Stability of dark solitary waves trapped in media with gain and loss

Yijiang Chen
Optical Sciences Centre, Australian National University, Canberra, Australia
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The stability of dark solitary waves trapped in self-defocusing media that include gain and two-photon absorption is investigated. The stationary propagation of the dark solitary waves is found to be stable to a symmetric perturbation but unstable to an asymmetric perturbation. This contrasts with their bright counterparts, which are unstable to all perturbations.

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Currently, there is an upsurge of research interest in optical solitons owing to their potential applications in optical signal processing and transmission [1]. Optical solitons as a whole consist of two species, referred to as bright and dark solitons. Analogous to their bright counterpart, dark solitons come further in two varieties: temporal and spatial [1]. The temporal dark soliton denotes the stable distortionless dark pulse traveling in the positive-group-velocity-dispersion region with the dispersion balanced by nonlinear phase modulation, while the spatial dark soliton means the stable self-guided dark beam propagating in a uniform self-defocusing nonlinear medium. Different as the physical origins are, the behavior of the two varieties is governed by the same mathematical equation, the nonlinear Schrödinger equation, in their corresponding normalized quantities [2]. These soliton solutions are, however, limited to the nonlinear media that are passive and lossless [2]. In reality, material loss may be inevitable, especially when nonlinearity is associated with two-photon absorption, which in fact often appears as a by-product of enhanced nonlinearity [3,4]. Its presence consequently leads to attenuation of a propagating pulse or light wave [5–8]. On the other hand, self-trapping may occur at Raman-Stokes frequencies [9–11] (or in an active nonlinear medium); an optical pulse or a self-guided beam then experiences a constant gain under conditions of strong Raman pumping with negligible depletion. Obviously, in the presence of either two-photon absorption or gain alone, the soliton solution or stationary propagation of a light wave in a medium is no longer possible. Naturally, an immediate question is whether the intensity-dependent two-photon absorption can be counteracted by constant gain to lead to the stationary propagation of a pulse or light beam when both two-photon absorption and constant gain are present. In a previous study [8], it was shown that this is indeed the case, i.e., two-photon absorption can be compensated by gain to yield the stationary solution. But a question remains as to whether such a stationary evolution of the dark solitary wave is stable to a perturbation since the stability characteristic of a solitary wave is crucial for its potential application [7]. The purpose of this Brief Report is to address the very question of stability of the dark solitary wave trapped in a nonlinear medium that includes two-photon absorption and gain.

The evolution of a light wave in a medium is governed by Maxwell's equations, which reduces to the Schrödinger equation under the assumption of the slowly varying approximation. Incorporated with two-photon absorption and gain effects, the modified nonlinear Schrödinger equation for a self-defocusing nonlinearity reads

$$i\frac{\partial e}{\partial z}+\frac{1}{2}\frac{\partial^2 e}{\partial x^2}+(-1+i\alpha_2)|e|^2 e-i\alpha e=0\ , \tag{1}$$

where the term involving α_2 accounts for the intensity-dependant two-photon absorption and that containing α represents the constant gain contribution. In the presence of α_2 or α alone, Eq. (1) does not admit any stationary solution. However when both $\alpha\neq 0$ and $\alpha_2\neq 0$, Eq. (1) yields [8]

$$e(x,z)=A\tanh(\nu x)\exp\{-i[\Gamma z+\gamma\ln\operatorname{sech}(\nu x)]\}\ , \tag{2}$$

where the solitary wave amplitude $A=\sqrt{\alpha/\alpha_2}$, the width $W=1/\nu=\sqrt{3\gamma/2\alpha}$, the guide index $\Gamma=2\alpha/3\gamma$, and $\gamma=\sqrt{9/4\alpha_2^2+2}-1.5/\alpha_2$. In the limit of $\alpha\to 0$ and $\alpha_2\to 0$, Eq. (2) reduces to the well-known dark-soliton solution with

$$\Gamma=\nu^2=A^2 \tag{3a}$$

of an arbitrary value. But for $\alpha\neq 0$ and $\alpha_2\neq 0$, the amplitude, the guide index, and the width are uniquely determined by the parameters α and α_2 and are related by

$$\Gamma=\nu^2=\frac{A^2}{2}+\left[\left(\frac{A^2}{2}\right)^2+\tfrac{2}{9}\alpha^2\right]^{1/2}>A^2\ . \tag{3b}$$

This expression compared with Eq. (3a) indicates that for a given amplitude A, the dark solitary wave trapped in the self-defocusing medium with gain and loss is more contracted ($W<1/A$ see Fig. 1) and its guide index is greater than the corresponding one trapped in the passive, lossless self-defocusing medium. Recall the relation [8] between the guide index Γ and the wave effective index $n_{\text{eff}}=\beta/k$ for a spatial dark solitary wave

$$\Gamma=n_0-n_{\text{eff}}^2/n_0$$

with β the wave propagation constant, k the wave num-

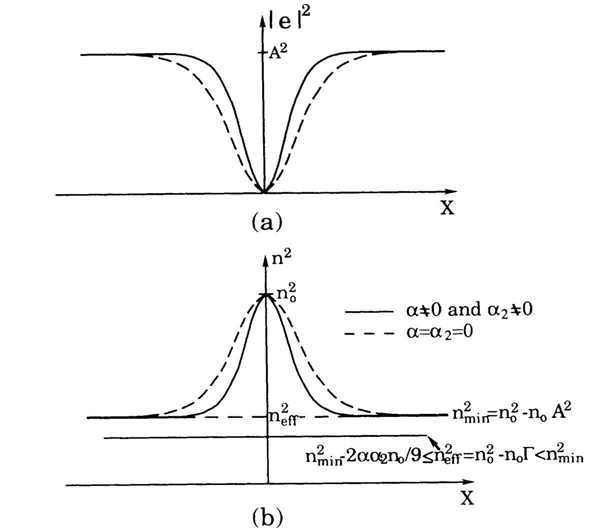

FIG. 1. Schematic illustration of the intensity profiles in (a) and the induced refractive-index profiles plus the effective wave indices $n_{\rm eff}$ in (b) of a spatial dark soliton (portrayed by the dashed curves for $\alpha=0$ and $\alpha_2=0$) and a spatial dark solitary wave (depicted by the solid curves for $\alpha\neq0$ and $\alpha_2\neq0$).

ber in free space, and n_0 the linear refractive index of the medium. By applying the relation to the spatial dark soliton ($\alpha=\alpha_2=0$), it was shown that the maximum induced refractive-index change is $n_0^2-n_{\rm min}^2=n_0^2-n_{\rm eff}^2$ for any Γ [12], i.e., the minimum refractive index $n_{\rm min}$ of the induced index profile for a dark soliton is equal to $n_{\rm eff}$ for any Γ (indicating its operation just at the cutoff) as shown in Fig. 1(b) (the dashed line). However, here the maximum induced refractive-index change for the dark solitary wave trapped in the medium with gain and loss is $n_0^2-n_{\rm min}^2=n_0A^2$ less than $n_0^2-n_{\rm eff}^2=n_0\Gamma$, i.e., the dark solitary wave is trapped below the cutoff [see the solid line in Fig. 1(b) showing $n_{\rm eff}<n_{\rm min}$]. This difference between $n_{\rm min}$ and $n_{\rm eff}$ is equal to

$$n_{\rm min}^2-n_{\rm eff}^2=n_0\left\{\left[\left[\frac{A^2}{2}\right]^2+\tfrac{2}{9}\alpha^2\right]^{1/2}-\frac{A^2}{2}\right\},$$

augmenting with increasing α and reaching the maximum value $\frac{2}{9}\alpha_2\alpha n_0$ as $A\to\infty$. The distinctions between the dark solitary wave in the medium with gain and loss (the solid curves) and that in the medium without gain and loss (the dashed curves) are outlined in Fig. 1.

The dark soliton (or the dark solitary wave trapped in a passive lossless self-defocusing medium) is a stable entity. Then what about the dark solitary wave trapped in a self-defocusing medium with gain and two-photon absorption? Can it evolve stably as in the case of the dark soliton? This question is to be addressed by numerical investigation on Eq. (1) using the beam propagation method. But first let us consider an analytical prediction by resorting to adiabatic approximation [8]. When gain and loss are small, it is legitimate to assume $A\approx1/W$, which leads to

$$e(x,z)=a\tanh(ax)\exp\left(-i\int_0^z a^2dz\right) . \qquad (4)$$

This expression, substituted into Eq. (1), yields an evolution equation governing a:

$$\frac{da}{dz}=\alpha a(1-a^2/A^2) . \qquad (5)$$

At $a=A=\sqrt{\alpha/\alpha_2}$, the stationary solution is recovered. Now allow a perturbation to deviate a from its equilibrium value A, Eq. (5) indicates $da/dz<0$ for $a>A$ and $da/dz>0$ for $a<A$. This means that a beam (or pulse) will radiate energy to adjust itself to the equilibrium with an expansion in the width when $a>A$, whereas it will absorb energy from the medium to expand toward the stationary solution with a narrowing in the width when $a<A$, i.e., the stationary solution of Eq. (1) is stable to a change in a or to a symmetric perturbation. This is corroborated by numerical stimulations. As examples, consider the initial excitations

$$e(x,0)=A\{\tanh(\nu x)\pm[\operatorname{sech}(x-2)-\operatorname{sech}(x+2)]\}\times\exp[-i\operatorname{sech}(\nu x)] \qquad (6)$$

with the second and third terms on the right-hand side accounting for symmetric perturbations. Figure 2 illustrates the evolution of the beams (or pulses). After a period of adaptation up to $z\approx50$, the beams evolve to the stationary solution and propagate stably thereafter. The greater the deviation of initial beams from the stationary solution, the longer the distance required to adjust to the steady state. Unfortunately, this stable evolution cannot last infinitely since the dark solitary wave of Eq. (2) is unstable to an asymmetric perturbation. This can be demonstrated by an initial asymmetric perturbation to the medium of Fig. 2

$$e(x,0)=A[\tanh(\nu x)+0.1\operatorname{sech}(x-2)]\times\exp[-i\operatorname{sech}(\nu x)] \qquad (7)$$

with the second term on the right-hand side standing for an asymmetric perturbation. As illustrated in Fig. 3, the dark hole experiences dissipation from the onset and at

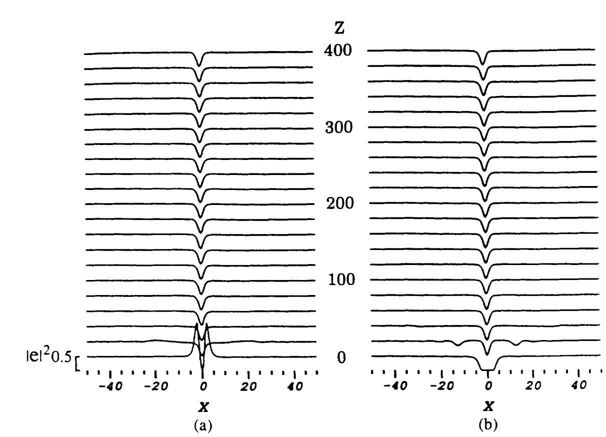

FIG. 2. Demonstration of the stability of the dark solitary wave of Eq. (1) to symmetric perturbations for $\alpha=0.1$ and $\alpha_2=0.2$. The initial excitation in (a) and (b) corresponds to the upper and lower signs in Eq. (6), respectively.

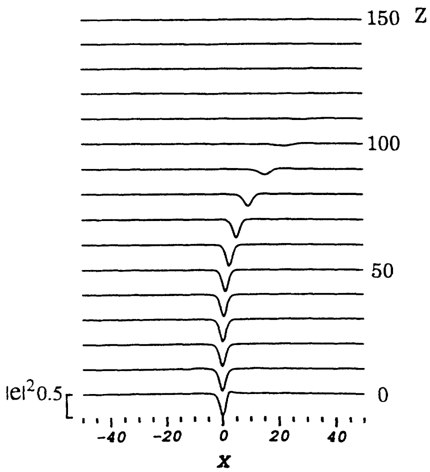

FIG. 3. Demonstration of the instability of the dark solitary waves of Eq. (1) to an asymmetric perturbation for $\alpha=0.1$ and $\alpha_2=0.2$ with the initial excitation of Eq. (7).

$z\approx 110$ it is completely absorbed into the background. This is because Eq. (1) also admits the plane-wave solutions

$$e(x,z)=\pm\sqrt{\alpha/\alpha_2}\exp[-i(\alpha/\alpha_2)z] \quad (8)$$

which are stable to any perturbations. Thus any numerical noise of asymmetric nature may eventually lead the stationary evolution in Fig. 2 to the plane-wave solution of Eq. (8).

Recall that the bright solitary wave trapped in a self-focusing medium is unstable to any perturbations [13] in contrast to the dark solitary wave examined here. This is due to the fact that a noise in the tail part (where $|e|\approx 0$) of the intensity profile of a bright solitary wave is readily amplified locally to the plane-wave solutions $e(x,z)=\pm\sqrt{\alpha/\alpha_2}\exp[i(\alpha/\alpha_2)z]$ as gain there predominates and thus spoils the stationary evolution. On the other hand, for the dark solitary wave, a noise in the tail or background of the intensity profile (where $|e|\approx\sqrt{\alpha/\alpha_2}$) once appearing is immediately suppressed since $|e(x,z)|>|e_{\text{equilibrium}}(x,z)|\approx\alpha/\alpha_2$ resulting from the noise leads to exaggeration of attenuation, driving $|e(x,z)|$ back to its equilibrium and $|e(x,z)|<|e_{\text{equilibrium}}(x,z)|$ gives the preference for gain, amplifying $|e(x,z)|$ to the stationary value. Note that the arguments here are based on the fact that loss is linearly proportional to the intensity as indicated in Eq. (1).

In conclusion, the dark solitary wave trapped in a self-defocusing medium that includes two-photon absorption and gain is investigated. The beam effective index of the spatial dark solitary wave is shown to be smaller than the minimum value of the induced refractive index profile, i.e., the beam propagates below the cutoff, in contrast to the spatial dark soliton (in the passive lossless medium) whose wave effective index is equal to the minimum value of the induced refractive-index profile (operating just at the cutoff). Most interestingly, it is found that the dark solitary wave is stable to a symmetric perturbation but unstable to an asymmetric perturbation. This contrasts with its bright counterpart, which is unstable to any perturbations.

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