Nonexponential decay of a stochastic one-channel system

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(Received 26 February 1991)

A general formula is presented that expresses the temporal evolution of compound systems. It connects the time dependence of the density matrix with the energy dependence of the scattering matrix. Using results of random matrix theory, we then study the decay behavior of stochastic compound systems with arbitrary coupling between bound states and decay channels. As an example we consider the case of one open channel and prove a nonexponential decay law for all times that asymptotically is of the form $t^{-3/2}$.

PACS number(s): 05.45.+b, 24.60.−k, 34.10.+x

I. INTRODUCTION

During the past few years the investigation of stochastic compound systems has found much interest. In particular, after the work of Verbaarschot, Weidenmüller, and Zirnbauer [1], a systematic study of the scattering properties of such systems with arbitrary number of open channels and arbitrary transmission coefficients became possible. In the present paper, we use the result of Ref. [1] in order to consider the decay of stochastic compound systems with one open channel. In particular, we show that for sufficiently large times t the decay of such systems follows a $t^{-3/2}$ law for *any* transmission coefficient. Although we were able to prove this analytically only in the limit $t\to\infty$, numerical studies show that an algebraic decay law of the form $(\text{const}+t)^{-3/2}$ qualitatively describes the behavior of the system for arbitrary times.

The present paper is organized as follows: A heuristic argument is given that makes the assertion plausible and shows that it is a consequence of the underlying Gaussian statistics of resonance decay amplitudes. Subsequently, a general formula (valid for any number of open channels) is derived that connects the decay function $P(t)$ of the compound system with the scattering matrix S. We then study the average of $P(t)$ over the ensemble of compound states with the help of the results of Ref. [1] and specialize the problem to one open channel.

II. A HEURISTIC ARGUMENT

Suppose we have a stochastic compound system coupled to one open channel, and suppose the transmission coefficient T is small so that the resonances are well isolated. Every resonance is of Lorentzian form and, consequently, shows an exponential decay behavior. Then, naively one could expect an exponential decay *on average,* too. A simple argument shows, however, that this is not necessarily the case. For this purpose we compute the probability $P(t)$ for the system to remain bound until time t if it was bound at $t=0$. Obviously, for isolated resonances, the decay of the compound system is given by the average of the (appropriately weighted) decay laws of the N individual resonances, i.e., $P(t)=N^{-1}\sum_\lambda w_\lambda \exp(-\Gamma_\lambda t)$ where λ runs over all resonance levels. The weight w_λ describes the probability of exciting a given resonance at $t=0$ and is given by the integrated cross section of that resonance. The latter one follows from the Lorentzian shape of an (isolated) resonance line: $w_\lambda\sim\int\Gamma_\lambda^2(E^2+\Gamma_\lambda^2)^{-1}dE=\pi\Gamma_\lambda$. Normalizing this expression by the averaged width $\bar\Gamma$ of all resonances we obtain $w_\lambda=\Gamma_\lambda/\bar\Gamma$. Furthermore, for stochastic compound systems in nuclear [2,3] as well as in molecular physics [4] the *distribution* ρ of the partial widths Γ_λ (coinciding in the one-channel case with the total ones) is known to be of the Porter-Thomas form: $\rho_{\rm PT}(\Gamma)=(2\pi\Gamma\bar\Gamma)^{-1/2}\exp(-\Gamma/2\bar\Gamma)$. (From the mathematical point of view this is the χ^2 distribution with one degree of freedom.) Using ergodicity, we can substitute the contribution of each resonance to $P(t)$ by an average over all possible Γ, the weight function being $\rho_{\rm PT}$. This leads to

$$P(t)=\bar\Gamma^{-1}\int_0^\infty\Gamma\rho_{\rm PT}(\Gamma)\exp(-\Gamma t)d\Gamma=(1+2t\bar\Gamma)^{-3/2}\,. \tag{1}$$

It turns out that, despite the fact that every resonance decays exponentially, their average (in a stochastic compound system weakly coupled to one open channel) shows an essentially *nonexponential* decay behavior, the asymptotic limit being $\sim t^{-3/2}$.

Obviously, the specific power of this algebraic law depends on the choice of the weight factors w_λ. We emphasize, however, that the w_λ used in Eq. (1) are natural if one realizes that there is no other way to populate the resonances than through (one of) the channel(s) by which they subsequently decay. This concept will also be introduced into the more rigorous random matrix model below. If one were to omit the w_λ—which leaves the physical process of populating the resonance states at $t=0$ undefined—the decay would be even slower, $P(t)$ being asymptotically $\sim t^{-1/2}$.

In the following, we give an exact derivation of this powerlike behavior. Remarkably, it is not restricted to the case of well-isolated resonances, but applies to arbitrary transmission coefficients.

III. TIME EVOLUTION OF COMPOUND SYSTEMS

We consider a quantum system consisting of $N \gg 1$ bound states $|\Phi_i\rangle$, $i=1,\ldots,N$ coupled to K continua (decay channels) $|\chi_c(E)\rangle$, $c=1,\ldots,K$. The total Hamiltonian of the system has the form

$$\mathcal{H}=\sum_{i,j=1}^{N}|\Phi_i\rangle H_{ij}\langle\Phi_j|+\sum_{c=1}^{K}\int dE|\chi_c(E)\rangle E\langle\chi_c(E)| + \sum_{c=1}^{K}\sum_{i=1}^{N}\int dE\,[|\Phi_i\rangle V_i^c\langle\chi_c(E)|+\text{H.c.}]\,, \quad (2)$$

where the H_{ij} are matrix elements of some Hermitian operator H. In keeping with Ref. [1], the energy dependence of the coupling vectors $V^c\equiv\{V_i^c\}_{i=1,\ldots,N}$ as well as possible threshold effects will be neglected throughout this paper. The coupling vectors V^c are supposed to be pairwise orthogonal (which means that we neglect direct reactions). Their norm, or, put differently, average coupling element $v_c^2=(1/N)\sum_{i=1}^{N}|V_i^c|^2$, is a measure of the coupling "strength" in the corresponding channel.

The probability $P(t)$ for the system to remain bound until time t is $P(t)=\text{tr}[\rho(t)]/\text{tr}[\rho(0)]$, where $\rho(t)$ is the density matrix of the system, restricted to the space of bound states. The initial configuration can be prepared by performing an instantaneous excitation of the system by a radiation pulse at $t=0$ [cf. Eq. (2)]; the bound states are populated only via their coupling to the channels. We therefore assume that $\rho(0)$ has nonvanishing elements only in the K-dimensional subspace spanned by the K coupling vectors V^c (the "coupling subspace"). It is impossible (in the very beginning of the time evolution) to populate a configuration of the "noncoupling subspace" by a pulse prepared from the channels. The ensuing structure of $\rho(0)$ allows for the connection between $P(t)$ and the scattering matrix established below.

These assumptions imply that the time evolution of the density matrix $\rho(t)$ is determined by the effective Hamiltonian [5,6] $H_{\text{eff}}=H-i\pi VV^+$, where $VV^+\equiv\sum_{c=1}^{K}V^cV^{c+}$, in a (quasi) unitary way:

$$\rho(t)=e^{iH_{\text{eff}}t}\rho(0)e^{iH_{\text{eff}}^+t}\,. \quad (3)$$

Equation (3) allows us to express $P(t)$ in terms of the scattering matrix $S_{ab}(E)$, $a,b=1,\ldots,K$ corresponding to the Hamiltonian (2). For this purpose, we introduce the Green function $G(E)=(E^+-H_{\text{eff}})^{-1}$, where $E^+=E+i0$, and make use of the Fourier representation

$$e^{-iH_{\text{eff}}t}=-(2\pi i)^{-1}\int dE\,e^{-iEt}G(E)\,, \quad (4)$$

valid for $t>0$. This leads to

$$P(t)=\frac{1}{4\pi^2}\int\int dE_1dE_2e^{i(E_2-E_1)t} \times\text{tr}[\rho(0)G^+(E_2)G(E_1)]\,, \quad (5)$$

where the cyclic permutability under the trace has been used.

Using the identity

$$G^+(E_2)[G^{-1}(E_1)-(G^+)^{-1}(E_2)]G(E_1) =G^+(E_2)-G(E_1) \quad (6)$$

and the fact that

$$G^{-1}(E)=E^+-H_{\text{eff}}=E^+-H+i\pi VV^+,$$

one sees that

$$G^+(E_2)G(E_1) =\frac{2\pi i[G^+(E_2)-G(E_1)]+4\pi^2G^+(E_2)VV^+G(E_1)}{2\pi i(E_1-E_2+2i0)}\,. \quad (7)$$

The right-hand side of Eq. (7) can be immediately expressed in terms of the S matrix corresponding to the Hamiltonian (2). In fact, omitting elastic phase shifts (which would drop out in the final expression anyway) S is given by the relation

$$S(E)=1-2\pi iV^+G(E)V \quad (8)$$

(1 denotes the unity operator in the channel space), and we obtain

$$V^+G^+(E_2)G(E_1)V=\frac{1}{2\pi i(E_1-E_2+2i0)} \times[S^+(E_2)S(E_1)-1]\,. \quad (9)$$

We now introduce a basis, where the first K vectors are given by the (normalized) coupling vectors V^c, $c=1,\ldots,K$ (this is possible, because the latter ones are pairwise orthogonal). In this basis $V_i^c=\delta_i^c(Nv_c^2)^{1/2}$, where δ_i^c is the Kronecker symbol. From Eq. (9) it follows that

$$[G^+(E_2)G(E_1)]_{ji}=\frac{[S^+(E_2)S(E_1)-1]_{ji}}{2\pi i(E_1-E_2+2i0)Nv_iv_j}\,, \quad i,j=1,\ldots,K\,. \quad (10)$$

Inserting this expression into Eq. (5), we finally obtain

$$P(t)=\frac{1}{8i\pi^3N} \times\int\int dE_1dE_2\frac{e^{i(E_2-E_1)t}}{E_2-E_1} \times\sum_{i,j=1}^{K}\frac{\rho_{ij}(0)}{v_iv_j}\left[\delta_{ij}-\sum_{k=1}^{K}S_{ki}(E_1)S_{kj}^*(E_2)\right]. \quad (11)$$

Here, the regularizing term $2i0$ has been omitted since the integrand is regular for arbitrary values of E_1 and E_2 (at $E_1=E_2$ this is guaranteed by the unitarity of S). Note that up to now no assumption concerning the nature of the underlying Hamiltonian H has been made. In the next section we consider the case that H is drawn from the Gaussian orthogonal ensemble (GOE).

IV. DECAY OF STOCHASTIC SYSTEMS

Equation (11) shows that the decay of a compound system is essentially determined by the Fourier transform of the two-point function $S(E_1)S^*(E_2)$. This allows us to investigate the decay of *stochastic* compound systems by using the results of Ref. [1] for the (GOE) averaged function $\overline{S_{ab}(E_1)S^*_{cd}(E_2)}\equiv F_{abcd}(E_1,E_2)$. We shall not give here the explicit form of F_{abcd}, but only make use of the following general properties [see Eq. (8.10) of Ref. [1]]: (i) F is translationally invariant: $F_{abcd}(E_1,E_2)=F_{abcd}(0,E_2-E_1)$; (ii) $F_{abad}=0$ if $b\neq d$; (iii) F is a threefold parameter integral, the integrand depending on (E_2-E_1) in an exponential way.

In order to exploit property (i), we introduce $\varepsilon=E_2-E_1$ and $\mathcal{E}=(E_1+E_2)/2$ and perform the trivial integration over $\mathcal{E}$: $\int d\mathcal{E}=ND$. Property (ii) ensures that the time evolution of the ensemble averaged probability $\overline{P(t)}$ depends only on the *diagonal* elements $\rho_{ii}(0)$ of the initial density matrix, i.e., on the initial occupation numbers. Finally, point (iii) together with Eq. (11) suggests that it will be advantageous to get rid of the factor $(E_2-E_1)^{-1}$ in Eq. (11) and, therefore, to consider $\overline{dP(t)/dt}$, rather than $\overline{P(t)}$ itself. The result for the ensemble averaged decay probability of a stochastic compound system reads

$$\overline{\frac{dP(t)}{dt}}=-\frac{1}{8\pi}\int d\varepsilon\, e^{i\varepsilon t}\sum_{i=1}^{K}\frac{\rho_{ii}(0)}{x_i}\sum_{k=1}^{K}\overline{S_{ki}(0)S^*_{ki}(\varepsilon)}\,,\qquad t>0 \tag{12}$$

where the dimensionless coupling constants $x_i=\pi^2 v_i^2/D$ have been introduced, and terms located at $t=0$ have been omitted.

Note that the results obtained so far are valid for an arbitrary number of open channels. In the next section, we exploit them in order to obtain the decay law in the special case of *one* open channel.

V. THE ONE-CHANNEL CASE

Equation (12) expresses $\overline{dP(t)/dt}$ effectively in terms of a twofold parameter integral, because after inserting the concrete expression for F, the integration over ε gives rise to a δ function which makes one of the parameter integrals trivial. In the one-channel case a further analytical treatment of this integral is possible.

Inserting Eq. (8.10) of Ref. [1] into Eq. (11) and using the parameter transformations of Ref. [7], one can show that

$$\overline{dP(t)/dt}=-\frac{1}{16x}\mathcal{D}_T\int_0^1 d\lambda\,\lambda(1-\lambda)(1-T\lambda)\times\int_0^\infty d\xi\,\delta\left[t-\frac{\pi}{D}(\xi+2\lambda)\right]\int_0^{\xi/4}dy\frac{\sqrt{\xi}(\lambda^2+\lambda\xi+\xi y)^{-2}}{[y(1+\xi+\xi y)(1+T\xi+T^2\xi y)]^{1/2}}\,. \tag{13}$$

Here, x is the coupling constant of the one open channel, $T=4x/(1+x)^2$ denotes the corresponding transmission coefficient, and $\mathcal{D}_T$ stands for the following differential operator with respect to T: $\mathcal{D}_T=4T^2[(1-T)\partial^2/\partial T^2-\partial/\partial T]$.

From Eq. (13), one can deduce the asymptotic behavior of $\overline{P(t)}$ at $t\to\infty$. Namely, for large t the second line of Eq. (13) tends to

$$\left[\frac{Dt}{\pi}\right]^{-5/2}T^{-1/2}\int_0^\infty dy\,(\lambda+y)^{-2}[y(1+y)(1+Ty)]^{-1/2}$$

which explicitly shows the powerlike asymptotic behavior. Performing, furthermore, the integration over λ, one finally obtains for the leading term at large t

$$\overline{P(t)}\approx\frac{1}{24x}\left[\frac{Dt}{\pi}\right]^{-3/2}\mathcal{D}_T\frac{I(T)}{\sqrt{T}}\,, \tag{14}$$

where

$$I(T)=\int_0^\infty dy\frac{(1+2y+2yT+3y^2T)\ln(1+1/y)-(2+T/2+3yT)}{\sqrt{y(1+y)(1+Ty)}}\,.$$

Returning to Eq. (13), one sees that (14) is valid for $t\gg\pi/D$, except for very small T. However, the case of small transmission coefficients has been extensively considered in Refs. [7–9]. In particular, applying Eq. (12) to Eq. (8) of Ref. [9], one immediately obtains that for small T

$$\overline{P(t)}\approx\frac{T}{4x}\left[1+T\frac{Dt}{\pi}\right]^{-3/2} \tag{15}$$

for $t\gtrsim 1/D$, which is in full agreement with the heuristic expectation (1) (for small T one has very accurately $DT=2\pi\overline{\Gamma}$).

Note that Eqs. (15) and (14) are compatible with each other. Indeed, taking into account that $I(T)$ behaves regularly at $T=0$ and that $I(0)=2$ (cf. appendix of Ref. [7]), one can check that the large t limit of (15) gives just the small T limit of (14).

Surprisingly, a functional form of the type (15) gives a

reasonable approximation to $\overline{P(t)}$ for *arbitrary* times t and *arbitrary* transmission coefficients T: A numerical calculation shows that the expression

$$\frac{1}{24x}\left[\frac{1}{T}+\frac{Dt}{\pi}\right]^{-3/2}\mathcal{D}_T\frac{I(T)}{\sqrt{T}}\ , \tag{16}$$

which has the correct asymptotic behavior (14) roughly approximates the exact function $P(t)$ as obtained by integrating Eq. (13) with the boundary condition $\overline{P(\infty)}=0$ (see Fig. 1). For small transmission coefficients T one sees that expression (16) is close to the exact value for all t. This is expected, since, for small T, expression (16) coincides with the right-hand side of Eq. (15). On the other hand, one sees that the simple expression (16) gives a qualitatively correct prediction up to $T=1$. This seems to reflect the fact that in the one-channel case the resonances are fairly well isolated for arbitrary coupling strength between bound states and continuum (see Refs. [10,11]).

Equation (16) indicates that, in the one-channel case considered here, the system decays *algebraically* already at times shorter than the average lifetime of a single resonance ($\hbar/\overline{\Gamma}$), without having a period of exponential decay. This conclusion is supported by recent work of Lewenkopf and Weidenmüller [11] who numerically considered the autocorrelation function $S(0)S^*(\varepsilon)$ as given by Eq. (8.10) of Ref. [1]. From Eq. (12), one immediately sees that an exponential decrease of $\overline{P(t)}$ is possible only if this correlator is of Lorentzian form: $S(0)S^*(\varepsilon)\sim 1/(1-i\varepsilon/\Gamma_{\text{corr}})$. Such a form is, however, clearly ruled out by their calculations: Despite the fact that each isolated resonance *is* of Lorentzian form, this does not hold for the average autocorrelation function.

VI. SUMMARY AND CONCLUSIONS

We have found a striking deviation from the exponential decay law for the average temporal behavior of stochastic quantum systems. This adds a further possible origin of nonexponentiality to the already known ones like the spreading of wave packets [12], the influence of the quantum-mechanical measurement process [13,14], interference effects between degenerate or close-lying resonances [6,15,16], or threshold and other effects of the boundedness of the spectrum [14,17,18].

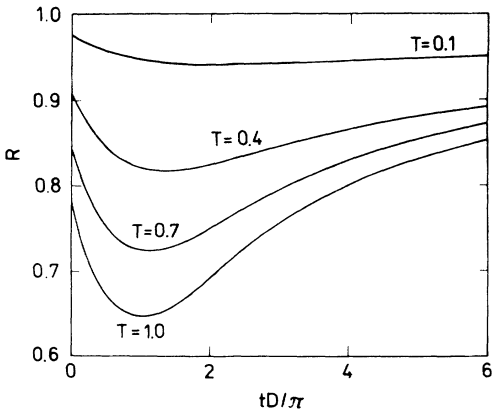

FIG. 1. Ratio R of the approximation (16) to the exact expression for $\overline{P(t)}$ [as obtained by integrating (13) over t] as function of the dimensionless time tD/π for different values of the transmission coefficient T. Only the region of small time is shown, where R most significantly differs from 1. Note the suppression of the origin.

We emphasize that none of these mechanisms can explain the algebraic decay law of the system considered here. We argue, on the contrary, that the particular form of the asymptotic result (14) is due to the specific statistical assumptions of the model: namely, a random Hamiltonian drawn from the GOE and the ensuing width distribution. In this sense, the obtained result offers a new test of the GOE via the time evolution of compound systems. Moreover, the $t^{-3/2}$ law obtained obviously hinges on the existence of *one* decay channel only. In general, for K open channels, the GOE generates a total width distribution of the form $\chi^2(\Gamma)$ with K degrees of freedom [19]. Instead of being located essentially around $\Gamma=0$ (as is the Porter-Thomas distribution), it approaches a $\delta(\Gamma-\overline{\Gamma})$ distribution, for $K\to\infty$. Then, repeating the considerations of the introduction, one is led to an exponential decay law (now on average) for arbitrary transmission coefficients. Again, a rigorous proof can be given, using the result of Ref. [1] together with Eq. (12) of the present paper [20].

The present considerations are not purely academic. At least in nuclear physics, there are experiments determining the average lifetime of states decaying into one or very few channels. For their interpretation, decay functions of the type of Eq. (15) have been used [21]. They have indeed been known to the "practitioners" [22] for a long time [23] since the case of isolated resonances is accessible to what we have called the heuristic argument. More possible applications are mentioned in Ref. [11].

Random matrix models have a close, not yet completely clarified relationship to what is called "quantum chaos." In this connection, a recent paper by Doron, Smilansky, and Frenkel [24] is of interest where the time behavior of a chaotic microwave cavity coupled to a single open waveguide mode was investigated. There, in semiclassical theory, an exponential decay law was found and fitted to the experimental data. The apparent contradiction to the time behavior expected from the present paper may possibly be resolved by carefully accounting for the number of open channels: The large and resonancelike absorption in the cavity wells described in [24] makes it questionable whether one is really dealing with a one-channel case in our sense. In particular, in the present formalism the existence of exactly one open channel requires $|S_{11}|^2=1$. In Ref. [24], this relation is obviously violated; i.e., one would have to introduce additional ("parasitic") channels in order to describe these absorption effects. In addition, we remark that algebraic decay laws are not alien to (partially) chaotic *classical* systems, as recent numerical studies have shown [25]. There, we have a picture quite similar to the present one: the averaged phase flow out of a whole region around a regular torus is algebraic, even though any *individual* trajectory spreads away exponentially.

By way of conclusion, our result represents an exact

answer to the problem of temporal behavior of open chaotic systems, within the specific model of a quantal system with a Gaussian Hamiltonian coupled to one open channel. We found an algebraic decay law for arbitrary coupling between bound states and continuum for practically all times.

ACKNOWLEDGMENTS

Numerous discussions with C. H. Lewenkopf, I. Rotter, T. H. Seligman, and H. A. Weidenmüller are gratefully acknowledged. One of us (F. M. D.) is indebted to the Max-Planck-Institut für Kernphysik for the hospitality offered to him.

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