to be determined, and $\breve{Q}_{k_0}$ is presumed invertible. Several applications of Eq. (15) lead to the following equivalent form for

$$\begin{bmatrix} \breve{A}^{\tau}_{k_0} & \breve{T}_{k_0} \\ \breve{C}^{\tau}_{k_0} & \breve{U}_{k_0} \end{bmatrix} \begin{bmatrix} \breve{0}_\partial & \breve{I}_\partial \\ -\breve{I}_\partial & \breve{0}_\partial \end{bmatrix} \begin{bmatrix} \breve{A}_{k_0} & \breve{C}_{k_0} \\ \breve{R}^{\tau}_{k_0} & \breve{S}^{\tau}_{k_0} \end{bmatrix} = \begin{bmatrix} \breve{0}_\partial & \breve{I} \\ -\breve{I}_\partial & \breve{0}_\partial \end{bmatrix}. \tag{54}$$

Note that the operator $\breve{Q}_{k_0}$ was absorbed; that is to say, it was arbitrary and undetermined in the proposed factorization Eq. (53). If we carry out the matrix multiplications in Eq. (54), four linear equations in the four unknown operators will be obtained. We obtain sufficient conditions for the existence of a solution to these equations by manipulating the operator algebra to be derived in subsequent sections: we suppose that (i) the operator product $\breve{A}_{k_0}\breve{C}^{\tau}_{k_0}$ is symmetric, that is

$$\breve{A}_{k_0}\breve{C}^{\tau}_{k_0}=\breve{C}_{k_0}\breve{A}^{\tau}_{k_0}=(\breve{A}_{k_0}\breve{C}^{\tau}_{k_0})^{\tau}, \tag{55}$$

and (ii) there is an operator $\breve{Z}$ that is symmetric,

$$\breve{Z}^{\tau}=\breve{Z}, \tag{56}$$

and such that the operators $\breve{Z}$ and $(\breve{A}_{k_0}\breve{Z}+\breve{C}_{k_0})$ are invertible,

$$\text{both } \breve{Z}^{-1} \text{ and } (\breve{A}_{k_0}\breve{Z}+\breve{C}_{k_0})^{-1} \text{ exist}. \tag{57}$$

When these criteria are satisfied, Eq. (54) has the solution

$$\begin{aligned} \breve{T}_{k_0}&=\breve{R}_{k_0}=-(\breve{A}_{k_0}\breve{Z}+\breve{C}_{k_0})^{-1}, \\ \breve{U}_{k_0}&=\breve{S}_{k_0}=\breve{Z}(\breve{A}_{k_0}\breve{Z}+\breve{C}_{k_0})^{-1}. \end{aligned} \tag{58}$$

The solution is not unique: in particular, for any constant λ, $\breve{R}_{k_0}+\lambda\breve{A}_{k_0}$, and $\breve{S}_{k_0}+\lambda\breve{C}_{k_0}, \breve{T}_{k_0}+\lambda\breve{A}^{\tau}_{k_0}, \breve{U}_{k_0}+\lambda\breve{C}^{\tau}_{k_0}$, also comprise a solution. We emphasize that the results Eqs. (54)–(58), as solutions to the problem of Eq. (53), were not derived by a formal procedure; beyond the requirement of employing the boundary conditions Eq. (39), and the knowledge (see Secs. V and VI and Appendix B) that Eqs. (56) and (57) apply, the results were derived with the aid of mathematical guesswork.

The matrix on the rhs of Eq. (54) establishes a nondegenerate, skew-symmetric bilinear form, which is an infinite dimensional *symplectic* geometry, on $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$; a symplectic transformation is a linear transformation on $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ leaving this form invariant. (See, for example, Ref. [45], p. 23 for a definition of finite-dimensional complex symplectic groups.) We note that when the results of Eq. (58) are substituted into Eq. (54), the first matrix factor on the lhs becomes the transpose of the third matrix factor; that is, the restrictions Eqs. (55)–(58) allows us to construct a matrix of dyadics, with $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$ as the first-row elements, which is a symplectic transformation operator on the direct-sum space $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$. An analogous result holds for the acoustic case, as can be inferred by manipulating an equation similar to Eq. (48), which is derived in a manner akin to Ref. [14], Eq. (37). There is in this sense a connection between reciprocity for a complete Green's function and a symplectic transformation in a suitable space of vector-valued functions on $\partial\Omega$.

V. THE RADIATION IMPEDANCE OPERATOR

In this section we shall define the radiation impedance operator $\breve{Z}^{+}_{k_0}$, find its inverse and transpose, and express it in terms of the primitive operators of Appendix A. We will then show that these results imply the existence of certain nonlinear operator identities among, as well as other properties of, the primitive operators.

Let Φ_{k_0} be an electromagnetic field, defined as in Eq. (8) for $\mathbf{r}\in\Omega^{\text{ex}}$, satisfying Maxwell's equations with no sources there, that is,

$$\mathcal{H}_{k_0}\Phi_{k_0}=0, \tag{59}$$

and satisfying the Silver-Müller conditions Eq. (33) for outgoing waves. Then according to Ref. [21], Eq. (4.19) and Theorem 4.27, the electromagnetic field in Φ^{ex} exists and is uniquely determined by prescribing any sufficiently well-behaved limiting tangential electric field on $\partial\Omega$. Uniqueness means that a zero tangential electric field implies that the electromagnetic field is identically zero in Ω^{ex}, while existence means that any tangential electric field is the limit of the electric part of at least one outgoing-wave solution Φ_{k_0} to Eq. (59); therefore, each tangential electric field gives rise to exactly one outgoing-wave solution. By the duality property of Maxwell's equations [Ref. [17], Eq. (6.151)] and of the radiation conditions (Ref. [21], Corollary 4.6), if $\Phi_{k_0}=(\mathbf{E}_{k_0},c\mathbf{B}_{k_0})^{\tau}$ is an outgoing-wave solution to Maxwell's equations, so is the dual solution

$$\Phi^{d}_{k_0}=(-c\mathbf{B}_{k_0},\mathbf{E}_{k_0})^{\tau}. \tag{60}$$

Accordingly, any vector field in $\mathcal{V}^{\partial\Omega}$ is also the limiting magnetic field belonging to exactly one outgoing-wave solution of Eq. (59). We have, therefore, established a one-to-one correspondence of $\mathcal{V}^{\partial\Omega}$ with itself, the first field in a corresponding pair being the tangential electric field, and the second the tangential magnetic field, extracted from the limiting fields on $\partial\Omega$ of an outgoing-wave solution to Maxwell's source-free equations in Ω^{ex}; moreover, the correspondence is obviously linear. By definition, therefore, there exists an invertible linear operator $\breve{Z}^{+}_{k_0}$, which we call the radiation impedance operator, which maps the second field into the first, as follows:

$$(\check{I}_{k_0}\mathbf{E}_{k_0})(\mathbf{r}_\partial)=-(\check{Z}^{+}_{k_0}\check{X}_\partial c\mathbf{B}_{k_0})(\mathbf{r}_\partial)\ , \tag{61}$$

where the $\check{X}_\partial$ operator is positioned for later convenience. We infer from the invertibility of $\check{Z}^{+}_{k_0}$ and from the duality property that

$$(\check{Z}^{+}_{k_0})^{-1}=-\check{X}_\partial\check{Z}^{+}_{k_0}\check{X}_\partial\ . \tag{62}$$

The transpose of $\check{Z}^{+}_{k_0}$ can be determined as follows. Let Φ_{1,k_0} and Φ_{2,k_0} be any pair of outgoing-wave solutions to Maxwell's equations in $\Omega^{\rm ex}$. A sequence of manipulations such as those that led to Eq. (48) can be made to yield, in the notation of Eq. (16),

$$(\mathbf{E}_{1,k_0};\check{X}_\partial c\mathbf{B}_{2,k_0})_{\partial\Omega}=(\mathbf{E}_{2,k_0};\check{X}_\partial c\mathbf{B}_{1,k_0})_{\partial\Omega}\ , \tag{63}$$

from which it follows, taking into account Eqs. (17), (60), and (61), that the radiation impedance operator is symmetric:

$$(\check{Z}^{+}_{k_0})^{\tau}=\check{Z}^{+}_{k_0}\ . \tag{64}$$

Let us now suppose that an electric surface current with density $\mathbf{J}_{k_0,e}$ is established in $\partial\Omega$. The outgoing-wave fields established in $\Omega^{\rm ex}$ can be computed by means of the free-space Green's function. According to Eqs. (31), (A12), and (A13), the exterior limiting tangential electric and magnetic fields on $\partial\Omega$ are

$$(\check{I}_\partial\mathbf{E}_{k_0})(\mathbf{r}_\partial+)=i\mu_0 c(2k_0)^{-1}(\check{X}_\partial\check{N}_{k_0}\check{X}_\partial\mathbf{J}_{k_0,e})(\mathbf{r}_\partial)\ , \tag{65}$$

$$(\check{I}_\partial c\mathbf{B}_{k_0})(\mathbf{r}_\partial+)=(\mu_0 c/2)[\check{X}_\partial(\check{M}_{k_0}+\check{I}_\partial)\mathbf{J}_{k_0,e}](\mathbf{r}_\partial)\ . \tag{66}$$

These equations hold for arbitrary $\mathbf{J}_{k_0,e}$; if this quantity is eliminated between Eqs. (65) and (66), we find that

$$(\check{I}_\partial\mathbf{E}_{k_0})(\mathbf{r}_\partial)=(i/k_0)[\check{N}_{k_0}\check{X}_\partial(\check{M}_{k_0}+\check{I}_\partial)^{-1}\check{X}_\partial c\mathbf{B}_{k_0}](\mathbf{r}_\partial)\ . \tag{67}$$

It follows from Eq. (61) that

$$\check{Z}^{+}_{k_0}=-(i/k_0)\check{X}_\partial\check{N}_{k_0}\check{X}_\partial(\check{M}_{k_0}+\check{I}_\partial)^{-1}\ . \tag{68}$$

A similar calculation with magnetic surface currents and outgoing fields yields the alternative form

$$\check{Z}^{+}_{k_0}=ik_0\check{X}_\partial(\check{M}_{k_0}+\check{I}_\partial)\check{X}_\partial\check{N}^{-1}_{k_0}\ , \tag{69}$$

consistent with Eq. (62).

It follows from Ref. [21], Theorems 4.23 and 4.37, that the operators $(\check{M}_{k_0}+\check{I}_\partial)\check{X}_\partial$ and $\check{N}_{k_0}$ have as a common null space the tangential magnetic field corresponding to any solution of the interior Maxwell problem for Ω with perfectly electrically conducting walls; if Ω is bounded, these magnetic fields are nontrivial only for a discrete set of real k_0 values, the cavity eigenfrequencies. At the same time, the operator $\check{Z}^{+}_{k_0}$ is well defined for all nonzero real wave numbers k_0; therefore, the singularities in the operator quotients of Eqs. (68) and (69) must cancel in some suitable sense as k_0 tends to one of the exceptional values.

Two additional forms for $\check{Z}^{+}_{k_0}$ can be inferred by taking the transpose of Eqs. (68) and (69), and using Eqs. (64), (A6), and (A7),

$$\check{Z}^{+}_{k_0}=(i/k_0)\check{X}_\partial(\check{M}_{k_0}-\check{I}_\partial)^{-1}\check{N}_{k_0}\check{X}_\partial, \tag{70}$$

$$\check{Z}^{+}_{k_0}=ik_0\check{N}^{-1}_{k_0}(\check{M}_{k_0}-\check{I}_\partial). \tag{71}$$

We can now derive certain nonlinear identities involving the primitive operators. It follows from Eqs. (69), (71), (A6), and (A7) that

$$\check{M}_{k_0}\check{N}_{k_0}=\check{N}_{k_0}\check{X}_\partial\check{M}_{k_0}\check{X}_\partial=(\check{M}_{k_0}\check{N}_{k_0})^{\tau}\ , \tag{72}$$

that is, the operator $\check{M}_{k_0}\check{N}_{k_0}$ is symmetric. It follows from Eqs. (68) and (71) that

$$\check{I}_\partial-\check{M}^{2}_{k_0}-k_0^{-2}\check{N}_{k_0}\check{X}_\partial\check{N}_{k_0}\check{X}_\partial=\check{0}_\partial\ . \tag{73}$$

Equation (73) can also be inferred from Eqs. (4.55′) and (4.56) of Ref. [21].

Equations (72) and (73) imply that if we define the linear operator $\mathring{\mathbf{P}}_{k_0}$ in $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ as

$$\mathring{\mathbf{P}}_{k_0}=\begin{bmatrix} -(\frac{1}{2})\check{X}_\partial(\check{I}_\partial+\check{M}_{k_0})\check{X}_\partial & -i(2k_0)^{-1}\check{X}_\partial\check{N}_{k_0} \\ i(2k_0)^{-1}\check{X}_\partial\check{N}_{k_0} & -(\frac{1}{2})\check{X}_\partial(\check{I}_\partial+\check{M}_{k_0})\check{X}_\partial \end{bmatrix}, \tag{74}$$

then $\mathring{\mathbf{P}}_{k_0}$ is a projection (idempotent) operator:

$$\mathring{\mathbf{P}}_{k_0}\mathring{\mathbf{P}}_{k_0}=\mathring{\mathbf{P}}_{k_0}\ . \tag{75}$$

That such a projection operator exists is also evident from the construction of the electromagnetic fields in $\Omega^{\rm ex}$ from the tangential fields given in Ref. [21], Theorem 4.5, for the tangential fields should be reproduced by taking the tangential limits of the exterior fields.

It follows from Eq. (31) that when the geometries of Ω and $\partial\Omega$ make the interior limits meaningful, the tangential surface field discontinuities and surface current densities satisfy

$$(\check{I}_\partial\check{E}_{k_0})(\mathbf{r}_\partial+)-(\check{I}_\partial\mathbf{E}_{k_0})(\mathbf{r}_\partial-)=+c\mu_0(\check{X}_\partial\mathbf{J}_{k_0,m})(\mathbf{r}_\partial)\ , \tag{76}$$

$$(\check{I}_\partial c\mathbf{B}_{k_0})(\mathbf{r}_\partial+)-(\check{I}_\partial c\mathbf{B}_{k_0})(\mathbf{r}_\partial-)=-c\mu_0(\check{X}_\partial\mathbf{J}_{k_0,e})(\mathbf{r}_\partial)\ . \tag{77}$$

This suggests that we define the self-inverse operator $\mathring{X}$ in $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ as a "current-to-tangential-field" transformation,

$$\mathring{X}\equiv\begin{bmatrix} \check{0}_\partial & \check{X}_\partial \\ -\check{X}_\partial & \check{0}_\partial \end{bmatrix}, \tag{78}$$

and the transformed projection operator is, using obvious definitions,

$$\mathring{P}'_{k_0}\equiv\mathring{X}\mathring{P}_{k_0}\mathring{X}=\mathring{I}-\mathring{P}^{\tau}_{k_0}\ . \tag{79}$$

Straightforward computations, along the lines of Appendix A, show that a vector field in $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ that is in the unit space (respectively, null space) of $\mathring{P}'_{k_0}$ is a direct sum of surface electric and magnetic currents that yield zero limiting tangential interior (exterior) fields; since the complete interior (exterior) fields can be constructed from their limiting interior (exterior) tangential fields—see Ref. [21], Theorem 4.1 (Theorem 4.5)—the complete electromagnetic fields are then also zero everywhere in Ω (respectively, $\Omega^{\rm ex}$).

We remark that several results associated with the theory of the radiation impedance operator in the electromagnetic case have analogues in the theory of the acoustic radiation impedance operator—see Ref. [14], in particular Secs. II C and II D. In a related vein, a recent review article [46] summarizes research on numerical work with nonreflecting boundary conditions in a number of physically distinct applications of scattering theory; nonreflecting boundary conditions for Maxwell's equations in three-dimensional space were not studied, however. We note also a connection with recent work in mathematics that was overlooked in Ref. [14]: following a name convention in a number of mathematical papers [47,48], we can construe the operator called $\check{Z}^{-1}_{k_0}$ in Ref. [14] as the "Dirichlet-to-Neumann operator" associated with the exterior acoustic radiation problem and boundary $\partial\Omega$, while the operators A and B determine implicitly the "Dirichlet-to-Neumann map" for the interior acoustic problem; the problem is simulated by Robin boundary conditions on $\partial\Omega$. The operator $\check{Z}^{+}_{k_0}$ associated with exterior electromagnetic radiation, and the operators $\check{A}_{k_0}$ and $\check{C}_{k_0}$, which determine the surface relation between the tangential electric and magnetic fields associated with the interior problem, evidently play an analogous role to the acoustic, "Dirichlet-to-Neumann," counterparts.

VI. THE TRANSITION OPERATOR

In this section we shall find an explicit expression, in terms of the operators $\check{A}_{k_0}$, $\check{C}_{k_0}$, and $\check{Z}^{+}_{k_0}$, for the T operator for electromagnetic-wave scattering with Leontovich boundary conditions on $\partial\Omega$. By specializing to $\check{C}_{k_0}=\check{0}_\partial$, we recover the T operator for the case that the obstacle is a perfect electrical conductor.

We note that if the source point is in Ω, the resulting electromagnetic field associated with $\Gamma^{0++}_{k_0}$ is a source-free, outgoing-wave solution to Maxwell's equations, and hence satisfies Eq. (61). Therefore we have, for $\mathbf{r}_2\in\Omega$, and for $\beta=e$ or m,

$$(\check{I}_\partial\Gamma^{0++}_{k_0,e\beta}+\check{Z}^{+}_{k_0}\check{X}_\partial\Gamma^{0++}_{k_0,m\beta})_{jk}(\mathbf{r}_{\partial 1};\mathbf{r}_2)=0\ . \tag{80}$$

Equations (51) and (64) now imply that

$$(\Gamma^{0++}_{k_0,\beta e}\check{I}_\partial+\Gamma^{0++}_{k_0,\beta m}\check{X}_\partial\check{Z}^{+}_{k_0})_{jk}(\mathbf{r}_1;\mathbf{r}_{\partial 2})=0\ , \tag{81}$$

whenever $\mathbf{r}_1\in\Omega$.

According to Eq. (39), the difference Green's function

$$(\Delta\Gamma)^{+\pm}_{k_0}\equiv\Gamma^{L+\pm}_{k_0}-\Gamma^{0+\pm}_{k_0} \tag{82}$$

comprises an outgoing wave, source-free signal in $\Omega^{\rm ex}$ with respect to the field point. The tangential, surface-limiting fields must therefore satisfy Eq. (61), which implies the following for the Green's dyadics with $\beta=e$ or m:

$$\check{I}_\partial\Gamma^{L+\pm}_{k_0,e\beta}+\check{Z}^{+}_{k_0}\check{X}_\partial\Gamma^{L+\pm}_{k_0,m\beta}=\check{I}_\partial\Gamma^{0+\pm}_{k_0,e\beta}+\check{Z}^{+}_{k_0}\check{X}_\partial\Gamma^{0+\pm}_{k_0,m\beta}\ . \tag{83}$$

These dyadics must also satisfy Eq. (40). We therefore have two equations for two unknown dyadics in each case ($\beta=e,m$), the solution of which is

$$\begin{aligned}\check{X}_\partial\Gamma^{L+\pm}_{k_0,e\beta}=&[-\check{Z}^{+}_{k_0}(\check{A}_{k_0}\check{Z}^{+}_{k_0}+\check{C}_{k_0})^{-1}\check{A}_{k_0}\check{X}_\partial+\check{X}_\partial]\\&\times(\check{I}_\partial\Gamma^{0+\pm}_{k_0,e\beta}+\check{Z}^{+}_{k_0}\check{X}_\partial\Gamma^{0+\pm}_{k_0,m\beta})\ ,\end{aligned} \tag{84}$$

$$\begin{aligned}\check{X}_\partial\Gamma^{L+\pm}_{k_0,m\beta}=&(\check{A}_{k_0}\check{Z}^{+}_{k_0}\check{C}_{k_0})^{-1}\\&\times\check{A}_{k_0}(\check{I}_\partial\Gamma^{0+\pm}_{k_0,e\beta}+\check{Z}^{+}_{k_0}\check{X}_\partial\Gamma^{0+\pm}_{k_0,m\beta})\ .\end{aligned} \tag{85}$$

An argument similar to that leading to Eq. (48), or Ref. [21], Theorem 4.5, shows that a source-free, outgoing-wave solution to Maxwell's equations in $\Omega^{\rm ex}$ can be reconstructed from its tangential limiting values on $\partial\Omega$. The difference Green's function Eq. (82) represents such a solution for any fixed source point, type, and orientation, while Eqs. (84), (85), (A2), and (A3) provide the surface fields. In $\mathcal{V}^{3\oplus 3}$ operator form, we have

$$\Theta_{\Omega^{\rm ex}}(\mathbf{r}_1)(\Delta\Gamma)^{+\pm}_{k_0}=-i\Gamma^{0++}_{k_0}\mathring{X}(\Delta\Gamma)^{+\pm}_{k_0}\ , \tag{86}$$

where $\mathbf{r}_1$ is the field point (left argument in the Green's functions). Following substitution of Eqs. (84) and (85) into the rhs of Eq. (86), and some algebra, we obtain

$$\begin{aligned}\Gamma^{L+\pm}_{k_0}=&[1-\Theta_\Omega(\mathbf{r}_1)]\Gamma^{0+\pm}_{k_0}+(i/2)\Gamma^{0++}_{k_0}\mathring{X}\Gamma^{0+\pm}_{k_0}\\&+\Gamma^{0++}_{k_0}\mathring{T}^{L+}_{k_0}\Gamma^{0+\pm}_{k_0}\ ,\end{aligned} \tag{87}$$

where a redundant $\Theta_{\Omega^{\rm ex}}(\mathbf{r}_1)$ multiplying $\Gamma^{L+\pm}_{k_0}$ has been dropped, and the operator $\mathring{T}^{L+}_{k_0}$ is defined as follows:

$$\mathring{T}^{L+}_{k_0}=-(i/2)\mathring{X}+\begin{bmatrix}-i(\check{A}_{k_0}\check{Z}^{+}_{k_0}+\check{C}_{k_0})^{-1}\check{A}_{k_0} & i(\check{A}_{k_0}\check{Z}^{+}_{k_0}+\check{C}_{k_0})^{-1}\check{C}_{k_0}\check{X}_\partial\\ -i\check{X}_\partial\check{Z}^{+}_{k_0}(\check{A}_{k_0}\check{Z}^{+}_{k_0}+\check{C}_{k_0})^{-1}\check{A}_{k_0} & i\check{X}_\partial\check{Z}^{+}_{k_0}(\check{A}_{k_0}\check{Z}^{+}_{k_0}+\check{C}_{k_0})^{-1}\check{C}_{k_0}\check{X}_\partial\end{bmatrix}\ . \tag{88}$$

In view of Eqs. (55) and (64), the operator $\mathring{T}^{L+}_{k_0}$ satisfies the analog in $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ space of the reciprocity condition Eq. (44).

In order to complete the reduction of Eq. (87) to the form of Eq. (38), more preliminary results are needed. If in the derivation of Eq. (48) we had chosen to integrate over Ω instead of Ω^{ex}, the requirement of outgoing-wave conditions could be dropped, and any pair of free-space Green's functions used. By this means we infer that

$$0=\Theta_\Omega(\mathbf{r}_1)\Gamma^{0+\pm}_{k_0}-\Gamma^{0++}_{k_0}\Theta_\Omega(\mathbf{r}_2)-i\Gamma^{0++}_{k_0}\mathring{X}\Gamma^{0+\pm}_{k_0}. \tag{89}$$

We define a modified Heaviside operator $\mathcal{H}^{\Omega}_{k_0}$, which is the standard operator modified by a Θ_Ω: if Φ and Ψ are any pair of $\mathcal{V}^{3\oplus 3}$ vector fields, then

$$(\Phi;\mathcal{H}^{\Omega}_{k_0}\Psi)_{\mathcal{V}^{3\oplus 3}}\equiv\int_\Omega d^3r\,\Phi^T\mathcal{H}_{k_0}\Psi\ . \tag{90}$$

We also define a Heaviside operator that differentiates to the left,

$$\tilde{\mathcal{H}}^{\Omega}_{k_0}\equiv\begin{bmatrix} k_0\delta_{jk} & i(\underset{\leftarrow}{\nabla}\times)_{jk} \\ -i(\underset{\leftarrow}{\nabla}\times)_{jk} & k_0\delta_{jk}\end{bmatrix}\Theta_\Omega(\mathbf{r})\ , \tag{91}$$

where the left arrow under the symbol ∇ indicates differentiation to the left when a matrix element of the operator is generated. Note that

$$(\Phi;\tilde{\mathcal{H}}^{\Omega}_{k_0}\Psi)_{\mathcal{V}^{3\oplus 3}}=(\Psi;\Pi\mathcal{H}^{\Omega}_{k_0}\Pi\Phi)_{\mathcal{V}^{3\oplus 3}}\ , \tag{92}$$

so that the operator

$$\tilde{\mathcal{H}}^{\Omega}_{k_0}+\mathcal{H}^{\Omega}_{k_0} \tag{93}$$

satisfies the reciprocity condition. In view of Eq. (32), moreover, we have

$$\Theta_\Omega(\mathbf{r}_1)\Gamma^{0+\pm}_{k_0}=\Gamma^{0++}_{k_0}\tilde{\mathcal{H}}^{\Omega}_{k_0}\Gamma^{0+\pm}_{k_0}\ , \tag{94}$$

$$\Gamma^{0++}_{k_0}\Theta_\Omega(\mathbf{r}_2)=\Gamma^{0++}_{k_0}\mathcal{H}^{\Omega}_{k_0}\Gamma^{0+\pm}_{k_0}\ . \tag{95}$$

We can now compute the T operator. We multiply both sides of Eq. (89) by the factor $(\frac{1}{2})$, and add corresponding sides of the resulting equation to Eq. (87). Upon applying Eqs. (94) and (95), we secure an expression of the form of Eq. (38), with

$$\mathcal{T}^{L+}_{k_0}=-\tfrac{1}{2}(\tilde{\mathcal{H}}^{\Omega}_{k_0}+\mathcal{H}^{\Omega}_{k_0})+\mathring{T}^{L+}_{k_0}\ . \tag{96}$$

When computing a matrix element of the operator $\mathcal{T}^{L+}_{k_0}$ with respect to a pair of vector fields in $\mathcal{V}^{3\oplus 3}$, we imagine that the last operator on the rhs of Eq. (96) has projection operators on either side that select from the adjacent vector field (presumed continuous in $\mathcal{E}^3$) its limiting tangential field on $\partial\Omega$; each limiting field belongs to $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$, so that a matrix element of the operator $\mathring{T}^{L+}_{k_0}$ with respect to, say, plane-wave states, is well defined. We note that in view of the properties of the $\mathring{T}^{L+}_{k_0}$ operator, the purely retarded complete Green's function $\Gamma^{L++}_{k_0}$ satisfies the reciprocity condition.

We can recover the case that $\Omega\cup\partial\Omega$ is filled by a perfect electrical conductor by setting $\breve{C}_{k_0}=\breve{0}_\partial$ and presuming $\breve{A}_{k_0}$ is invertible in Eq. (88). The resulting T operator is

$$\mathcal{T}^{E+}_{k_0}=-\tfrac{1}{2}(\tilde{\mathcal{H}}^{\Omega}_{k_0}+\mathcal{H}^{\Omega}_{k_0})+\mathring{T}^{E+}_{k_0}\ , \tag{97}$$

where

$$\mathring{T}^{E+}_{k_0}=\begin{bmatrix}-i(\breve{Z}^{+}_{k_0})^{-1} & -(i/2)\breve{X}_\partial \\ -(i/2)\breve{X}_\partial & \breve{0}_\partial\end{bmatrix}. \tag{98}$$

With the derivation of Eqs. (88) and (96)–(98) complete, we have achieved the principal objectives of this paper. Certain applications and a further discussion of the general theory established herein are subjects of a planned future publication [22].

VII. SCATTERING AMPLITUDES; THE CASE OF ZERO SCATTERING

We shall now conclude the paper by first, extracting the scattering amplitude and the differential scattering cross section for electromagnetic-wave scattering in terms of certain matrix elements of the T operator, and second, demonstrating that there exist, at least in a mathematical sense, Leontovich boundary conditions on an obstacle such that the scattered wave is identically zero at a given frequency. It is argued on physical grounds that realization of zero electromagnetic-wave scattering boundary conditions in the time domain, which is at all frequencies, conflicts with causality.

We shall consider only the purely outgoing-wave Green's function $\Gamma^{L++}_{k_0,\alpha\beta,jk}(\mathbf{r}_1;\mathbf{r}_2)$, and let the distance r_2 of the source point from the origin become large. Then the original free-space signal in the vicinity of the obstacle, that is the first summand on the rhs of Eq. (39), looks—except for a slowly varying modulation—like a set of plane-wave states (column vectors), all with propagation direction $\hat{\lambda}=-\hat{r}_2$; only two of the six column vectors are independent, however. Moreover, as the field point $\mathbf{r}_1$ also recedes, we expect to obtain something like the scattering amplitude times $r_1^{-1}\exp(ik_0r_1)$ from the second summand on the rhs of Eq. (39) (see Ref. [49], Eq. (15)). Let us define the six-by-six polarization matrix $Q(\hat{\lambda})$ to have matrix elements

$$\begin{aligned}&[Q(\hat{\lambda})]_{ee,jk}=[Q(\hat{\lambda})]_{mm,jk}\equiv\delta_{jk}-\lambda_j\lambda_k\ ,\\ &[Q(\hat{\lambda})]_{me,jk}=-[Q(\hat{\lambda})]_{em,jk}\equiv\epsilon_{jpk}\lambda_p\ ,\end{aligned} \tag{99}$$

where $\hat{\lambda}$ is any unit three-vector. Then we have

$$\begin{aligned}&\Gamma^{0++}_{k_0,\eta\beta,qk}(\mathbf{r}_4;-\hat{\lambda}r_2)\\ &\quad\underset{r_2\to\infty}{\sim}-k_0(4\pi r_2)^{-1}\exp(ik_0r_2)\exp(ik_0\hat{\lambda}\cdot\mathbf{r}_4)\\ &\qquad\times[Q(\hat{\lambda})]_{\eta\beta,qk}\ ,\end{aligned} \tag{100}$$

$$\Gamma^{0++}_{k_0,\alpha\zeta,jp}(\hat{\kappa}r_1;\mathbf{r}_3) \underset{r_1\to\infty}{\sim} -k_0(4\pi r_1)^{-1}\exp(ik_0r_1)\exp(-ik_0\hat{\kappa}\cdot\mathbf{r}_3) \times[\Pi\mathcal{Q}(-\hat{\kappa})^T\Pi]_{\alpha\zeta,jp} \; . \tag{101}$$

The factor $-k_0(4\pi r_2)^{-1}\exp(ik_0r_2)$ on the rhs of Eq. (100), and hence of the asymptotic form of Eq. (39), is dropped; what remains of the rhs of Eq. (39) are scattering wave functions formed from initial, free-space plane waves (of unit amplitude, if the source current is perpendicular to $\hat{\mathbf{r}}_2$). We infer from the limiting form of these wave functions as $r_1\to\infty$, with $\hat{\mathbf{r}}_1=\hat{\kappa}$ as the final propagation direction, that the six-by-six scattering amplitude matrix has the expression

$$\langle\alpha j\hat{\kappa}\|\mathcal{A}_{k_0}\|\beta k\hat{\lambda}\rangle=-k_0(4\pi)^{-1}\sum_{\zeta,p}\sum_{\eta,q}[\Pi\mathcal{Q}(-\hat{\kappa})^T\Pi]_{\alpha\zeta,jp}\left[\int_{\partial\Omega}dA_3\int_{\partial\Omega}dA_4\exp(-ik_0\hat{\kappa}\cdot\mathbf{r}_{\partial 3})\mathring{T}^{L+}_{k_0,\zeta\eta,pq}(\mathbf{r}_{\partial 3};\mathbf{r}_{\partial 4})\times\exp(ik_0\hat{\lambda}\cdot\mathbf{r}_{\partial 4})\right][\mathcal{Q}(\hat{\lambda})]_{\eta\beta,qk} \; . \tag{102}$$

We have accounted in Eq. (102) for the fact that, in view of Eqs. (59) and (92), both T-operator constituents $\tilde{\mathcal{H}}^{\partial\Omega}_{k_0}$ and $\mathcal{H}^{\partial\Omega}_{k_0}$ give zero contribution to the scattering amplitude. Only the dyadic constituent with $\alpha=\beta=e$ needs to be carried along, as the remaining dyadics can be obtained from it by operating with $(\hat{\kappa}\times)$ on the left and/or $-(\times\hat{\lambda})$ on the right; note that this dyadic is the scattering amplitude matrix of Ref. [49], and satisfies the condition of reciprocity, as defined in Ref. [49], Eq. (22). Let the real unit vectors $\hat{\epsilon}$ and $\hat{\epsilon}'$ represent the final and initial electric linear polarization vectors, respectively, with $\hat{\epsilon}\cdot\hat{\kappa}=0$ and $\hat{\epsilon}'\cdot\hat{\lambda}=0$. Them the differential cross section for scattering with initial direction $\hat{\lambda}$ and initial linear polarization $\hat{\epsilon}'$, into final direction $\hat{\kappa}$ with linear polarization $\hat{\epsilon}$, is [compare Ref. [17], Eq. (9.81)]

$$\frac{d\sigma_{k_0}}{d\omega(\hat{\kappa})}(\hat{\kappa}\hat{\epsilon}\leftarrow\hat{\lambda}\hat{\epsilon}')=\left|\sum_{j,k}\epsilon_j\langle ej\hat{\kappa}\|\mathcal{A}_{k_0}\|ek\hat{\lambda}\rangle\epsilon'_k\right|^2 , \tag{103}$$

where $d\omega(\hat{\kappa})$ is the differential of solid angle around direction $\hat{\kappa}$.

We note a consequence of the structure of the projection operator Eq. (74) and of the Leontovich boundary conditions Eq. (35): it is possible to choose the operators $\check{A}_{k_0}$ and $\check{C}_{k_0}$ so that the obstacle gives rise to no scattered wave at all. This physical property is in a sense opposite to the blackbody property: in the former case, the tensor scattering (S) matrix of Ref. [49] is the unit matrix (zero scattering amplitude), while in the latter case the S matrix is, for some linear subspace of the space of spherical electromagnetic waves with wave number k_0, the zero matrix (no outgoing waves, i.e., total absorption of any ingoing wave). We extract the α,β tangential operator component of the projection operator of Eq. (74) as

$$\check{P}_{k_0,\alpha\beta}\equiv(\mathring{P}_{k_0})_{\alpha\beta} \; , \tag{104}$$

and take the operators $\check{A}'_{k_0}$ and $\check{C}'_{k_0}$ such that they satisfy conditions to be described presently. Now let

$$\check{A}_{k_0}=\check{A}'_{k_0}\check{P}_{k_0,ee}-\check{C}'_{k_0}\check{X}_\partial\check{P}_{k_0,me} \; , \tag{105}$$

$$\check{C}_{k_0}=(\check{A}'_{k_0}\check{P}_{k_0,em}-\check{C}'_{k_0}\check{X}_\partial\check{P}_{k_0,m,m})\check{X}_\partial \; . \tag{106}$$

Then by virtue of Eqs. (A6), (A7), (72), and (73), the reciprocity condition Eq. (55) is satisfied by the pair $\check{A}_{k_0},\check{C}_{k_0}$, whatever be $\check{A}'_{k_0}$ and $\check{C}'_{k_0}$. Moreover, Eqs. (70) and (71) imply that for arbitrary $\check{A}'_{k_0}$ and $\check{C}'_{k_0}$, we have

$$\check{A}_{k_0}\check{Z}^+_{k_0}+\check{C}_{k_0}=\check{A}'_{k_0}\check{Z}^+_{k_0}+\check{C}'_{k_0} \; . \tag{107}$$

In a generic scattering problem, we have in Ω^{ex} (i) the total field $(\mathbf{E}^T,c\mathbf{B}^T)^\tau$, which is the superposition of (ii) the initial field $(\mathbf{E}^0,c\mathbf{B}^0)^\tau$ and (iii) the scattered field $(\mathbf{E}^+,c\mathbf{B}^+)^\tau$. Field (ii) also represents a solution to Maxwell's source-free equations Eq. (59) in an open set that includes $\Omega\cup\partial\Omega$; therefore, its surface values on $\partial\Omega$ are annihilated by the operator $\mathring{P}_{k_0}$. Field (iii) is a solution of Eq. (59) of outgoing-wave types, so that its surface tangential values are reproduced by the operator $\mathring{P}_{k_0}$. The field (i) must satisfy Eq. (35), which now reduces to

$$\check{A}'_{k_0}\mathbf{E}^+-\check{C}'_{k_0}\check{X}_\partial c\mathbf{B}^+=\mathbf{0}_\partial \; . \tag{108}$$

The scattered field satisfies Eq. (61), moreover, so that we now infer

$$(\check{A}'_{k_0}\check{Z}^+_{k_0}+\check{C}'_{k_0})\check{X}_\partial\mathbf{B}^+=\mathbf{0}_\partial \; . \tag{109}$$

If we could restrict the operators $\check{A}'_{k_0}$ and $\check{C}'_{k_0}$ so that $\check{I}_\partial\mathbf{B}=\mathbf{0}_\partial$ is the only solution to Eq. (109), there would be no scattered wave, by the uniqueness theorem; we shall discuss these restrictions in Appendix B. Boundary conditions of this type, if they exist physically, would simulate empty space for the interior Ω: the obstacle would be invisible to electromagnetic radiation with wave number k_0. It is not known if a suitable combination of active and passive mechanisms can be manufactured so as to simulate to desired accuracy Leontovich boundary conditions of the type generated by Eqs. (105) and (106) for even a single wave number of the k_0 spectrum.

We remark that there exist analogous sets of Robin boundary conditions that yield zero scattering amplitude at a given frequency in the acoustic case [14,50].

In connection with boundary conditions that yield zero electromagnetic scattering, let us give brief consideration to scattering processes in the time (ct) domain, such that

SBC's of the generic Leontovich form Eq. (35) are satisfied throughout the frequency (k_0) domain. By virtue of our physical hypotheses concerning the nature of the obstacle—see the closing paragraphs of Sec. III—disturbances within the obstacle cannot propagate faster than the speed of light in vacuum. The T operator represents the response, in the form of an electromagnetic current, to the stimulus of an impinging field; therefore, the transformation of the operator of Eq. (88) to the time domain yields a signal that originates at a point stimulus in space-time and that must be zero outside the forward light cone with respect to the stimulus. Physical considerations may led to the imposition of a more restrictive causality requirement, that is, that the response to a signal must propagate at least in part along the surface of the obstacle at a speed less than the speed of light, so that the geodesic distance, rather than the Euclidean distance, between two points on $\partial\Omega$ should be applied to the computation of the maximum propagation speed. These causality requirements on the T operator in the time domain entail limitations on the constituent operators $\check{A}_{k_0}$ and $\check{C}_{k_0}$, taken collectively across the k_0 spectrum. The mathematical problem of translating the causality requirements into useful restrictions on the k_0 dependence of $\check{A}_{k_0}$ and $\check{C}_{k_0}$ in Eq. (35) remains to be investigated. It is plausible on physical grounds, however, that it is not possible to simulate empty space, as in Eqs. (105) and (106), across the spectrum in a causal structure: For, an electromagnetic wave in empty space travels at the speed of light in $\mathcal{E}^3$, so that the surface currents on $\partial\Omega$ would have to rearrange themselves at a speed corresponding to straight-line transmission in $\mathcal{E}^3$; signals in material obstacles are inevitably delayed somewhat with respect to light signals in empty space, so that the surface currents on $\partial\Omega$, whether real or fictitious, will lose the race with light, and will be unable to respond so as to prevent a scattered wave from being generated.

APPENDIX A: FREE-SPACE GREEN'S FUNCTIONS; SURFACE LIMITS AND PRIMITIVE OPERATORS

The free-space Green's functions can be obtained by Fourier-transform methods from Eq. (32), with the choice of outgoing or ingoing waves at infinity being realized in the well-known manner. The results, which we shall state without a derivation, are equivalent to the formulas of Papas (Ref. [51], pp. 22 and 23, Eqs. (16), (20), (24), and (25)), whose electromagnetic potentials and "antipotentials" are present only implicitly here. It is convenient to employ the scalar free-space Green's functions $G^{0\pm}_{k_0}$, which are

$$G^{0\pm}_{k_0}(\mathbf{r}_1;\mathbf{r}_2)=-(4\pi)^{-1}\exp(\pm ik_0|\mathbf{r}_1-\mathbf{r}_2|)\ , \tag{A1}$$

where the upper "+" (lower "−") superscript sign corresponds to outgoing (ingoing) radiation. The Green's dyadics of the rhs of Eq. (34) are as follows:

$$G^{0\pm}_{k_0,ee,jk}(\mathbf{r}_1;\mathbf{r}_2)=G^{0\pm}_{k_0,mm,jk}(\mathbf{r}_1;\mathbf{r}_2)$$
$$=(k_0\delta_{jk}+k_0^{-1}\nabla_{1j}\nabla_{1k})G^{0\pm}_{k_0}(\mathbf{r}_1;\mathbf{r}_2)\ , \tag{A2}$$

$$G^{0\pm}_{k_0,me,jk}(\mathbf{r}_1;\mathbf{r}_2)=-G^{0\pm}_{k_0,em,jk}(\mathbf{r}_1;\mathbf{r}_2)$$
$$=-i(\nabla_1\times)_{jk}G^{0\pm}_{k_0}(\mathbf{r}_1;\mathbf{r}_2)\ , \tag{A3}$$

where the differentiations are to be carried out after integration over $\mathbf{r}_2$. The problem of the treatment of the singularities in the dyadics of Eq. (A2), that is, of the evaluation of the fields at points where the source current densities are not locally zero, has been discussed at length—see Refs. [30] and [52], and references given therein. Note that the purely outgoing-wave Green's function defined by Eqs. (34), (A2), and (A3) satisfies the reciprocity condition Eq. (44).

We recapitulate for the reader's convenience the definitions and properties of certain operators, which map $\mathcal{V}^{\partial\Omega}$ into itself, and which we call "primitive" operators by analogy with the operators of Ref. [14], Sec. III A. Let $\mathbf{a}(\mathbf{r}_{\partial 2})$ be a tangential current distribution on $\partial\Omega$. We define [Ref. [21], Eqs. (2.82) and (2.85)]

$$(\check{M}_{k_0}\mathbf{a})(\mathbf{r}_{\partial 1})\equiv-2\int_{\partial\Omega}dA_2\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})\times\{\nabla_1\times[G^{0+}_{k_0}(\mathbf{r}_1;\mathbf{r}_{\partial 2})\mathbf{a}(\mathbf{r}_{\partial 2})]|\}\bigg|_{\mathbf{r}_1=\mathbf{r}_{\partial 1}}\ , \tag{A4}$$

$$(\check{N}_{k_0}\mathbf{a})(\mathbf{r}_{\partial 1})\equiv-2\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})\times\lim_{\mathbf{r}_1\to\mathbf{r}_{\partial 1}\pm}\left[\nabla_1\times\nabla_1\times\int_{\partial\Omega}dA_2G^{0+}_{k_0}(\mathbf{r}_1;\mathbf{r}_{\partial 2})[\hat{\mathbf{n}}(\mathbf{r}_{\partial 2}\times\mathbf{a}(\mathbf{r}_{\partial 2})]\right]\ . \tag{A5}$$

It is shown in Ref. [21], p. 64, that the operator $\check{N}_{k_0}$ is well defined by Eq. (A5), in that the limit is the same whether $\partial\Omega$ is approached from Ω^{ex} or from Ω.

The primitive operators have the following transposed forms (Ref. [21], Eq. (2.83) and p. 64):

$$\check{M}^{\tau}_{k_0}=\check{X}_\partial\check{M}_{k_0}\check{X}_\partial\ , \tag{A6}$$

$$\check{N}^{\tau}_{k_0}=\check{N}_{k_0}\ . \tag{A7}$$

Given a vector field $\mathbf{a}\in\mathcal{V}^{\partial\Omega}$, we define an associated vector potential $\mathbf{A}(\mathbf{r}_1)$ for $\mathbf{r}_1\in\Omega\cup\Omega^{\text{ex}}$ (Ref. [21], Eq. (2.64)),

$$\mathbf{A}(\mathbf{r}_1)\equiv-\int_{\partial\Omega}dA_2G^{0+}_{k_0}(\mathbf{r}_1;\mathbf{r}_{\partial 2})(\check{I}_\partial\mathbf{a})(\mathbf{r}_{\partial 2})\ , \tag{A8}$$

where the redundant $\check{I}_\partial$ is introduced for clarity. It can be shown that (Ref. [21], Eqs. (2.71) and (2.85))

$$\lim_{\mathbf{r}_1\to\mathbf{r}_{\partial 1}\pm}\{\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})\times[\nabla_1\times\mathbf{A}(\mathbf{r}_1)]\}=\tfrac{1}{2}[(\check{M}_{k_0}\pm\check{I}_\partial)\mathbf{a}](\mathbf{r}_{\partial 1})\ , \tag{A9}$$

$$\lim_{\mathbf{r}_1\to\mathbf{r}_{\partial 1}\pm}\{\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})\times[\nabla_1\times\nabla_1\times\mathbf{A}(\mathbf{r}_1)]\}=-\tfrac{1}{2}(\check{N}_{k_0}\check{X}_\partial\mathbf{a})(\mathbf{r}_{\partial 1})\ . \tag{A10}$$

We suppose that the electric and magnetic currents are confined to tangential surface currents on $\partial\Omega$, in otherwise empty space. We want to find the limits, as the field point approaches $\partial\Omega$ from $\Omega\cup\Omega^{\rm ex}$, of the tangential electric and magnetic fields generated by these currents; the fields are presumed computed from the currents with the free-space Green's function $\Gamma^{0++}_{k_0}$. These limits can be given in terms of the primitive operators $\check{M}_{k_0}$ and $\check{N}_{k_0}$. The scalar Green's function satisfies the scalar Helmholtz equation when the field point is apart from any sources; this property has the effect in the present circumstances of allowing us to make the following replacement in Eq. (A2):

$$k_0\delta_{jk}+k_0^{-1}\nabla_{1j}\nabla_{1k}\to k_0^{-1}[\nabla_1\times(\nabla_1\times)]_{jk}\ . \tag{A11}$$

Accordingly, Eqs. (34), (A2), (A3), (A7), and (A8) now imply that the dyadic components of the free-space Green's function have the following operators as limiting tangential projections:

$$\check{I}_\partial\lim_{\mathbf{r}_L\to\mathbf{r}_{\partial L}\pm}\Gamma^{0++}_{ee}\check{I}_\partial=\check{I}_\partial\lim_{\mathbf{r}_L\to\mathbf{r}_{\partial L}\pm}\Gamma^{0++}_{mm}\check{I}_\partial$$
$$=-(2k_0)^{-1}(\check{X}_\partial\check{N}_{k_0}\check{X}_\partial)\ , \tag{A12}$$

$$\check{I}_\partial\lim_{\mathbf{r}_L\to\mathbf{r}_{\partial L}\pm}\Gamma^{0++}_{me}\check{I}_\partial=-\check{I}_\partial\lim_{\mathbf{r}_L\to\mathbf{r}_{\partial L}\pm}\Gamma^{0++}_{em}\check{I}_\partial$$
$$=-(i/2)[\check{X}_\partial(\check{M}_{k_0}\pm\check{I}_\partial)]\ , \tag{A13}$$

where the "L" subscript means that the left-hand argument of the Green's function is the last to approach $\partial\Omega$, whether from $\Omega^{\rm ex}$ (upper sign) or from Ω (lower sign).

The primitive operators satisfy certain nonlinear operator identities, which are Eqs. (72) and (73), in Sec. V.

APPENDIX B: MATHEMATICS OF LEONTOVICH BOUNDARY CONDITIONS

In this appendix we shall establish a set of conditions on the operators $\check{A}_{k_0}$ and $\check{C}_{k_0}$ of Eq. (35), which conditions, when satisfied, are sufficient to guarantee the existence of the inverse to the operator $(\check{A}_{k_0}\check{Z}^{+}_{k_0}+\check{C}_{k_0})$; the existence of this inverse was taken for granted in the derivation of Sec. VI. A brief discussion of some sufficient conditions on the operators $\check{A}'_{k_0}$ and $\check{C}'_{k_0}$ of Eqs. (105) and (106), such that the operator of Eq. (107) has an inverse, concludes the appendix.

It is now convenient to define a sesquilinear inner product for an ordered pair of tangential vector fields $\mathbf{E}_1\in\mathcal{V}^{\partial\Omega}$ and $\mathbf{E}_2\in\mathcal{V}^{\partial\Omega}$, as follows:

$$\langle\mathbf{E}_1|\mathbf{E}_2\rangle_{\partial\Omega}\equiv(\mathbf{E}_1^*;\check{I}_\partial\mathbf{E}_2)_{\partial\Omega}=\langle\mathbf{E}_2|\mathbf{E}_1\rangle^*_{\partial\Omega}\ . \tag{B1}$$

The $\mathbf{E}_1,\mathbf{E}_2$ matrix element for a tangential operator $\check{Y}$ is now defined, with a further use of Dirac notation, as

$$\langle\mathbf{E}_1|\check{Y}|\mathbf{E}_2\rangle_{\partial\Omega}\equiv(\mathbf{E}_1^*;\check{Y}\mathbf{E}_2)_{\partial\Omega}\ . \tag{B2}$$

The adjoint of a tangential operator $\check{Y}$ with respect to this type of inner product is called its Hermitian conjugate $\check{Y}^\dagger$, and is defined as that operator for which

$$\langle\mathbf{E}_1|\check{Y}^\dagger|\mathbf{E}_2\rangle_{\partial\Omega}=\langle\mathbf{E}_2|\check{Y}|\mathbf{E}_1\rangle^*_{\partial\Omega}\ , \tag{B3}$$

for any pair of sufficiently well behaved, but otherwise arbitrary, vector fields in $\mathcal{V}^{\partial\Omega}$. The operator $\check{Y}$ is called *Hermitean* if $\check{Y}^\dagger=\check{Y}$.

We shall make use of the following result. Let $\mathrm{Im}(k_0)\geq 0$ with $k_0\neq 0$, and let $\Phi_{k_0}=(\mathbf{E}_{k_0},c\mathbf{B}_{k_0})^\tau$ be a solution to Eq. (59) in $\Omega^{\rm ex}$, such that the outgoing-wave conditions of Eq. (33) are satisfied. Then it follows from Ref. [21], Theorems 4.17 and 4.3, that if (using dyadic notation)

$$\mathrm{Re}\left[|k_0|^2\int_{\partial\Omega}dA\,c\mathbf{B}_{k_0}\cdot\check{X}_\partial\cdot\mathbf{E}_{k_0}\right]\leq 0\ , \tag{B4}$$

then the electromagnetic field is identically zero in $\Omega^{\rm ex}$. In view of Eq. (62), and the fact that $\check{X}^\dagger_\partial=-\check{X}_\partial$, we have

$$\mathrm{Re}\left[\int_{\partial\Omega}dA(c\check{X}_\partial\mathbf{B}_{k_0})^*\cdot\check{Z}^{+}_{k_0}\cdot(\check{X}_\partial c\mathbf{B}_{k_0})\right]\leq 0 \tag{B5}$$

implies that the tangential vector field $\check{X}_\partial c\mathbf{B}_{k_0}$ is identically zero on $\partial\Omega$. That is to say, if we define the Hermitian part $H[\check{Z}^{+}_{k_0}]$ of an operator $\check{Z}^{+}_{k_0}$ as

$$H[\check{Z}^{+}_{k_0}]\equiv\tfrac{1}{2}(\check{Z}^{+}_{k_0}+\check{Z}^{+\dagger}_{k_0})\ , \tag{B6}$$

we infer from Eqs. (B5) and (62) that the Hermitian operators

$$H[\check{Z}^{+}_{k_0}]\text{ and }H[(\check{Z}^{+}_{k_0})^{-1}]\text{ are positive definite}\ . \tag{B7}$$

In order to show that Eqs. (84) and (85) are meaningful, we must show that the functional equation

$$(\check{A}_{k_0}\check{Z}^{+}_{k_0}+\check{C}_{k_0})\mathbf{E}=\mathbf{F} \tag{B8}$$

is uniquely solvable for $\mathbf{E}\in\mathcal{V}^{\partial\Omega}$ with any given $\mathbf{F}\in\mathcal{V}^{\partial\Omega}$. Uniqueness means that the only solution to the equation

$$(\check{A}_{k_0}\check{Z}^{+}_{k_0}+\check{C}_{k_0})\mathbf{E}=\mathbf{0}_\partial \tag{B9}$$

is $\mathbf{E}=\mathbf{0}_\partial$. A solution to Eq. (B8) will always exist if no vector field in $\mathcal{V}^{\partial\Omega}$ is orthogonal to every lhs of Eq. (B8), where $\mathbf{E}$ ranges over all of $\mathcal{V}^{\partial\Omega}$. Equivalently, we require that the only solution to the adjoint functional equation

$$(\check{Z}^{+\tau}_{k_0}\check{A}^{\tau}_{k_0}+\check{C}^{\tau}_{k_0})\mathbf{S}=\mathbf{0}_\partial \tag{B10}$$

is $\mathbf{S}=\mathbf{0}_\partial$; either form of adjoint, that is Eq. (17) or Eq. (B3), can be used to the same effect.

Let $\check{\Lambda}$ be a Hermitean operator acting in $\mathcal{V}^{\partial\Omega}$ and let ρ and σ be real numbers. Then we have the operator identity

$$H[(\rho^2\breve{A}^{\dagger}_{k_0}+\sigma^2\breve{Z}^{+\dagger}_{k_0}\breve{C}^{\dagger}_{k_0})\breve{\Lambda}(\breve{A}_{k_0}\breve{Z}^{+}_{k_0}+\breve{C}_{k_0})]$$
$$-H[(\rho^2\breve{A}^{\dagger}_{k_0}\breve{\Lambda}\breve{A}_{k_0}+\sigma^2\breve{C}^{\dagger}_{k_0}\breve{\Lambda}\breve{C}_{k_0})\breve{Z}^{+}_{k_0}]$$
$$\equiv\rho^2H[\breve{C}^{\dagger}_{k_0}\breve{\Lambda}\breve{A}_{k_0}]+\sigma^2\breve{Z}^{+\dagger}_{k_0}H[\breve{C}^{\dagger}_{k_0}\breve{\Lambda}\breve{A}_{k_0}]\breve{Z}^{+}_{k_0}\ . \tag{B11}$$

This result suggests the following: we constrain the operators $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$ such that there exist Hermitian $\breve{\Lambda}$ and real ρ,σ such that

$$\rho^2\breve{A}^{\dagger}_{k_0}\breve{\Lambda}\breve{A}_{k_0}+\sigma^2\breve{C}^{\dagger}_{k_0}\breve{\Lambda}\breve{C}_{k_0}=\breve{I}_\partial\ , \tag{B12}$$

and such that

$$H[\breve{C}^{\dagger}_{k_0}\breve{\Lambda}\breve{A}_{k_0}]\ \text{is positive semidefinite}\ . \tag{B13}$$

The existence of a nonzero solution $\mathbf{E}$ to Eq. (B9) now leads to a contradiction, since equating the $\mathbf{E},\mathbf{E}$ matrix element each side of Eq. (B11), computed according to the prescription of Eq. (B2), yields a negative lhs and non-negative rhs. Hence, there is at most one solution of Eq. (B8) for any given $\mathbf{F}$; we still want to determine that there is at least one solution.

In order to show that Eq. (B10) has only the trivial solution, we shall employ an argument that invokes the further conditions (i) that the Hermitean operator

$$\breve{A}_{k_0}\breve{A}^{\dagger}_{k_0}+\breve{C}_{k_0}\breve{C}^{\dagger}_{k_0}\ \text{is positive definite}\ , \tag{B14}$$

or, equivalently, that no nonzero vector field in $\mathcal{V}^{\partial\Omega}$ is mapped into the zero field by both $\breve{A}^{\tau}_{k_0}$ and $\breve{C}^{\tau}_{k_0}$, and (ii) that the reciprocity criterion Eq. (55) is satisfied.

Let us now assume that $\mathbf{S}$ is a nonzero solution of Eq. (B10), and impose electric and magnetic surface currents $-(c\mu_0)^{-1}\breve{A}^{\tau}_{k_0}\mathbf{S}$ and $(c\mu_0)^{-1}\breve{X}_\partial\breve{C}^{\tau}_{k_0}\mathbf{S}$, respectively. We compute the electromagnetic fields in $\Omega\cup\Omega^{\text{ex}}$ with $\Gamma^{0++}_{k_0}$; the limiting exterior tangential fields on $\partial\Omega$ are, following applications of Eqs. (A12), (A13), (70), (71), and (64),

$$(\breve{I}_\partial\mathbf{E}_{k_0})(\mathbf{r}_\partial+)=\tfrac{1}{2}\breve{X}_\partial(\breve{M}_{k_0}-\breve{I}_\partial)\breve{X}_\partial(\breve{Z}^{+\tau}_{k_0}\breve{A}^{\tau}_{k_0}+\breve{C}^{\tau}_{k_0})\mathbf{S}-\breve{C}^{\tau}_{k_0}\mathbf{S}\ , \tag{B15}$$

$$(\breve{I}_\partial c\mathbf{B}_{k_0})(\mathbf{r}_\partial+)=(2ik_0)^{-1}\breve{X}_\partial\breve{N}_{k_0}(\breve{Z}^{+\tau}_{k_0}\breve{A}^{\tau}_{k_0}+\breve{C}^{\tau}_{k_0})\mathbf{S}+\breve{X}_\partial\breve{A}^{\tau}_{k_0}\mathbf{S}\ . \tag{B16}$$

By the working hypothesis, the first summand on the rhs both of Eq. (B15) and of Eq. (B16) is zero. We have, accordingly,

$$(\breve{I}_\partial\mathbf{E}_{k_0})(\mathbf{r}_\partial+)=-\breve{C}^{\tau}_{k_0}\mathbf{S}\ , \tag{B17}$$

$$(\breve{I}_\partial c\mathbf{B}_{k_0})(\mathbf{r}_\partial+)=\breve{X}_\partial\breve{A}^{\tau}_{k_0}\mathbf{S}\ . \tag{B18}$$

The latter are the limiting tangential values of an outgoing-wave solution to the free-field Maxwell equations, which satisfy Eq. (35), in view of Eq. (55). The uniqueness theorem proved earlier in this appendix now implies that the exterior field must be identically zero. The rhs's both of Eq. (B17) and of Eq. (B18) are therefore equal to $\mathbf{0}_\partial$, and we have discovered a nonzero $\mathbf{S}^*$ that is mapped into the zero field by $\breve{A}^{\dagger}_{k_0}$ and by $\breve{C}^{\dagger}_{k_0}$, contrary to Eq. (B14). We infer that Eq. (B10) has only the trivial solution. That is, under the restrictions Eqs. (55), (B12), (B13), and (B14), the operator $(\breve{A}_{k_0}\breve{Z}^{+}_{k_0}+\breve{C}_{k_0})$ is invertible.

The exterior impedance boundary value problem of Ref. [21], Chap. 4.1, is a special case of the Leontovich boundary value problem treated here: We choose $\breve{A}_{k_0}=\psi(\mathbf{r}_\partial)\breve{I}_\partial$, $\breve{C}_{k_0}=\breve{I}_\partial$, $\sigma\neq0$, and $\breve{\Lambda}=[\rho^2|\psi(\mathbf{r}_\partial)|^2+\sigma^2]^{-1}\breve{I}_\partial$; then Eqs. (55), (B12), and (B14) are satisfied, and Eq. (B13) reduces to the condition $\mathrm{Re}\psi(\mathbf{r}_\partial)\geq0$ for all $\mathbf{r}_\partial\in\partial\Omega$, which subsumes the criterion Eq. (4.79) of Theorem 4.45 in Ref. [21].

Gyromagnetic materials, such as ferrites, have nonsymmetric magnetic permeability tensors [53], and hence can give rise to violations of reciprocity in electromagnetic-wave scattering; moreover, it is plausible that scattering by an obstacle made of such materials could be simulated in the exterior region by a suitable choice of the $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$ operators in Eq. (35), but such that Eq. (55) is invalid. I have not been able to establish $\breve{Z}^{+}_{k_0}$-independent existence criteria for solutions to Eq. (B8) under these circumstances; the almost redundant existence condition that

$$(\breve{A}_{k_0}\breve{Z}^{+}_{k_0}+\breve{C}_{k_0})(\breve{A}_{k_0}\breve{Z}^{+}_{k_0}+\breve{C}_{k_0})^{\dagger}\ \text{is positive definite} \tag{B19}$$

may be difficult to verify.

It remains to address the question of restricting the operators $\breve{A}'_{k_0}$ and $\breve{C}'_{k_0}$ of Eqs. (105) and (106) so that the "if" statement following Eq. (109) can be realized, at least mathematically. In view of Eq. (107), if Eqs. (B12) and (B13) are satisfied with the operators $\breve{A}'_{k_0}$ and $\breve{C}'_{k_0}$ in place of $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$, respectively, then the argument following Eq. (B13) applies, which implies that the kernel of the operator $(\breve{A}'_{k_0}\breve{Z}^{+}_{k_0}+\breve{C}'_{k_0})$ is zero; that is, there is no scattering according to the argument following Eq. (108). In order to secure the stronger result that $(\breve{A}'_{k_0}\breve{Z}^{+}_{k_0}+\breve{C}'_{k_0})$ has an inverse, so that the T operator exists, we note that the argument in the paragraphs including Eqs. (B14)–(B18) carries through with $\breve{A}'_{k_0}$ and $\breve{C}'_{k_0}$ replacing $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$, provided that we make the additional restriction that

$$\breve{A}'_{k_0}(\breve{C}'_{k_0})^{\tau}=\breve{C}'_{k_0}(\breve{A}'_{k_0})^{\tau}\ . \tag{B20}$$

The latter restriction resembles, but is not needed for, the reciprocity condition Eq. (55), as noted following Eq. (106).

APPENDIX C: LEONTOVICH BOUNDARY CONDITIONS AS A SIMULATION OF A COMPLEX OBSTACLE

A concrete problem of electromagnetic-wave scattering in the frequency domain typically involves a scatterer that is geometrically complex, and is made up of materials with inhomogeneous and possibly anisotropic constitutive parameters ϵ,μ,σ. A strategy for the treatment of such problems is the division of space into an interior region, where the solution of Maxwell's equations can be approached by a refined numerical method, and an exterior region, where the desired solution has the analytically simple structure of an outgoing scattered wave (which is to be determined) superimposed on a known incident wave. The theory presented in the main body of this paper has the capability of facilitating the treatment of such problems, given that the response of the obstacle to an impinging, time-harmonic electromagnetic wave can be simulated in the exterior region by an equivalence class of nonlocal Leontovich boundary conditions, as defined by Eq. (35), such that Eq. (B8) can be solved uniquely whatever be its rhs. The purpose of this appendix is to make the conjectured existence of a suitable pair of operators $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$ plausible in general by giving a constructive procedure for their determination in the special cases that the obstacle is entirely manufactured of a substance having uniform, isotropic constitutive parameters. In the argument we shall employ a generalization of the projection operator formalism. The appendix concludes with the sketch of a method of treatment of scattering from a special class of obstacles, which are simple geometrically, and comprise an impenetrable part and a cavity part, such that a subset of $\partial\Omega$ can be taken as the aperture to the cavity, and the cavity is filled with a substance of the type just mentioned.

We consider the scatterers Ω that are smooth-surfaced and bounded, with uniform, isotropic electric permittivity ϵ, magnetic permeability μ, and conductivity σ. We define the index of refraction n and complex effective wave number κ for the interior region as follows:

$$n=[\epsilon\mu/(\epsilon_0\mu_0)]^{1/2}>0\ , \tag{C1}$$

$$\kappa=k_0 n[1+i\sigma/(k_0\epsilon c)]^{1/2}\ . \tag{C2}$$

In Eq. (C2), the square root on the rhs is to be taken to be that root with positive real part, whether $k_0>0$ or $k_0<0$ [and hence $\mathrm{Im}(\kappa)\geq 0$]. Maxwell's equations Eq. (31) in the medium now generalize to

$$\begin{bmatrix}(\mu_0/\mu)(\kappa^2/k_0) & -i(\mu_0/\mu)\nabla\times\\ i\nabla\times & k_0\end{bmatrix}\begin{bmatrix}\mathbf{E}_{k_0}\\ c\mathbf{B}_{k_0}\end{bmatrix}=\begin{bmatrix}-ic\mu_0\mathbf{J}_{k_0,e}\\ -ic\mu_0\mathbf{J}_{k_0,m}\end{bmatrix}. \tag{C3}$$

We now consider that all space is filled with this same medium; computation of a Green's function for this latter problem is now straightforward, say, by transforming to wave-vector space, doing the required algebra, and transforming back to position space. We take the scalar Green's function $G^S_{k_0}$ in the medium (denoted by "S" for substance) to be

$$G^S_{k_0}(\mathbf{r}_1;\mathbf{r}_2)=-(4\pi|\mathbf{r}_1-\mathbf{r}_2|)^{-1}\exp(i\kappa|\mathbf{r}_1-\mathbf{r}_2|)\ . \tag{C4}$$

Then the matrix Green's function for electromagnetic waves in the space-filling medium is

$$\Gamma^S_{k_0}(\mathbf{r}_1;\mathbf{r}_2)=\begin{bmatrix}(\mu/\mu_0)(k_0/\kappa^2) & 0\\ 0 & 1/k_0\end{bmatrix}\delta^3(\mathbf{r}_1-\mathbf{r}_2)+\begin{bmatrix}(\mu/\mu_0)(k_0/\kappa^2)\nabla_1\times(\nabla_1\times) & i\nabla_1\times\\ -i(\mu/\mu_0)\nabla_1\times & (1/k_0)\nabla_1\times(\nabla_1\times)\end{bmatrix}G^S_{k_0}(\mathbf{r}_1;\mathbf{r}_2)\ , \tag{C5}$$

in an obvious notation. The first matrix in Eq. (C5) will be omitted in what follows, as it gives a nonzero result only within the support of the source current distribution [i.e., the rhs of Eq. (C3)], which for our purposes is confined to $\partial\Omega$; we will be interested only in the limiting values of the fields as the field point approaches $\partial\Omega$ from points in Ω or in Ω^{ex}.

We can now construct a generalization of the operator formalism of Appendix A and Sec. V from the constituents of Eq. (C5). We shall obtain a generalized projection operator analogous to Eq. (74), the null space of which is a tangential field distribution on $\partial\Omega$, which is associated with an electromagnetic field that is well behaved and satisfies the source-free Eq. (C3) in Ω. We define operators $\breve{M}^S_{k_0}$ and $\breve{N}^S_{k_0}$ following Eqs. (A4) and (A5):

$$(\breve{M}^S_{k_0}\mathbf{a})(\mathbf{r}_{\partial 1})\equiv-2\int_{\partial\Omega}dA_2\hat{n}(\mathbf{r}_{\partial 1})\times\{\nabla_1\times[G^S_{k_0}(\mathbf{r}_1;\mathbf{r}_{\partial 2})\mathbf{a}(\mathbf{r}_{\partial 2})]\}|_{\mathbf{r}_1-\mathbf{r}_{\partial 1}}\ , \tag{C6}$$

$$(\breve{N}^S_{k_0}\mathbf{a})(\mathbf{r}_{\partial 1})\equiv-2\hat{\mathbf{n}}(\mathbf{r}_{\partial 1})\times\lim_{\mathbf{r}_1\to\mathbf{r}_{\partial 1}\pm}\left[\nabla_1\times\nabla_1\times\int_{\partial\Omega}dA_2G^S_{k_0}(\mathbf{r}_1;\mathbf{r}_{\partial 2})[\hat{\mathbf{n}}(\mathbf{r}_{\partial 2})\times\mathbf{a}(\mathbf{r}_{\partial 2})]\right]\ . \tag{C7}$$

The analogues of Eqs. (A6) and (A7), respectively, hold for these operators,

$$(\breve{M}^S_{k_0})^\tau=\breve{X}_\partial\breve{M}^S_{k_0}\breve{X}_\partial\ , \tag{C8}$$

$$(\breve{N}^S_{k_0})^\tau=\breve{N}^S_{k_0}\ . \tag{C9}$$

The Silver-Müller asymptotic conditions Eq. (33) for outgoing waves (denoted with a superscript "+") generalize to

$$\begin{gathered}\lim_{r\to\infty}\{r[(c\mathbf{B}^+_{k_0})(\mathbf{r})-(\kappa/k_0)(\hat{\mathbf{r}}\times\mathbf{E}^+_{k_0})(\mathbf{r})]\}=\mathbf{0}\ ,\\ \lim_{r\to\infty}\{r[\mathbf{E}^+_{k_0}(\mathbf{r})+(k_0/\kappa)(\hat{\mathbf{r}}\times c\mathbf{B}^+_{k_0})(\mathbf{r})]\}=\mathbf{0}\ .\end{gathered} \tag{C10}$$

We presume that, as in Sec. V, either tangential field distribution on $\partial\Omega$, $\breve{I}_\partial\mathbf{E}^+_{k_0}(\mathbf{r}_\partial+)$ or $\breve{I}_\partial c\mathbf{B}^+_{k_0}(\mathbf{r}_\partial+)$, uniquely determines the complete outgoing-wave exterior field. Accordingly, there must exist an operator $\breve{Z}^S_{k_0}$ such that

$$(\check{I}_\partial \mathbf{E}^{+}_{k_0})(\mathbf{r}_\partial +) = -(\check{Z}^{S}_{k_0}\check{X}_\partial c\mathbf{B}^{+}_{k_0})(\mathbf{r}_\partial +) \ . \tag{C11}$$

An argument similar to that which led to Eq. (64) implies that

$$(\check{Z}^{S}_{k_0})^{\tau} = \check{Z}^{S}_{k_0} \ . \tag{C12}$$

The source-free version of Eq. (C3) implies that if $\Phi_{k_0} = [\mathbf{E}_{k_0}; c\mathbf{B}_{k_0}]^{\tau}$ is a solution, then so is the dual $[(k_0/\kappa)^2 c\mathbf{B}_{k_0}; -\mathbf{E}_{k_0}]^{\tau}$; note also that Eq. (C10) is satisfied by the second field if it is by the first. Hence Eq. (62) generalizes to

$$(\check{Z}^{S}_{k_0})^{-1} = -(\kappa/k_0)^2 \check{X}_\partial \check{Z}^{S}_{k_0} \check{X}_\partial \ . \tag{C13}$$

We can infer the interior and exterior surface-limiting fields associated with a given surface current distribution from the Green's function of Eq. (C5), the definitions Eqs. (C6) and (C7), and limiting processes like those that led to Eqs. (A12) and (A13). Also, Eq. (C3) implies that we should take for the current-to-tangential-field transformation the operator

$$\begin{bmatrix} 0 & \pm\mu_0 c\check{X}_\partial \\ \mp\mu c\check{X}_\partial & 0 \end{bmatrix} , \tag{C14}$$

where the upper (lower) signs correspond to the exterior (interior) limiting tangential fields; Eq. (C14) generalizes Eq. (78). Remembering the factor $(-i)$ on the rhs of Eq. (C3), we infer the generalization $\mathring{P}^{S}_{k_0}$ of the projection operator of Eq. (74):

$$\mathring{P}^{S}_{k_0} = \begin{bmatrix} -\tfrac{1}{2}\check{X}_\partial(\check{M}^{S}_{k_0} + \check{I}_\partial)\check{X}_\partial & -[ik_0/(2\kappa^2)]\check{X}_\partial \check{N}^{S}_{k_0} \\ [i/(2k_0)]\check{X}_\partial \check{N}^{S}_{k_0} & -\tfrac{1}{2}\check{X}_\partial(\check{M}^{S}_{k_0} + \check{I}_\partial)\check{X}_\partial \end{bmatrix} . \tag{C15}$$

The operator $\mathring{P}^{S}_{k_0}$ acts as the unit operator on the tangential values of outgoing-wave solutions to Eq. (C3) in Ω^{ex}, and annihilates the tangential limiting values of solutions of Eq. (C3) that are regular in Ω. We infer from the former property and from Eq. (C12) [and consistent with Eq. (C13)] the following four expressions for $\check{Z}^{S}_{k_0}$, generalizing Eqs. (68)–(71), respectively:

$$\check{Z}^{S}_{k_0} = -(ik_0/\kappa^2)\check{X}_\partial \check{N}^{S}_{k_0} \check{X}_\partial (\check{M}^{S}_{k_0} + \check{I}_\partial)^{-1} \tag{C16}$$

$$= ik_0 \check{X}_\partial (\check{M}^{S}_{k_0} + \check{I}_\partial)\check{X}_\partial (\check{N}^{S}_{k_0})^{-1} \tag{C17}$$

$$= (ik_0/\kappa^2)\check{X}_\partial (\check{M}^{S}_{k_0} - \check{I}_\partial)^{-1} \check{N}^{S}_{k_0} \check{X}_\partial \tag{C18}$$

$$= ik_0 (\check{N}^{S}_{k_0})^{-1} (\check{M}^{S}_{k_0} - \check{I}_\partial) \ . \tag{C19}$$

Equations (72) and (73) generalize to

$$\check{M}^{S}_{k_0} \check{N}^{S}_{k_0} = \check{N}^{S}_{k_0} \check{X}_\partial \check{M}^{S}_{k_0} \check{X}_\partial = (\check{M}^{S}_{k_0} \check{N}^{S}_{k_0})^{\tau} , \tag{C20}$$

$$\check{I}_\partial - (\check{M}^{S}_{k_0})^2 - \kappa^{-2} \check{N}^{S}_{k_0} \check{X}_\partial \check{N}^{S}_{k_0} \check{X}_\partial = \check{0}_\partial \ . \tag{C21}$$

The linear manifold of regular solutions to the Maxwell equations Eq. (C3) in the domain Ω does not change if the medium filling the exterior region Ω^{ex} is replaced by a vacuum. Hence the interior limiting values of the tangential fields are still annihilated by the $\mathring{P}^{S}_{k_0}$ of Eq. (C15). The corresponding exterior limiting fields can be inferred from the continuity of $\check{I}_\partial \mathbf{E}_{k_0}$ and $\check{I}_\partial \mathbf{H}_{k_0} = (1/\mu)\check{I}_\partial \mathbf{B}_{k_0}$ across $\partial\Omega$. Hence the modified projection operator $\mathring{P}^{S'}_{k_0}$, where

$$\mathring{P}^{S'}_{k_0} = \mathring{D}^{-1} \mathring{P}^{S}_{k_0} \mathring{D} \ , \tag{C22}$$

$$\mathring{D} = \begin{bmatrix} \check{I}_\partial & 0 \\ 0 & (\mu/\mu_0)\check{I}_\partial \end{bmatrix} , \tag{C23}$$

will annihilate the exterior limiting tangential fields corresponding to regular interior solutions in the new circumstance that the medium fills only Ω while Ω^{ex} contains a vacuum. Analogous to Eqs. (105) and (106) now, any suitable linear combination of the first-row and second-row operators in $\mathring{P}^{S'}_{k_0}$ provides a representative pair of operators $\check{A}_{k_0}$ and $\check{C}_{k_0}$ that express the Leontovich boundary conditions Eq. (35) for the particular obstacle.

We note that while the projection operator $\mathring{P}^{S'}_{k_0}$ annihilates tangential fields that correspond to regular interior solutions in Ω, it does not act as the unit operator on outgoing-wave states in a vacuum. An operator, which we call $\mathring{P}_{k_0}(\check{Z}^{+}_{k_0})$, can easily be defined which does have both properties; we define a general class of such operators $\mathring{P}_{k_0}(\check{Z})$ which are functions of a generic radiation impedance operator $\check{Z}$, as

$$\mathring{P}_{k_0}(\check{Z}) \equiv \begin{bmatrix} \check{Z}(\check{A}_{k_0}\check{Z} + \check{C}_{k_0})^{-1}\check{A}_{k_0} & -\check{Z}(\check{A}_{k_0}\check{Z} + \check{C}_{k_0})^{-1}\check{C}_{k_0}\check{X}_\partial \\ \check{X}_\partial(\check{A}_{k_0}\check{Z} + \check{C}_{k_0})^{-1}\check{A}_{k_0} & -\check{X}_\partial(\check{A}_{k_0}\check{Z} + \check{C}_{k_0})^{-1}\check{C}_{k_0}\check{X}_\partial \end{bmatrix} . \tag{C24}$$

The operator $\mathring{P}_{k_0}(\check{Z})$ annihilates tangent vector fields $\Phi = [\check{I}_\partial \mathbf{E}_{k_0}; c\check{I}_\partial \mathbf{B}_{k_0}]^{\tau} \in \mathcal{V}^{\partial\Omega \oplus \partial\Omega}$ that satisfy Eq. (35), and acts as the unit operator on tangential fields that satisfy a variant of Eq. (61),

$$\check{I}_\partial \mathbf{E}_{k_0} = -\check{Z}\check{X}_\partial c\mathbf{B}_{k_0} \ . \tag{C25}$$

We note that the product of two projection operators of the type of Eq. (C24) is again a projection operator,

$$\mathring{P}_{k_0}(\check{Z}_1)\mathring{P}_{k_0}(\check{Z}_2) = \mathring{P}_{k_0}(\check{Z}_1) \ . \tag{C26}$$

Accordingly, the null spaces of the two operators coincide; this result also can be inferred by noting that the two operators differ by left multiplication by an invertible operator that maps $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ into itself. In physical terms, this result corresponds to the circumstance that the linear space of regular interior (i.e., in Ω) solutions to Maxwell's equations does not depend on the response of the exterior region to electromagnetic fields. The operator formalism therefore achieves a partial decoupling of the exterior and interior Maxwell problems for scattering from a physical obstacle.

We remark that the approach to scattering taken herein can be construed as a completion and generalization of the integral equation approach to scattering, as described in, say, Refs. [21,54–56].

We conclude this appendix by delineating an approach to a more complex scattering problem, which approach employs some of the methods and results just discussed. The principal objective of what follows is to obtain a representative pair of operators $\breve{A}_{k_0}, \breve{C}_{k_0}$ for Leontovich boundary conditions for the case that the obstacle's surface is partly a perfect electrical conductor and partly the aperture to a cavity; in the present case, the cavity is presumed filled with a material medium with uniform constitutive parameters ϵ, μ, and σ. As the details of the computation are somewhat lengthy, we shall not work out an expression for computing $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$, but merely sketch out the main steps. We shall not address the problem of computing the operators $(\breve{A}_{k_0}\breve{Z}^{+}_{k_0}+\breve{C}_{k_0})^{-1}\breve{A}_{k_0}$ and $(\breve{A}_{k_0}\breve{Z}^{+}_{k_0}+\breve{C}_{k_0})^{-1}\breve{C}_{k_0}$, which are needed to obtain the T operator of Eq. (88).

We suppose that the half space $z>0$ corresponds to Ω^{ex} and is a vacuum, while the region $z<0$ is the obstacle Ω and is divided in two parts, with cylindrical geometry: let $\Pi\subset\mathcal{E}^2$ be such that whenever $(x,y)\in\Pi$ and $z<0$, a medium with the characteristics of Eqs. (C1) and (C2) is present, while if $(x,y)\notin\Pi$ and $z<0$, a perfect electrical conductor is present. The planar geometry for $\partial\Omega$ means that an explicit, albeit singular, form for the operator $\breve{Z}^{+}_{k_0}$ of Eqs. (68)–(71) exists, since the operator $\breve{M}_{k_0}$ of Eq. (A4) is the zero operator when $\partial\Omega$ is a plane (see also Ref. [22]).

The interface $\partial\Omega=\mathcal{E}^2$ comprises the union of the region II, which we shall presume is a simply connected, compact set, with the (we presume) smooth bounding curve $\partial\Pi$, and with the corresponding exterior region $\Pi^{\mathrm{ex}}\equiv\mathcal{E}^2-\Pi-\partial\Pi$. For $\mathbf{r}_\partial\in\Pi^{\mathrm{ex}}$, we obtain the "$E$" case Leontovich boundary conditions of Eq. (36). In order to infer nontrivial $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$ operators for tangential fields defined on the whole of $\partial\Omega$, we proceed as follows. We obtain an expression for a complete set of waveguide modes for electromagnetic waves propagating in either direction $\pm\hat{e}_z$ in a waveguide with cross section Π in the (x,y) plane, having a perfectly conducting wall at $\partial\Pi$ for all z (that is, infinite in both z directions) and filled with the given medium. The reduction of the Maxwell problem in this case to the solution of certain eigenvalue-eigenfunction problems for the two-dimensional Laplace operator in the domain Π is discussed in, say, Ref. [57], Vol. 1, Chap. 2. It proves to be the case that homogeneous Dirichlet and homogeneous Neumann boundary conditions on $\partial\Pi$ are associated, respectively, with the longitudinal electric field for transverse magnetic (TM) waves and the longitudinal magnetic field associated with transverse electric (TE) waves; as we have presumed that Π is simply connected, there are no TEM waves, but the method proposed here is straightforwardly generalized to the case that Π is multiply connected.

Having obtained, at least formally, a complete set of waveguide solutions, we extract the corresponding transverse (say, tangent to the plane $z=0$) electric and magnetic fields associated with each. We represent such a transverse field jointly as four-component entity

$$\Psi^{\perp}=(E_x(x,y),E_y(x,y),cB_x(x,y),cB_y(x,y))^{\tau}\ ;$$

we denote the components as $\Psi^{\perp}_{\alpha}(x,y)$, $\alpha=1,2,3,4$, respectively. Let $\Psi^{\perp,TX}_{k_0,p\sigma}$ represent complete sets of these transverse fields; in the symbol $X=M$ or $X=E$, while $p=1,2,3,\ldots$ denotes the mode of the corresponding longitudinal electric field (for $X=M$) or magnetic field (for $X=E$), and $\sigma=-1\,(\sigma=+1)$ corresponds to propagation, that is to say exponentially decreasing magnitude, in the direction $-\hat{e}_z$ $(+\hat{e}_z)$. We define a sesquilinear inner product between two transverse fields as follows, using Dirac notation:

$$\langle\Psi^{\perp,TX}_{k_0,p\sigma}|\Psi^{\perp,TX'}_{k_0,p'\sigma'}\rangle \equiv\int_{\Pi}\sum_{\alpha=1}^{4}[\Psi^{\perp,TX}_{k_0,p\sigma,\alpha}(x,y)]^{*}\Psi^{\perp,TX'}_{k_0,p',\sigma',\alpha}(x,y)dx\,dy\ . \tag{C27}$$

It proves to be the case that these transverse fields are almost orthogonal: the rhs of Eq. (C27) is proportional to $\delta_{XX'}\delta_{pp'}$, but fields with different propagation direction $\sigma\neq\sigma'$ fail to be orthogonal. It is now easy to construct, by means of an infinite sum, a projection operator that annihilates the transverse fields associated with waves that propagate in the $-\hat{e}_z$ direction and acts as the unit operator on transverse fields associated with $+\hat{e}_z$ propagating waves. This construction is greatly facilitated by the partial separability of Maxwell's equations in the particular geometry, and by the orthogonality properties of the transverse fields associated with different modes. From this projection operator we can infer representative operators $\breve{A}^{\Pi}_{k_0}$ and $\breve{C}^{\Pi}_{k_0}$ with domain and range of definition the linear space of two-component tangent vector fields on Π, and which yield a zero result when applied as in Eq. (35) to those, and only to those, tangential electric and magnetic fields that belong to waveguide modes propagating in the direction $-\hat{e}_z$ in the cavity.

We can now combine results to infer a representative pair of operators that jointly express the Leontovich boundary condition Eq. (35) across the whole of $\partial\Omega$. We have

$$\begin{aligned}\breve{A}_{k_0}&=\Theta_{\Pi^{\mathrm{ex}}}\breve{I}_{\partial}\Theta_{\Pi^{\mathrm{ex}}}+\Theta_{\Pi}\breve{A}^{\Pi}_{k_0}\Theta_{\Pi}\ ,\\ \breve{C}_{k_0}&=(\mu/\mu_0)\Theta_{\Pi}\breve{C}^{\Pi}_{k_0}\Theta_{\Pi}\ ,\end{aligned} \tag{C28}$$

where Θ_{Π} and $\Theta_{\Pi^{\rm ex}}$ are the unit step operators on the respective subsets of $\partial\Omega=\mathscr{E}^2$, and the limiting tangential fields on which $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$ operate are taken from the vacuum values in $\Omega^{\rm ex}$.

A question remains concerning Eq. (C28), which is, does the discontinuity in the properties of the obstacle across $\partial\Pi$ at $z=0$ occasion δ-function contributions to $\breve{A}_{k_0}$ or $\breve{C}_{k_0}$, the domain and range of which are localized to this one-dimensional subset of $\partial\Omega$? It is implausible that terms of this degree of singularity will appear in $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$, on the following grounds. As noted in the discussion following Eq. (35), Eq. (35) can be construed as a mapping of $\mathcal{V}^{\partial\Omega\oplus\partial\Omega}$ into $\mathcal{V}^{\partial\Omega}$, the kernel of which mapping is just the linear space of tangential fields corresponding to regular solutions of Maxwell's equations within the obstacle Ω. Suppose that the waveguide were continued indefinitely in both directions $z\rightarrow\mp\infty$ with cross section Π, but with the part $z>0$ being a vacuum and $z<0$ being filled with the given medium. The construction that led to Eq. (C28) shows that the kernel of the mapping Eq. (35) defined by the given operators $\breve{A}_{k_0}$ and $\breve{C}_{k_0}$ is just the space of tangential electromagnetic fields associated with waveguide solutions that propagate in the $-\hat{e}_z$ direction inside the medium. This property should not change if all of the region $z>0$ is replaced by a vacuum, the more so in view of the circumstance that a singularity in the tangential electric or magnetic field with the strength of a δ function appears to lead to a nonintegrable singularity in the field energy density around the lip of the waveguide.

We remark that although it is not necessary to obtain a projection operator of the type of Eq. (C24) in order to infer a representative pair of operators $\mathring{A}_{k_0}$ and $\breve{C}_{k_0}$ for a particular obstacle, the tangential operators that appear in the transition operator—see Eq. (88)—are just those needed to construct the projection operator of Eq. (C24) (with $\breve{Z}=\breve{Z}^{\,+}_{k_0}$), and conversely. In fact, Eq. (88) can be written as follows:

$$\mathring{T}^{\,L+}_{k_0}=-(i/2)\mathring{X}+i\mathring{X}\mathring{P}_{k_0}(\breve{Z}^{\,+}_{k_0})\;, \tag{C29}$$

where we have used Eqs. (78) and (C24). This result can be interpreted by noting that the transition operator has the function of creating currents in $\partial\Omega$ that cancel the original free-space wave within Ω and generate a suitable outgoing scattered wave in $\Omega^{\rm ex}$.

APPENDIX D: A CIRCUIT ANALOG

In order to help clarify the physical content of the formalism in this paper, we shall describe a linear network analog to the scattering problem, such that time-harmonic voltages and currents will replace time-harmonic electric and magnetic fields, and finite matrices will replace linear-functional operators.

The setup is sketched in Fig. 1. The network comprises two detachable parts, the "exterior" part $\Omega^{\rm ex}$ and an "interior" part Ω_ν, where ν belongs to a discrete or continuous set; the two parts are connected at a number N ports.

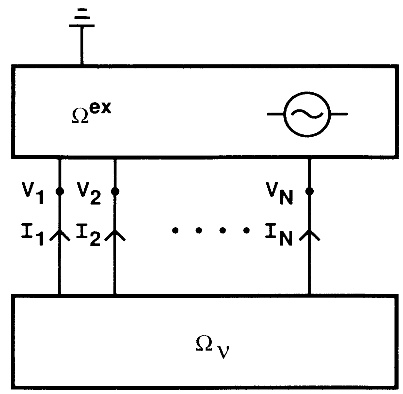

FIG. 1. Network analog to scattering.

The exterior network $\Omega^{\rm ex}$ has the following characteristics: first, it is fixed as to its internal wiring and impedances; second, it has one or more adjustable internal sources of time-harmonic voltage, all with a common frequency, and each with an associated constant, output-independent, internal impedance; and third, it is grounded and otherwise connected so that when the internal voltages are all set to zero, the passive response to $\Omega^{\rm ex}$ to an input of currents $\mathbf{I}^+=(I_1^+,I_2^+,\ldots,I_N^0)^\tau$ at the ports is the voltages $\mathbf{V}^+=(V_1^+,V_2^+,\ldots,V_N^+)^\tau$, respectively; that is,

$$V_p^+=\sum_{q=1}^{N} Z_{pq}I_q^+\;, \tag{D1}$$

where the symmetric $N\times N$ impedance matrix Z_{pq} is presumed to be invertible.

The N-ported internal network Ω_ν can have a variable structure, as signified by the index ν, and is generally not grounded, so that the relationship between the outflowing currents $\mathbf{I}^\nu$ and terminal voltages $\mathbf{V}^\nu$ is not necessarily one to one, but is given by a relationship of the form

$$\sum_{q=1}^{N} A_{pq}^{\nu}V_q^{\nu}+\sum_{q=1}^{N} C_{pq}^{\nu}I_q^{\nu}=0\;. \tag{D2}$$

Equation (D2) represents an equivalence class of such conditions, as simultaneous left multiplication of A^ν and C^ν by an invertible Y yields a physically equivalent set of "impedance boundary conditions" on Ω_ν. The matrix A_{pq}^{ν} is singular if Ω^ν is not grounded, since $V_1^\nu=V_2^\nu=\cdots=V_N^\nu$ means that no currents flow; the matrix C^ν is singular if Ω_ν short circuits two or more ports. We nevertheless require A^ν and C^ν to be nontrivial jointly, which for our purposes means that the inverse matrix $(A^\nu Z+C^\nu)^{-1}$ must exist; sufficiency conditions for the existence of the inverse can be developed along the lines of Appendix B. When the matrix C_{pq}^{ν} is invertible we can arrange for it to be the identity matrix δ_{pq}, so that A_{pq}^{ν} is then the admittance matrix for the network Ω_ν. The reciprocity property will hold provided that

$$A^\nu(C^\nu)^\tau=[A^\nu(C^\nu)^\tau]^\tau\;. \tag{D3}$$

Suppose that boxes Ω^{ex} and Ω_0 are connected as indicated in Fig. 1, with the internal voltages in Ω^{ex} set to some fixed magnitudes and phases. It is observed that the port voltages and currents have the values $\mathbf{V}^0$ and $\mathbf{I}^0$. Now suppose that Ω_0 is replaced by Ω_1 with no changes in the circuit or source EMF's in Ω^{ex}. We call the new voltages and currents at the ports $\mathbf{V}^1$ and $\mathbf{I}^1$; these can be inferred from the original values $\mathbf{V}^0$ and $\mathbf{I}^0$, and from the matrices Z, A^1, and C^1, as follows. We apply the principle of superposition to Ω^{ex}, and subtract the corresponding quantities of the first problem from the second; this difference problem is of the passive type, with all internal EMF's set to zero, and with voltages $\mathbf{V}^+=\mathbf{V}^1-\mathbf{V}^0$ at, and currents $\mathbf{I}^+=\mathbf{I}^1-\mathbf{I}^0$ into, the ports. These are related by Eq. (D1), while the quantities $\mathbf{V}^1$ and $\mathbf{I}^1$ are related by Eq. (D2). We combine these results to infer that

$$\mathbf{I}^+=-(A^1Z+C^1)^{-1}(A^1\mathbf{V}^0+C^1\mathbf{I}^0)\ , \tag{D4}$$

$$\mathbf{V}^+=Z\mathbf{I}^+\ . \tag{D5}$$

We remark that representative matrices A^0 and C^0 are often not required—the analysis of the network $\Omega^{ex}\cup\Omega^0$ contains this information implicitly.

We can consider Ω_0 to represent a system such that solution of the combined circuit equations for $\Omega^{ex}\cup\Omega_0$ is relatively easy, while that for the system $\Omega^{ex}\cup\Omega_1$ is difficult. For example, we can take Ω_0 to be an open circuit (i.e., take $A^0=0$, $C^0_{pq}=\delta_{pq}$) or a short circuit (i.e., take $A^0_{pq}=\delta_{pq}-\delta_{1q}$ and $C^0_{pq}=\delta_{p1}$), while Ω_1 can be a complex network. The above formalism shows that a partial decoupling is possible: we must analyze the three circuits $\Omega^{ex}\cup\Omega_0$, Ω^{ex}, and Ω_1 in order to find $\mathbf{V}^0$ and $\mathbf{I}^0$, the impedance matrix Z, and nontrivial representatives of the matrices A^1 and C^1, respectively. Then it is a matter of algebra of $N\times N$ matrices and N-component vectors to infer the boundary values $\mathbf{V}^+$ and $\mathbf{I}^+$ of the analogue of the scattered wave, and hence the new values of voltages and currents throughout Ω^{ex}.

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