Excitation of solitons by an external resonant wave with a slowly varying phase velocity

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A mechanism is proposed for the excitation of solitons in nonlinear dispersive media. The mechanism employs an external pumping wave with a varying phase velocity, which provides a continuous resonant excitation of a nonlinear wave in the medium. Two different schemes of a continuous resonant growth (continuous phase locking) of the induced nonlinear wave are suggested. The first of them requires a definite time dependence of the pumping-wave phase velocity and is relatively sensitive to the initial wave phase. The second employs the dynamic autoresonance effect and is insensitive to the exact time dependence of the pumping-wave phase velocity. It is demonstrated analytically and numerically, for a particular example of a driven Korteweg–de Vries (KdV) equation with periodic boundary conditions, that as the nonlinear wave grows, it transforms into a soliton, which continues growing and accelerating adiabatically. A fully nonlinear perturbation theory is developed for the driven KdV equation to follow the growing wave into the strongly nonlinear regime and describe the soliton formation.

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I. INTRODUCTION

Consider a medium that admits propagation of solitons. The question that we are going to address in this work is the following: is it possible to use an external driver for an efficient, selective excitation of solitons in such a medium? Mathematically, a driver means a perturbation term in a "soliton equation." Generally, such a term must be small enough in order not to "deform" the soliton solution too much. On the other hand, we need an efficient excitation. We are going to show that this aim can be achieved if the driver presents a small-amplitude pumping wave, which resonantly acts on a medium.

Various schemes of resonant excitation of waves by external pumping waves have been extensively studied in numerous applications. The plasma wave excitation by two copropagating laser waves, whose frequency and wave-number differences match the frequency and the wave number of the plasma wave (the so-called beat-plasma-wave excitation) presents an important example of such a process. For this particular case, Rosenbluth and Liu [1] showed that the resonant growth is saturated at a relatively small amplitude, which scales like $\epsilon^{1/3}$ (ϵ is a small parameter characterizing the laser drive). The saturation occurs as the excitation enters the weakly nonlinear stage, and it results from dephasing between the pumping and excited waves. This result was generalized by Vainberg, Meerson, and Sasorov [2], who considered resonant excitation of waves from a low (even zero) level for four other typical driven nonlinear wave equations [Korteweg–de Vries (KdV), modified KdV, sine-Gordon and nonlinear Schrödinger equations]. They showed that the resonance is generally achieved, when the frequency and wave number of the pumping wave satisfy the dispersion relation of the linear waves of the medium. Also, they found that in all the cases considered, the evolution equations for the amplitude and phase of the fundamental mode are of the Rosenbluth-Liu type.

In order to go beyond the weakly nonlinear saturation of the induced wave, one has to find a way of continuous energy transfer from the pumping to excited wave, which would continue into the fully nonlinear stage. It can be achieved, if the driving force has a character of noise with a sufficiently broad power spectrum density. It was shown [3] that in this case the wave growth continues (on the average), and can lead to the soliton formation, as new harmonics of the broadband noise driving wave enter the resonance interaction. However, the efficiency of this statistical mechanism is relatively low.

A more efficient excitation can be achieved if we properly "tailor" with time the pumping wave frequency, i.e., employ a "chirped" pumping pulse. The chirp form should be chosen to make up for the nonlinear frequency shift of the excited wave and preserve the phase locking between the waves. There are two possible schemes of such a chirping.

In the first of them, we are "tuned" to specific initial conditions (initial amplitude of the excited wave and initial relative phase between the pumping and excited waves) and require that an *exact* resonance is preserved forever. This condition gives a concrete formula for the time dependence of the frequency. Such a scheme (which can be called "the rigid frequency chirping") presents a space-time generalization of many schemes encountered in charged-particle accelerators (see, e.g., Ref. [4]).

The second scheme employs a *sufficiently slow* frequency chirping *with an arbitrary form* ("the loose frequency chirping"), proceeding in the right direction and compensating for the nonlinear frequency shift *on the average*. This scheme also generalizes a number of schemes, encountered in charged-particle acceleration schemes [5-7] and other applications [8,9], where it was called the dynamic autoresonance.

Once the continuous phase locking is achieved, the induced wave will grow into a strongly nonlinear stage. We are interested in the cases when nonlinear media admit existence of solitons. Whether solitons develop may depend on additional constraints imposed on the system. We shall consider a driven KdV equation in a system with periodic boundary conditions (for example, a ring resonator). The periodicity plays an important role in the present mechanism of the soliton formation because of two additional integrals of motion [3] (see below). Our aim is to show, both analytically and numerically, that the continuous nonlinear growth of the phase-locked induced wave in such a system necessarily leads to the soliton formation.

The organization of the paper is the following. In Sec. II we present a perturbation theory, which describes the continuous growth of a weakly nonlinear wave due to the frequency chirping and generalize the results of Vainberg, Meerson, and Sasorov [2]. The evolution equations for the amplitude and phase of the fundamental, obtained in Sec. II, are used to present the two above-mentioned chirping schemes, rigid and loose. The continuous wave growth predicted by the weakly nonlinear theory necessitates the development of an adequate, fully nonlinear perturbation theory. In Sec. III we develop such a theory, which is then used in Sec. IV to follow the excited wave into the strongly nonlinear stage and describe the soliton formation. In Sec. V we perform direct numerical simulations with the driven KdV equation and compare the numerical results with the theory. Section VI presents a brief summary of the results.

II. WEAKLY NONLINEAR THEORY AND TWO SCHEMES OF A CONTINUOUS WAVE GROWTH

A. Basic equations

Consider the KdV equation driven by a small-amplitude traveling wave with a slowly time-dependent (chirped) frequency:

$$u_t+uu_x+u_{xxx}=-\epsilon\sin[kx-\Phi(t)]\ , \tag{1}$$

where $\Phi(t)$ is the external wave phase, $\dot{\Phi}(t)=\omega(t)$ is the frequency, and k is the wave number. The small positive parameter ϵ describes a weak coupling between the external wave and the medium. Equation (1) is written in the reference frame, moving with the "acoustic" speed of the medium [10,11].

The number of parameters in Eq. (1) can be reduced, if we transform to a new variable $kx=x'$. The transformed equation coincides with Eq. (1) (but with $k=1$), if we replace $k^3t\to t$, $k^{-2}u\to u$, and $k^{-5}\epsilon\to\epsilon$. We shall use this scaling later, when comparing the results of our theory with numerical simulations.

We are interested in the wave excitation from a small initial level, $u(x,t=0)\ll 1$. Therefore, at the initial stage, we can neglect the nonlinear term in Eq. (1). The dispersion relation for undriven ($\epsilon=0$) linear perturbations of the media has the form of $\Omega=-\kappa^3$, where Ω and κ are the frequency and wave number of the sinusoidal perturbations. If, for a given wave number κ of the external wave the initial value of the driving frequency $\omega(0)$ is close to the resonant value $-\kappa^3$, the resonant growth of the induced sinusoidal wave starts [2]. In the case of ω=const the wave growth saturates at relatively small wave amplitudes, $u\ll 1$. Therefore the nonlinear term in Eq. (1) can be treated perturbatively for all times, and the fundamental mode saturation is accompanied by generation of the (relatively small) second harmonic, while the higher harmonics are negligible [2]. In the case of a continuous wave growth we are interested in now, such a weakly nonlinear theory can work only for a limited time. However, it is instructive to briefly outline the initial, small-amplitude stage of the excitation.

Following Ref. [2], we are looking for an approximate, weakly nonlinear solution of Eq. (1) in the following form:

$$u(x,t)=a(t)\sin[\xi+\phi(t)]+v(\xi,a,\phi)+w(\xi,a,\phi)+\cdots\ , \tag{2}$$

where $a(t)$ and $\phi(t)$ are the slowly varying amplitude and phase of the fundamental, $\xi=kx-\Phi(t)$ is the "fast" variable, v and w are the second and third harmonics, respectively, $w\ll v\ll a$. We substitute Eq. (2) into Eq. (1), keep only the terms of a leading order and find equations for a and ϕ from the condition of the absence of the secular growth of the third harmonic:

$$\dot{a}=-\epsilon\cos\phi\ , \tag{3}$$

$$\dot{\phi}=\omega(t)+k^3-\frac{a^2}{24k}+\frac{\epsilon}{a}\sin\phi\ . \tag{4}$$

Once $a(t)$ and $\phi(t)$ are known, the second harmonic correction v is also easily found:

$$v(\xi,a,\phi)=-\frac{1}{12}\left[\frac{a}{k}\right]^2\cos(2\xi+2\phi)\ . \tag{5}$$

Equations (3) and (4) will coincide with those found in Ref. [2] if we consider the case of a constant frequency $\omega=-k^3$. In this case Eqs. (3) and (4) are autonomous, therefore integrable. Figure 1 shows the phase trajectories of Eqs. (3) and (4) for the case of $k=1$, $\epsilon=0.1$, and $\omega=-k^3=-1$. Both phase-locked and phase-unlocked trajectories are shown. (It is convenient to work with both the positive and the negative amplitudes.) There is a special limiting trajectory (starting at $a=0$, $\phi=0$), for which $a=(96k\epsilon\sin\phi)^{1/3}$, therefore the maximum attainable amplitude of the fundamental is $(96k\epsilon)^{1/3}$. For this special trajectory, the time $T_{\rm nl}$ it takes the amplitude a to complete one oscillation cycle ("nonlinear period") is [2]

$$T_{\rm nl}=\epsilon^{-2/3}2^{13/3}3^{-7/12}F(\pi\setminus 15^\circ)\approx 22.2\epsilon^{-2/3}\ , \tag{6}$$

where F is the elliptic integral of the first kind.

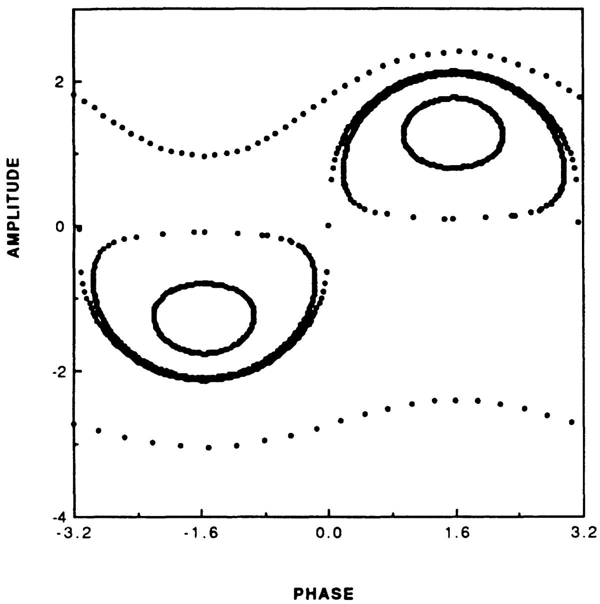

FIG. 1. Phase portrait (ϕ,a) of system of equations (3) and (4) for $\epsilon=0.1$, $k=1$, and $\omega=\text{const}=-1$.

Now let us return to the case of a time-dependent frequency $\omega(t)$ and consider the two schemes of continuous excitation.

B. Rigid frequency chirping

Let us select a "seed" wave with a fixed initial amplitude a_* and phase ϕ_* and require that the phase remains constant: $\phi(t)=\phi_*$. This immediately follows a linear growth of a with time:

$$a(t)=a_0(t)=a_*-\epsilon t\cos\phi_* .$$

(For the *growth* to occur it is necessary that $\cos\phi_*<0$.) Now, from Eq. (4) we easily find the specific $\omega(t)$ dependence which provides the constancy of ϕ and linear growth of a:

$$\omega(t)=-k^3+\frac{1}{24k}(a_*-\epsilon t\cos\phi_*)^2 -\epsilon\sin\phi_*(a_*-\epsilon t\cos\phi_*)^{-1} . \quad (7)$$

As $a_0(t)$ becomes of the order of unity, Eqs. (3) and (4) are no longer valid, and we need a fully nonlinear theory to describe the subsequent wave evolution.

It is clear from Eq. (7) that the rigid chirping scheme is sensitive to the initial phase and amplitude of the wave. Therefore it is interesting to find out whether it works when there are small deviations of the initial amplitude and phase from the prescribed values a_* and ϕ_*. We are looking for solutions of Eqs. (3) and (4) in the form of $a(t)=a_0(t)+\delta a(t)$ and $\phi(t)=\phi_*+\delta\phi(t)$, and linearizing Eqs. (3) and (4) with respect to small deviations $\delta a(t)\ll a_0(t)$ and $\delta\phi(t)\ll\phi_*$. We obtain the following set of linear equations:

$$\frac{d(\delta a)}{dt}=\epsilon\sin\phi_*\delta\phi , \quad (8)$$

$$\frac{d(\delta\phi)}{dt}=b(t)\delta a+\epsilon a_0^{-1}(t)\cos\phi_*\delta\phi , \quad (9)$$

where

$$b(t)=-\frac{1}{12}k^{-1}a_0^{-2}(t)[a_0^3(t)+12k\epsilon\sin\phi_*] . \quad (10)$$

Taking the time derivative of Eq. (8) and using Eqs. (8) and (9) to eliminate $\delta\phi$ and $d(\delta\phi)/dt$, we arrive at the following second-order equation:

$$\frac{d^2(\delta a)}{dt^2}-\epsilon[a_0(t)]^{-1}\cos\phi_*\frac{d(\delta a)}{dt}-\epsilon b(t)\sin\phi_*\delta a=0 . \quad (11)$$

It follows from Eqs. (10) and (11) that the sufficient condition for stability is $\sin\phi_*>0$ (remember that $\cos\phi_*<0$). In this case Eq. (11) describes a linear oscillator with a time-dependent oscillation frequency and dissipation. In the course of time, small oscillations of the amplitude a around the linearly growing solution $a_0(t)$ will be damped. Therefore, if the test wave phase ϕ_* satisfies simultaneously the two inequalities, $\cos\phi_*<0$ and $\sin\phi_*>0$ (i.e., if $\pi/2<\phi_*<\pi$), the excitation process is always stable: not only the "seed" wave, but also its close neighbors on the phase plane (ϕ,a) are excited. Numerical calculations enable us to reinforce this statement: even relatively distant neighbors (whose initial phases ϕ_0 belong to the same interval $\pi/2<\phi_0<\pi$) get phase locked and excited to large amplitudes. Figure 2 shows two examples of such a process. Here a rigid frequency chirping prescribed by specific initial conditions provides an efficient excitation for other initial conditions as well. It is seen that the wave phase quickly gets phase locked close to ϕ_* and then performs small oscillations around ϕ_*. Meanwhile, the wave amplitudes in the two examples grow almost linearly with time, and the two plots of the wave amplitudes versus time are almost indistinguishable. In addition, we found that, in many cases, "seed" wave phases lying *outside* the favorable interval $\pi/2<\phi_*<\pi$ also provide an efficient excitation. In these cases, the phase ϕ quickly jumps to a regime of oscillations around some value belonging to the favorable interval. Simultaneously, the wave amplitude is growing on the average (see Fig. 3).

C. Loose frequency chirping

Let us assume now that we are starting from an exact resonance, $\omega(t=0)=-k^3$, and select an *arbitrary* phase-locked trajectory not too close to the limiting trajectory (see Fig. 1). If we *slowly* increase the frequency $\omega(t)$ (i.e., do it on a time scale much longer than the "nonlinear period" T_{nl}, encountered in the problem with $\omega=\text{const}$), we shall permit the nonlinear oscillations of a, shown in Fig. 1. In addition to these oscillations, however, the amplitude a will experience a slow upward drift. In other words, the wave will grow *on the average.* The average amplitude behavior at large times is universal for all the phase-locked trajectories and has the following form:

$$a = 2(3k)^{1/2}[\omega(t)+k^3]^{1/2} , \quad (12)$$

which follows from the dynamic autoresonance condition $\omega(t)+k^3-a^2/24k=0$. This mechanism is obviously insensitive to the exact form of $\omega(t)$ (once it is slow enough, $\dot{\omega}T_{\rm nl}/\omega \ll 1$), and it must work equally well for the majority of initially phase-locked trajectories. For a given ϵ, some optimal chirping rate can always be found: too low a chirping rate means inefficient excitation, while too high one leads to the phase unlocking and termination of the excitation. Figure 4 shows an example of excitation in the loose chirping case. We put $k=1$, started with a small amplitude and chose a simple linear chirp $\omega(t)=-1+\alpha t$. It can be seen from Fig. 4 that, at large times, the time-average amplitude grows like $(24\alpha t)^{1/2}$ in agreement with Eq. (12). Calculations with other initial

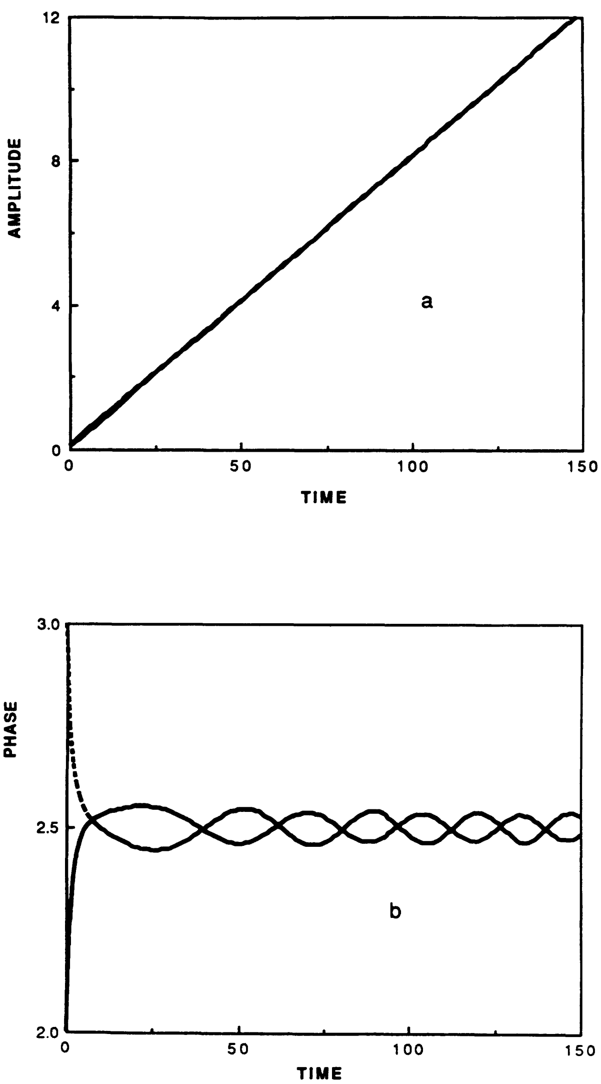

FIG. 2. Rigid chirping scheme as described by the weakly nonlinear equations (3) and (4). Shown are the amplitude $a(a)$ and phase $\phi(b)$ vs time. The parameters are the following: $\epsilon=0.1$, $k=1$, $\omega(t)$ is given by Eq. (7) with $a_*=0.1$ and $\phi_*=2.5$, which corresponds to the favorable interval $(\pi/2,\pi)$. The initial conditions are $a_0=a_*$, $\phi_0=2.0$ (solid line) and 3.0 (dashed line). The amplitude plots for the two initial conditions are indistinguishable.

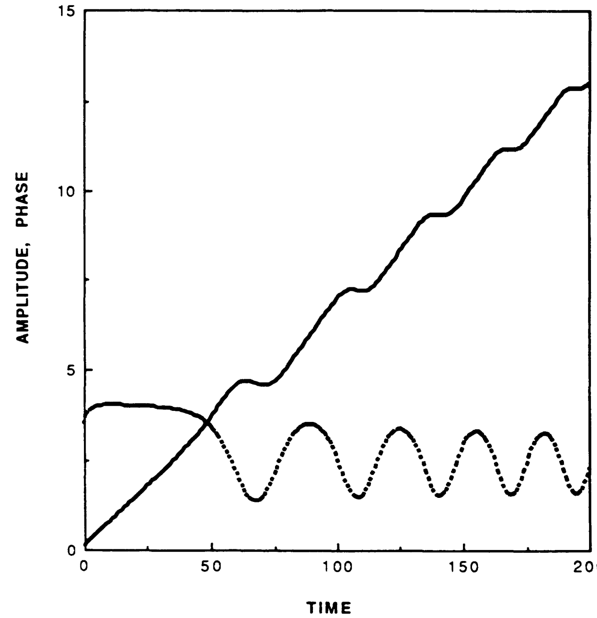

FIG. 3. Rigid chirping scheme as described by the weakly nonlinear equations (3) and (4). Shown are the amplitude a (solid line) and phase ϕ (dashed line) vs time. The "seed" wave phase $\phi_*=4.0$ and initial phase $\phi_0=3.5$ do not belong to the "favorable" interval $(\pi/2,\pi)$.

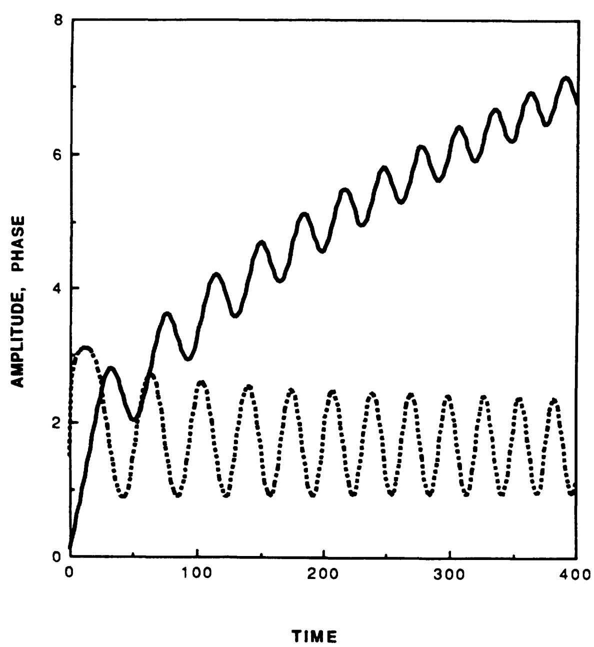

FIG. 4. Loose chirping scheme as described by the weakly nonlinear equations (3) and (4). Shown are the amplitude a (solid line) and phase ϕ (dashed line) vs time. Parameters are the following: $\epsilon=0.1$, $k=1$, $\omega(t)=-1+\alpha t$, and $\alpha=0.005$. The initial conditions are $a_0=0.1$ and $\phi_0=1.5$.

phases and other forms of $\omega(t)$ give similar results. Therefore the wave excitation in this regime is insensitive to the exact *form* of the chirp, and depends only on its *rate*.

In summary, we have shown in this section that properly varying with time the external wave frequency, we can achieve a continuous wave growth. As the wave grows, it reaches large amplitudes. Therefore a weakly nonlinear theory breaks down, and an adequate fully nonlinear theory must be developed.

III. FULLY NONLINEAR PERTURBATION THEORY: EVOLUTION EQUATIONS

Our aim is to develop a perturbation theory for Eq. (1), which would use the smallness of ϵ, but hold true for any amplitude of the excited nonlinear wave.

Let us start with an arbitrary traveling-wave solution of the unperturbed KdV equation. Such a solution, $u(x,t)=u^0(\xi), \xi=x-Vt$, satisfies the following equation:

$$-Vu^0+\tfrac{1}{2}(u^0)^2+u^0_{\xi\xi}=C=\text{const} . \tag{13}$$

Its general bounded solution presents a "cnoidal" wave [10,11], which we write down in the following form:

$$u^0(\xi)=A\,\mathrm{dn}^2_m\left[\left[\frac{A}{12}\right]^{1/2}(\xi-x_0)\right]+\gamma , \tag{14}$$

where A is the wave amplitude, $V=(A/3)(2-m^2)+\gamma$ is the velocity of the wave, dn_m is the Jacobian elliptic function [12] with the modulus m, and γ is an arbitrary constant (a constant pedestal). Generally, solution (14) is characterized by three independent constants: A, γ, and m. The cnoidal wave is characterized by its average-over-period value,

$$\bar{u}=\gamma+A\frac{E}{K} \tag{15}$$

[$K=K(m)$ and $E=E(m)$ are the complete elliptic integrals of the first and second kind, respectively], and by its spatial period,

$$\lambda=4K\left[\frac{3}{A}\right]^{1/2} . \tag{16}$$

Constant C in Eq. (13) is determined by the three independent constants and equal to

$$C=-V\gamma+\frac{\gamma^2}{2}-(1-m)\frac{A^2}{6} . \tag{17}$$

The modulus $0<m<1$ of function dn serves as a measure of the wave nonlinearity. If $m \ll 1$, the wave form is close to a cosine, while m approaching unity corresponds to a sech^2-like soliton solution [10,11].

Now we return to the perturbed problem and look for the solution to Eq. (1) in the form of a general (cnoidal) traveling wave *with slowly varying parameters* plus a small correction:

$$u(x,t)=A(t)\mathrm{dn}^2_{m(t)}\left[\left[\frac{A(t)}{12}\right]^{1/2}[\xi-x_0(t)]\right]+\gamma(t)+\sum_{n=1}^{N^*}W^{(n)}(x,t) . \tag{18}$$

The main, "adiabatic" part of the solution, $u^0(\xi,t)=A\,\mathrm{dn}^2(\ )+\gamma$, depends on the "fast" variable, which now becomes

$$\xi=x-\int_0^t V(t')dt$$

and the "slow" variable t. The *small* correction to the slowly varying cnoidal wave is sought for perturbatively, in the form of a power series of ϵ. Once the form of the solution is prescribed, we should obtain the evolution equations for $A(t)$, $m(t)$, $\gamma(t)$, and $x_0(t)$. It appears, however, that only two evolution equations are actually needed, if our problem is spatially periodic. First, if $L=2\pi/k$ is the spatial period of the problem, then we have to demand $\lambda=L$, which gives an additional algebraic relation between the parameters $A(t)$, $m(t)$, and $\gamma(t)$. Second, it can be easily checked that the average-over-period value

$$\frac{1}{L}\int_{-L/2}^{L/2}u(x,t)dx$$

remains constant even in the presence of the perturbation [3]. In particular, if we start excitation from the zero level, $u(x,0)=0$, then the average-over-period value (16) must be equal to zero for all times. Therefore we have one more algebraic relation between the parameters. These two relations reduce the number of parameters characterizing the cnoidal wave to two, the first of them $x_0(t)$ and the second any of the parameters $A(t)$, $m(t)$, and $\gamma(t)$. It is convenient to choose the modulus $m(t)$ as the second parameter, because in the case of continuous wave growth, m approaches unity, which means the formation of a soliton.

The evolution equations for $m(t)$ and $x_0(t)$ in every order of ϵ can be obtained from the necessary conditions of boundedness of the functions $W^{(n)}(x,t)$, which have the form of orthogonality relations. (Similar perturbation theories were developed earlier for single-soliton and multisoliton solutions [13].) Substituting Eq. (18) into Eq. (1), and linearizing the obtained equation with respect to $W^{(n)}$, we obtain the following relationships in each order of ϵ:

$$\frac{\partial}{\partial\xi}(-VW^{(n)}+u^0W^{(n)}+W^{(n)}_{\xi\xi})=H^{(n)} , \tag{19}$$

where functions $H^{(n)}(\xi,t)$ do not contain function $W^{(n)}$. For example, in the first order

$$H^{(1)}=-u^0_t-\epsilon\sin\left[k\xi+k\int_0^t V(t')dt'-\Phi(t)\right] . \tag{20}$$

Integrating Eq. (19) once, we obtain

$$-VW^{(n)}+u^{(0)}W^{(n)}+W^{(n)}_{\xi\xi}=\int_{-L/2+x_0}^{\xi}H^{(n)}d\xi' . \tag{21}$$

(For definiteness, we consider Eq. (1) on interval $[-L/2,L/2]$ and require that periodic functions $W^{(n)}$ vanish at $\xi=-L/2+x_0$.) The *homogeneous* equation, associated with Eq. (21), has the following general solution:

$$W^{(n)}=C_1u_\xi^0+C_2u_\xi^0\int_{-L/2+x_0}^{\xi}(u_{\xi'}^0)^{-2}d\xi' , \tag{22}$$

where $C_{1,2}$ are arbitrary constants. Therefore we can write down the following general solution of Eq. (21):

$$W^{(n)}=C_1u_\xi^0+C_2u_\xi^0\int_{-L/2+x_0}^{\xi}(u_{\xi'}^0)^{-2}d\xi'+u_\xi^0\int_{-L/2+x_0}^{\xi}(u_{\xi'}^0)^{-2}d\xi'\int_{-L/2+x_0}^{\xi'}u_{\xi''}^0d\xi''\int_{-L/2+x_0}^{\xi''}H^{(n)}d\xi''' . \tag{23}$$

Function u_ξ^0 takes the zero values at two points of the interval $[-L/2,L/2]$: $\xi=x_0$ and $\xi=x_0+L/2$ or $x_0-L/2$. Therefore, as can be easily seen from Eq. (23), boundedness of functions $W^{(n)}$ necessitates $C_2=0$, and

$$\int_{-L/2}^{L/2}u_\xi^0d\xi\int_{-L/2}^{\xi}H^{(n)}d\xi'\equiv\int_{-L/2}^{L/2}d\xi\, u^0H^{(n)}=0 \tag{24}$$

and

$$\int_{-L/2}^{0}u_\xi^0d\xi\int_{-L/2}^{\xi}H^{(n)}d\xi'=0 \tag{25}$$

where we have shifted the variable, $\xi-x_0\to\xi$, and used the periodicity of function u_ξ^0 (from now on ξ means $\xi-x_0$). Conditions (24) and (25) give, in each order of the perturbation theory, the evolution equations for m and x_0 we are looking for. In the following we obtain the explicit form of these equations in the first order of ϵ. Prior to that, however, let us simplify Eqs. (24) and (25). Using Eqs. (18) and (20), we rewrite Eq. (24) as

$$\begin{aligned}\frac{\partial}{\partial t}&\int_{-L/2}^{L/2}\tfrac{1}{2}(u^0)^2d\xi\\ &=-\epsilon\int_{-L/2}^{L/2}\sin[kx(\xi)-\Phi(t)]u^0(\xi)d\xi\\ &=-\epsilon A\sin\mu\int_{-L/2}^{L/2}\mathrm{dn}^2\left[\sqrt{A/12}\,\xi\right]\cos k\xi\, d\xi\end{aligned} \tag{26}$$

where we have introduced the relative phase of the external and excited waves:

$$\mu=k\int_0^t V(t')dt'-\Phi(t)-kx_0(t) . \tag{27}$$

Note that Eq. (26) can also be obtained by a direct "energetic" approach. Indeed, if we multiply Eq. (1) by u, integrate both sides with respect to x over the period L and substitute the slowly evolving cnoidal wave solution into the (small) right-hand side of the equation, we shall immediately obtain Eq. (26).

Now let us integrate Eq. (25) by parts. We have

$$\int_{-L/2}^{0}u^0H^{(1)}d\xi-u^0(0)\int_{-L/2}^{0}H^{(1)}d\xi=0 . \tag{28}$$

It is convenient to represent $H^{(1)}$ as a sum of symmetric and antisymmetric parts with respect to argument ξ: $H^{(1)}=H_s^{(1)}+H_a^{(1)}$. Using the symmetry and zero average of function u^0 and once again employing Eq. (24), we see that only the antisymmetric part,

$$H_a^{(1)}=-u_\xi^0\dot{x}_0-\epsilon\cos\mu\sin k\xi , \tag{29}$$

contributes to the integrals entering Eq. (28). Therefore Eq. (28) can be rewritten as

$$\begin{aligned}\dot{x}_0&\left[\int_{-L/2}^{0}u^0u_\xi^0d\xi-u^0(0)\int_{-L/2}^{0}u_\xi^0d\xi\right]\\ &=-\epsilon\cos\mu\int_{-L/2}^{0}[u^0(\xi)-u^0(0)]\sin k\xi\, d\xi .\end{aligned} \tag{30}$$

Now we have to calculate the integrals entering Eqs. (26) and (30). The integral on the left-hand side of Eq. (26) can be calculated with the help of Eq. (13). Integrating Eq. (13) over ξ and using the periodicity of u^0 and the fact that the average of u^0 over the period is zero, we arrive at the following expression:

$$\tfrac{1}{2}\int_{-L/2}^{L/2}(u^0)^2d\xi=CL=A^2L\left[(2-m)\left[\frac{E}{3K}\right]-\frac{EK}{2}-\frac{1-m}{6}\right] . \tag{31}$$

On the right-hand side of Eq. (26) appears a standard integral, which one encounters when expanding function dn^2 in the Fourier series:

$$\begin{aligned}\int_{-L/2}^{L/2}u^0(\xi)\cos k\xi\, d\xi&=A\int_{-L/2}^{L/2}\mathrm{dn}^2(\sqrt{A/12}\,\xi)\cos k\xi\, d\xi\\ &=\frac{2\pi^2q\sqrt{12A}}{K(1-q^2)} ,\end{aligned} \tag{32}$$

where $q=\exp(-\pi K'/K)$, $K'=K(1-m)$. Now, taking the time derivative of Eq. (31) and substituting the result into Eq. (26), we obtain, after some algebra, the evolution equation for $m(t)$:

$$\dot{m}=\frac{\epsilon\pi^2L^2mq\sin\mu}{24(1-q^2)EK(E-K)[K-E/(1-m)]} . \tag{33}$$

Now we proceed to Eq. (30). Taking the integrals on the left-hand side and substituting the values of $u^0(0)=A$ and $u^0(-L/2)=A(1-m)$, we reduce the left-hand side to $\dot{x}_0A^2m^2/2$. The right-hand side integral in Eq. (30) can be rewritten as

$$\begin{aligned}\int_{-L/2}^{0}&[u^0(\xi)-u^0(0)]\sin k\xi\, d\xi\\ &=-\sqrt{12/A}\,m\int_{-K}^{0}\mathrm{sn}^2\xi\sin\left[\frac{2\pi\xi}{K}\right]d\xi .\end{aligned} \tag{34}$$

It is sufficient (see Sec. IV) to evaluate this integral, taking into account only the first two Fourier harmonics of function sn^2. The result has the following form:

$$\int_{-K}^{0} \mathrm{sn}^2\xi \sin\left[\frac{2\pi\xi}{K}\right] d\xi = \frac{2K}{\pi m}\left[1-\frac{E}{K}+\frac{4\pi^2 q^2}{K^2(1-q^4)}\right]+\mathcal{O}(q^4) . \quad (35)$$

The equation for $\mu(t)$ is now directly obtained from Eqs. (27), (30), (34), and (35):

$$\dot{\mu}=kV(m)-\omega(t) - \frac{\epsilon L^2 \cos\mu}{12K^2 m^2}\left[1-\frac{E}{K}+\frac{4\pi^2 q^2}{K^2(1-q^4)}\right] . \quad (36)$$

Equations (33) and (36) form a closed set. It can be checked that they are reduced to Eqs. (3) and (4) for the fundamental Fourier harmonic of the cnoidal wave, if we proceed to the limit of $m \ll 1$ and substitute $\mu=-\phi+\pi/2$.

Once $m(t)$ and $\mu(t)$ [and therefore $x_0(t)$] are found, we can return, if necessary, to Eq. (23) and find the first-order correction $W^{(1)}$ to the slowly evolving cnoidal wave.

IV. FULLY NONLINEAR PERTURBATION THEORY: CONTINUOUS WAVE GROWTH AND SOLITON FORMATION

Equations (33) and (36) resemble those arising in various applications referring to nonlinear resonance phenomena. It is clearly seen from Eqs. (33) and (36) that the continuous wave growth (the increase of m with time) requires phase locking and, therefore, proximity to the Cherenkov-type resonance $\omega=kV$ between the phase velocities of the pump wave and induced wave. Similar to the weakly nonlinear stage of excitation, there are two possible schemes of the frequency chirping: rigid and loose. A proper chirping makes it possible to phase lock the waves and provide a continuous excitation into the strongly nonlinear regime. As m approaches unity, the cnoidal wave must transform into a soliton with a "pedestal":

$$u^s = A\,\mathrm{sech}^2\left[\left[\frac{A}{12}\right]^{1/2}(x-x_0)\right]-4L^{-1}(3A)^{1/2} , \quad (37)$$

where $A=3V=12L^{-2}\ln^2[16/(1-m)]$. (We have used the well-known asymptotic relations for dn, E and K at $m \to 1$). The increase of m means simultaneous amplification, acceleration, and narrowing of the nonlinear wave because of the specific relations between the wave amplitude, phase velocity, and width in the KdV equation.

Figure 5 shows the numerical solutions of Eqs. (33) and (36) for the rigid frequency chirping. We found it too cumbersome to prescribe the rigid chirping form from Eqs. (33) and (36) and employed in Eqs. (33) and (36) the simple formula (7), which followed from the weakly nonlinear theory. However, in contrast to calculations of Sec. II C, we took $k=2\pi/12.5$ (the value we are using in the next section when numerically solving the driven KdV equation). From the dimensional analysis following Eq. (1), we know that we should replace ϵ by $\epsilon'=k^{-5}\epsilon$, a by $a'=k^{-6}a$, and multiply function u (and hence the amplitude of the fundamental, maxima and minima, pedestal, etc.) by a factor k^{-2}, if we want to return to the case of $k=1$, but retain the same "physical" parameters. As initial conditions we chose those of the "seed" wave itself, $a_0=a_*$ and $\phi_0=\phi_*$. Figure 5 shows that even under such an inexact chirping form, a significant wave amplitude is reached, and parameter m approaches unity, which means soliton formation. Later, phase unlocking occurs, and the resonant growth terminates because of

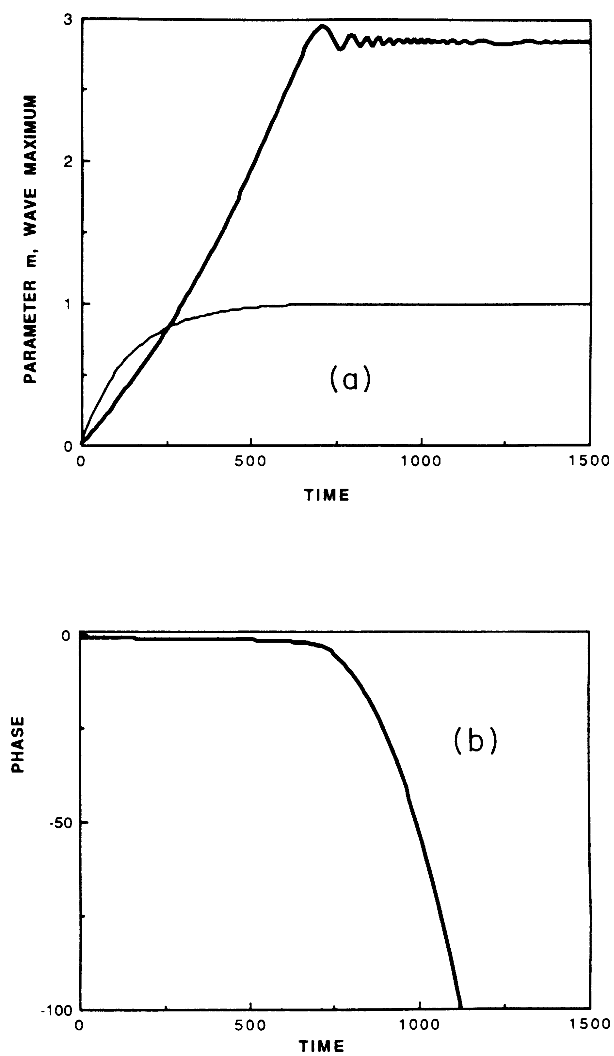

FIG. 5. Rigid chirping scheme as described by the fully nonlinear equations (33) and (36). $\epsilon=0.1$, $k=2\pi/12.5$, $\omega(t)$ is given by Eq. (7) with $a_*=0.01$ and $\phi_*=2.5$. Shown in (a) are the parameter m (thin line) and the wave maximum $A+\gamma$, calculated from m (thick line). (b) shows the phase μ. The initial conditions are $a_0=a_*$, $\phi_0=\phi_*$.

the inexact rigid chirping form used. We checked separately that the initial stage of the excitation is described very well by the weakly nonlinear theory of Sec. III. Note that as m grows, the term proportional to ϵ in Eq. (36) becomes very small, which justifies using only two Fourier harmonics of function sn^2 in the evaluation of integral (35).

Figure 6 refers to the case of a loose frequency chirping. Again, we chose a linear frequency chirping and started from the exact resonance with the linear modes of the medium: $\omega(t)=-k^3+\alpha' t$. We see from Fig. 6 that the simple linear chirping provides a continuous phase locking between the waves and a persistent growth of the induced wave. Parameter m is growing, on the average, and approaching unity, which means formation of a soliton. At this stage, Eqs. (33) and (36) can be simplified. Using asymptotic relations for the complete elliptic integrals at $m \to 1$ and transforming from variable m to the soliton amplitude A, we can write down the following equations for the adiabatic evolution of the *soliton* parameters:

$$\dot{A}=-\epsilon\frac{4\sqrt{3}\pi^2}{L^2\sqrt{A}}\sinh^{-1}\frac{2\sqrt{3}\pi^2}{L\sqrt{A}}\sin\mu\ , \tag{38}$$

$$\dot{\mu}=\frac{A}{3}-\omega\ , \tag{39}$$

where we have neglected the small term, proportional to ϵ, in Eq. (39). Equation (38) can be further simplified if the excitation continues to very large soliton amplitudes, so that $A \gg 12\pi^4/L^2$. In this case we have simply $\dot{A}=-2\epsilon\sin\mu$. Therefore, if phase μ is locked in the interval $\pi<\mu<2\pi$, the soliton amplitude grows linearly with time, the growth rate being twice as large as the growth rate of the fundamental in the initial stage of the process [see Eq. (3)]. Simultaneously, the soliton is accelerated: its velocity grows linearly with time. Equations (38) and (39) can also be obtained directly from Eq. (1) in the framework of the *soliton* perturbation theory.

V. NUMERICAL SIMULATIONS

In order to check the predictions of the theory, developed in Secs. II–IV, and directly follow the soliton formation, we solved Eq. (1) numerically. For this purpose we developed a spectral code, described in the Appendix. Equation (1) has been solved on the interval $-6.25<x<6.25$ ($L=12.5$ and $k=2\pi/12.5=0.502\,65$). In all runs we used $\epsilon=3.209\times10^{-3}$ which corresponds to

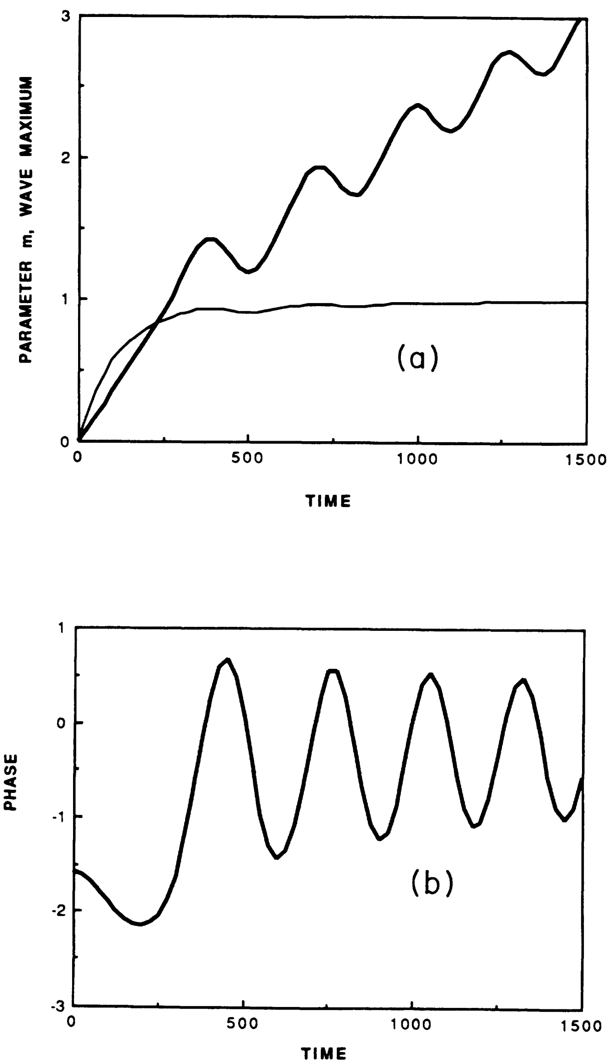

FIG. 6. Loose chirping scheme as described by the fully nonlinear equations (33) and (36) with $\epsilon=0.1$, $k=2\pi/12.5$, $\omega(t)=-k^3+\alpha k^6 t$, and $\alpha=0.01$. Shown in (a) are the parameter m (thin line) and the wave maximum $A+\gamma$, calculated from m (thick line). (b) shows the phase μ. The initial conditions correspond to the zero amplitude and phase $\mu_0=-\pi/2$.

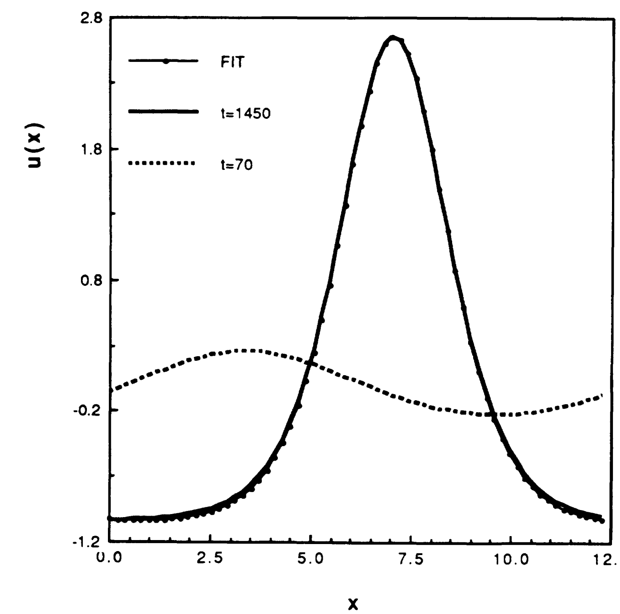

FIG. 7. Numerical solution of Eq. (1) for function $u(x,t)$ with the zero initial condition for two successive time moments: $t=70$ and 1450. The parameters are the same as in Fig. 6. Also shown is the soliton solution (37). The amplitude of the "fitting soliton" has been calculated as the difference between the maximum and minimum of the numerical solution.

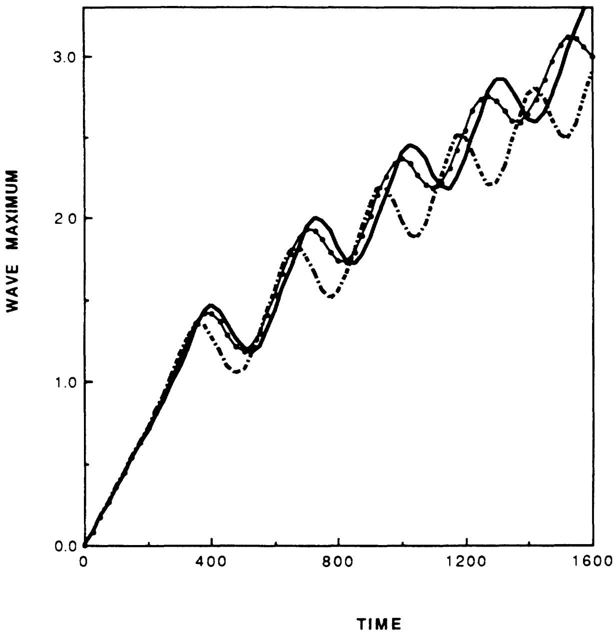

FIG. 8. The wave maximum vs time in the loose chirping scheme as found from (a) numerical simulations (solid line), (b) fully nonlinear perturbation theory (solid line with dots), and (c) weakly nonlinear perturbation theory (dash-dotted line). The parameters and initial conditions are the same as in Figs. 6 and 7.

$\epsilon=0.1$ in the dimensionless ($k=1$) version of Eq. (1).

We performed simulations for both the rigid and loose frequency chirping regimes, starting with the zero initial condition. Figure 7 refers to the loose chirping and shows function $u(x)$, found numerically, at two different time moments: $t=70$, when the induced wave is almost sinusoidal, and $t=1450$, where it is already almost indistinguishable from the soliton solution (37) shown in the same figure. The same simple chirping form $\omega(t)=-k^3+\alpha' t$ was used. Figure 8 refers to the same case and shows the wave maximum versus time as found from (a) numerical simulations, (b) fully nonlinear perturbation theory, and (c) weakly nonlinear perturbation theory. The latter was calculated taking into account the second harmonic from Eq. (5). A good agreement between the fully nonlinear theory and simulation for all times is clearly seen. Also, the weakly nonlinear theory agrees very well with both the simulations, and fully nonlinear theory at the initial stage of the process.

Figure 9 refers to the rigid frequency chirping and shows the wave maximum versus time for the same chirping form and initial conditions as those used in Fig. 5. Also shown is the wave maximum versus time, as predicted by the fully nonlinear theory. The agreement is very good. At the final stage of the excitation the wave form is very close to the soliton (37) (we do not show the corresponding figure, because it is very similar to Fig. 7).

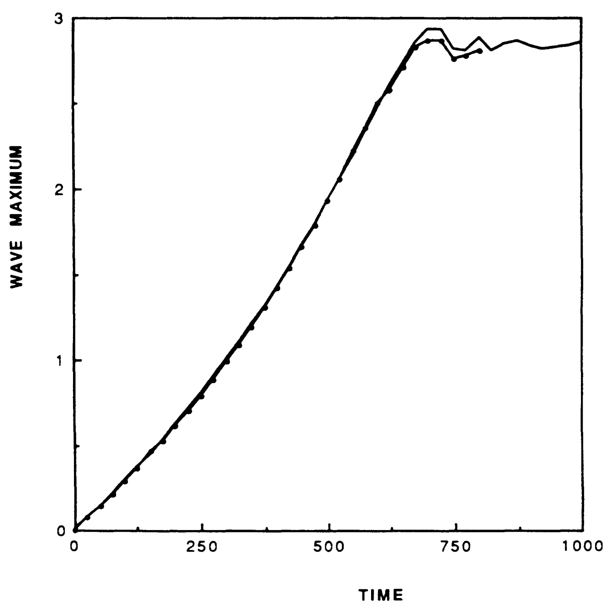

FIG. 9. The wave maximum vs time in the rigid chirping scheme as found from (a) numerical simulations (solid line with dots) and (b) fully nonlinear perturbation theory (solid line). The chirping form and initial conditions are the same as in Fig. 5.

VI. SUMMARY

We have shown analytically and numerically that a proper variation (chirping) of the external wave frequency makes it possible to achieve a continuous growth of induced nonlinear dispersive waves. Under the constraint of a spatial periodicity, the growing wave transforms into a soliton, which continues amplifying and accelerating.

We have proposed two excitation schemes, which we call the rigid and loose frequency chirpings. The former requires a definite chirping form, suitable to a given "seed" wave. The latter admits an arbitrary (monotonous) chirping form. The rigid chirping scheme can act much faster than the loose chirping scheme. However, it is generally more sensitive to the initial conditions: they must belong to a relatively small vicinity of the given "seed" wave. In contrast, the loose chirping regime requires only that the initial conditions lie within the phase-locking region of the relevant phase space and that the frequency variation proceed in the right direction and be sufficiently slow.

Both schemes have analogs in particle accelerators. It is interesting that not only the soliton, but also a general cnoidal wave behaves in this excitation process like a particle with a variable mass, accelerated via the (slowly time-dependent) Cherenkov resonance with an external wave.

In the present work, we limited ourselves to the particular case of a driven KdV equation. However, we expect similar mechanisms of soliton excitation to act in other soliton equations as well, which promises various applications.

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APPENDIX

For numerical solution of Eq. (1), we have developed a code which is similar to the second-order split-step code used earlier [14]. Equation (1) can be represented in the spectral form:

$$\frac{\partial W_k}{\partial t} - ik^3 W_k = F_k(W_k) \ . \tag{A1}$$

Here

$$W_k = \int_{-\infty}^{+\infty} \exp(-ikx) u(x,t) dx$$

is the Fourier transform of $u(x,t)$, and

$$F_k(W_k) = -\int_{-\infty}^{+\infty} \left[\frac{1}{2}\frac{\partial u^2}{\partial x} + \epsilon \sin[kx - \Phi(t)] \right] \exp(-ikx) dx$$
$$= -\tfrac{1}{2} ik B_k(W_k) + \left[\frac{\epsilon}{2i} \right] \{\delta(k-k_0)\exp[-i\Phi(t)] - \delta(k+k_0)\exp[+i\Phi(t)]\} , \tag{A2}$$

where

$$B_k(W_k) = \int_{-\infty}^{+\infty} \exp(-ikx) u^2 dx = \int_{-\infty}^{+\infty} W_k W_{k-k'} dk' \ . \tag{A3}$$

Equation (A1) has the following formal solution:

$$W_k(t+\tau) = \exp[-ik^3(t+\tau)] W_k(0) + \int_0^{t+\tau} \exp[ik^3(t+\tau-t')] F_k(W_k(t')) dt' \ ,$$

where τ is the chosen time step and $W_k(0)$ is the initial condition. This equation can be rewritten as

$$W_k(t+\tau) = \exp(ik^3\tau) W_k(t) + \exp[ik^3(t+\tau)] \int_t^{t+\tau} \exp(-ik^3 t') F_k(W_k(t')) dt' \ . \tag{A4}$$

To construct a numerical code based on the exact expression (A4), we expand the integral entering Eq. (A4) in terms of the small parameter τ. It is convenient to use only those expansions of the integral which include even powers of τ, in order to preserve the Hamiltonian properties of Eq. (1) in the numerical code.

Let us consider the following expression:

$$W_k(t-\tau) = \exp(-ik^3\tau) W_k(t) + \exp[ik^3(t-\tau)] \int_t^{t-\tau} \exp(-ik^3 t') F_k(W_k(t')) \ . \tag{A5}$$

Subtracting Eq. (A5) from Eq. (A4), we obtain after simple algebra

$$W_k(t+\tau) - W_k(t-\tau) = 2i \sin(k^3\tau) W_k(t) + Z_k \ , \tag{A6}$$

where

$$Z_k = \int_0^{\tau} \{\exp[-ik^3(t'-\tau)] F_k(W_k(t'+t)) + \exp[-ik^3(\tau-t')] F_k(W_k(t-t'))\} dt' \ . \tag{A7}$$

Now, expanding F_k in the vicinity of $\tau=0$,

$$F_k(W_k(t \pm t')) = F_k(W_k(t)) \pm \dot{F}_k(W_k(t)) t' + \mathcal{O}(\tau^2)$$

and evaluating the first derivative as

$$\dot{F}_k = \frac{F_k(W_k(t+\tau)) - F_k(W_k(t-\tau))}{2\tau} + \mathcal{O}(\tau^2)$$

we obtain Z_k in the following form:

$$Z_k = 2\frac{\sin k^3\tau}{k^3} F_k(W_k(t)) + ik^{-3} \left[1 - \frac{\sin(k^3\tau)}{\tau k^3} \right] [F_k(W_k(t+\tau)) - F_k(W_k(t-\tau))] + \mathcal{O}(\tau^3) \ . \tag{A8}$$

For brevity, we denote

$$Y_k = W_k - ik^{-3} \left[1 - \frac{\sin(k^3\tau)}{k^3\tau} \right] F_k(W_k) \ . \tag{A9}$$

Then we obtain the following expression:

$$Y_k(t+\tau)-Y_k(t-\tau)=2i\sin(k^3\tau)Y_k(t)+2\frac{\sin k^3\tau}{k^3\tau}F_k(W_k(t)) \ . \tag{A10}$$

Now we can use relations (A9), (A10), and (A2) as a predictor-corrector code: the predictor is

$$\begin{aligned}Y_k(t+\tau)=&Y_k(t-\tau)+2i\sin(k^3\tau)Y_k(t)\\&-i\frac{\sin^2 k^3\tau}{k^6\tau}\{kB_k(W_k(t))-\epsilon[e^{-i\Phi(t)}\delta(k+k_0)-e^{+i\Phi(t)}\delta(k-k_0)]\}+\mathcal{O}(\tau^3) \ ,\end{aligned} \tag{A11}$$

and the corrector is

$$W_k(t+\tau)-\left[1-\frac{\sin k^3\tau}{k^3\tau}\right]\frac{B_k(W_k(t+\tau))}{k^2}=Y_k(t+\tau) \ . \tag{A12}$$

We replace the Fourier transform by its discrete version on the interval $[-L/2,L/2]$. Then, assuming periodic boundary conditions for Eq. (1) with the period L, we can calculate the discrete version of convolution (A3) by means of two successive fast Fourier transforms (FFT's). For each time step, we have to calculate only two FFT's for the predictor and two FFT's for the corrector. We can use the following straightforward procedure:

$$\begin{aligned}&W_k^{(n+1)}=Y_k+\frac{1}{k^2}\left[1-\frac{\sin k^3\tau}{k^3\tau}\right]B_k(W_k^{(n)}) \ ,\\&W_k^{(0)}=Y_k \ .\end{aligned} \tag{A13}$$

In fact, a single iteration (A13) already provides an accuracy $\mathcal{O}(\tau^3)$, because

$$\frac{1}{k^2}\left[1-\frac{\sin k^3\tau}{k^3\tau}\right]\sim\tau^2k^4$$

for $k^3\tau \ll 1$.

The code has the following favorable properties.

(a) It is exactly conservative: the Jacobian of mapping (A11) equals unity.

(b) It is *linearly* stable for any value of the time step τ since its employs the *exact solution of linearized* Eq. (1).

(c) It requires only two iterations of the right-hand side of (A6) for obtaining an accuracy $\mathcal{O}(\tau^3)$.

(d) It exactly conserves the additional integral of Eq. (1):

$$\int_{-L/2}^{L/2}u(x,t)dx=0 \ . \tag{A14}$$

Property (d), which is very convenient, immediately follows from relation

$$\int_{-L/2}^{L/2}u(x,t)dx=W_{k=0}(t) \ .$$

Indeed, from Eqs. (A9) and (A10) we have for $k=0$:

$$W_0(t+\tau)=W_0(t-\tau)=W_0(t)=\text{const} \ .$$

Therefore, for the zero initial conditions we have $W_0(t)=W_0(0)=0$, which proves (A14).

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