Exact solutions of the nonlinear Schrödinger equation for the normal-dispersion regime in optical fibers

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We describe a method for obtaining exact solutions of the nonlinear Schrödinger equation for describing pulse propagation in optical fibers in the normal-group-velocity-dispersion regime. The method is based on the construction of a certain complete integrable finite-dimensional dynamical system whose solutions determine the exact solutions of the nonlinear Schrödinger equation.

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Optical solitons in fibers are pulses that propagate without any change in pulse shape or intensity. Because of their remarkable stability properties, optical solitons are now at the center of an active research field of nonlinear wave propagation in optical fibers. This research field started with the result obtained in [1,2] that under appropriate combinations of pulse shape and intensity, the effects of the intensity-dependent refractive index of the fiber exactly compensate for the pulse-spreading effects of group-velocity-dispersion. For the anomalous-group-velocity-dispersion regime $(\partial^2\omega/\partial k^2>0)$ which occurs in typical single-mode silica-based fibers for wavelengths $\lambda>1.27$ μm, the fundamental soliton is called a bright pulse [1] and the propagation of these bright solitons has been studied intensively and verified experimentally [3]. For the normal-group-velocity-dispersion regime $(\partial^2\omega/\partial k^2<0)$ the theory [2] and numerical simulations [4,5] predict that the solitons are dark pulses (i.e., a dip occurs at the center of the pulse). The generation of dark solitons in single-mode optical fibers was also demonstrated [6–8]. Recently a soliton-transmission technique that makes positive use of the existence of slight fiber loss, called dynamic soliton communication, was used to send optical solitons over long distances [9]. It was demonstrated that digitally coded optical solitons at a bit rate of 10 Gbit/s can be successfully transmitted over 300 km using erbium-doped fiber amplifiers. We mention also the works of several very active research groups in the field of the theory of pulse propagation in optical fibers in both the picosecond and the femtosecond regime [10–23].

The propagation of optical pulses in monomode optical fibers exhibiting Kerr-law nonlinearities is described well by the dimensionless nonlinear Schrödinger equation (NLSE):

$$i\psi_t+\alpha\psi_{xx}+2|\psi|^2\psi=0\ , \tag{1}$$

where ψ represents a normalized complex amplitude of the pulse envelope, t is a normalized distance along the fiber, x is the normalized time with the frame of reference moving along the fiber at the group velocity, $\alpha=+1$ corresponds to the anomalous-dispersion region, where bright solitons can exist, and $\alpha=-1$ corresponds to the normal-dispersion region where dark solitons occur. The NLSE is one of the complete integrable nonlinear equations and the solutions may be obtained by different methods, e.g., by using the inverse scattering method [24–30]. A large number of exact analytical solutions for the higher-order NLSE were found by using Lie-group theory [31–33]. Another way of obtaining solutions of the NLSE is the Darboux-transformation method [34].

Recently an alternative method of obtaining exact solutions of the NLSE for describing pulse propagation in optical fibers in the anomalous dispersion regime was given [35]. This method is based on the following linear relationship between the real $u(x,t)$ and the imaginary $v(x,t)$ parts of the complex amplitude pulse envelope $\psi(x,t)$:

$$u(x,t)-a_0(t)v(x,t)-b_0(t)=0 \tag{2}$$

with the coefficients a_0 and b_0 which depend only on the "time" variable t. The method is essentially the construction of a certain system of ordinary differential equations the solutions of which determine the solutions of the NLSE (1).

In this paper, following the method developed in [35], we obtain exact solutions of the NLSE describing pulse propagation in monomode optical fibers in the normal group-velocity dispersion region, i.e., for $\alpha=-1$ in the NLSE (1). By using the linear relationship (2) between the unknown functions $u(x,t)$ and $v(x,t)$ we will construct a certain dynamical system, the solutions of which determine the exact solutions of the NLSE (1) with $\alpha=-1$. We obtain a three-parameter family of solutions of the NLSE (1) which are expressed in terms of the Jacobi elliptic functions and the incomplete elliptic integral of the third kind. In the general case the solutions are double periodic in the "time" variable t and periodic in the "spatial" variable x.

In what follows we will use the same notation as in [35] for the sake of any easy comparison of solutions corresponding to $\alpha=1$ and -1.

We introduce the new unknown functions $Q(x,t)$, $\delta(t)$, and $\varphi(t)$ through $a_0(t)=\cot\varphi(t)$, $b_0(t)=-\delta(t)/\sin\varphi(t)$, and $u(x,t)=Q(x,t)\cos\varphi(t)-\delta(t)\sin\varphi(t)$, such that we have the following representation for the unknown function $\psi(x,t)$:

$$\psi(x,t)=[Q(x,t)+i\delta(t)]e^{i\varphi(t)} . \quad (3)$$

By introducing (3) in the NLSE with $\alpha=-1$ and taking the real and imaginary parts we are left with the following system of differential equations:

$$Q_{xx}+\delta_t+\varphi_t Q-2\delta^2 Q-2Q^3=0 , \quad (4)$$

$$Q_t-\varphi_t\delta+2\delta Q^2+2\delta^3=0 . \quad (5)$$

Here the differential equation (4) has a prime integral:

$$Q_x^2-Q^4-(2\delta^2-\varphi_t)Q^2+2\delta_t Q=h(t) , \quad (6)$$

where $h(t)$ is a function which depends only on the "time" variable t.

The condition of compatibility of the system of the differential equations (4) and (5), i.e., $Q_{xt}=Q_{tx}$ gives the following system of three ordinary differential equations:

$$\varphi_{tt}+8\delta\delta_t=0 , \quad (7)$$

$$\delta_{tt}-4\delta h+\delta\varphi_t^2-4\delta^3\varphi_t+4\delta^5=0 , \quad (8)$$

$$h_t-2\delta\delta_t\varphi_t+4\delta^2\delta_t=0 . \quad (9)$$

The dynamical system (7)–(9) corresponding to Eq. (1) has the following three prime integrals:

$$\varphi_t+4\delta^2=W , \quad (10)$$

$$h-W\delta^2+3\delta^4=H , \quad (11)$$

$$\delta_t^2+(W^2-4H)\delta^2-8W\delta^4+16\delta^6=D . \quad (12)$$

With the ansatz (3), the functions φ, δ, and h depend only on the "time" variable t, therefore W, H, and D are constants (x independent).

Next, with the help of the substitution $z(t)=\delta^2(t)$, we obtain

$$z_t^2=-64z^4+32Wz^3-4(W^2-4H)z^2+4Dz . \quad (13)$$

Now let $\alpha_0=0$, α_1, α_2, and α_3 be the roots of the polynomial on the right-hand side (rhs) of Eq. (13). These roots are connected with the prime integrals W, H, and D via the Viète relations:

$$\begin{aligned} &W=2(\alpha_1+\alpha_2+\alpha_3),\\ &H=\alpha_1^2+\alpha_2^2+\alpha_3^2-2(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1),\\ &D=16\alpha_1\alpha_2\alpha_3 . \end{aligned} \quad (14)$$

Next Eqs. (6) and (13) become, respectively,

$$\begin{aligned} Q_x^2=&Q^4-(W-6z)Q^2\\ &-3[(\alpha_1-z)(\alpha_2-z)(\alpha_3-z)]^{1/2}Q\\ &-(3z^2-Wz-h) , \end{aligned} \quad (15)$$

$$z_t^2=-64z(z-\alpha_1)(z-\alpha_2)(z-\alpha_3) . \quad (16)$$

The polynomial on the rhs of Eq. (15) has the form $P_1(Q)P_2(Q)$ where

$$\begin{aligned} P_{1,2}(Q)=&Q^2\pm2(\alpha_3-z)^{1/2}Q+\alpha_3-\alpha_1-\alpha_2+z\\ &\pm2[(\alpha_1-z)(\alpha_2-z)]^{1/2} . \end{aligned} \quad (17)$$

The simplest solutions of Eq. (16) are the constant functions $z=0$, α_1, α_2, and α_3 which give the stationary solutions of Eq. (1). Similarly, Eq. (15) has the solutions $Q=Q_i(t)$, $i=1,\ldots,4$. where $Q_i(t)$ are the roots of the polynomial on the rhs of Eq. (15). For these particular solutions the function ψ depends only on the "time" variable t.

From Eq. (16) it is easy to see that at least one of the roots α_i is positive and in the following we suppose that $\alpha_3\geq 0$.

Because the functions $Q(x,t)$ and $z(t)$ are real we have two cases to analyze: case A, $\alpha_1\leq\alpha_2\leq\alpha_3$ are real numbers, and case B, $\alpha_3\geq 0$, α_2 and α_2 are complex numbers.

(i) Case A. The discriminants $D_\pm$ of the polynomials $P_{1,2}(Q)$ are

$$D_\pm=\begin{cases}-[(z-\alpha_1)^{1/2}\pm(z-\alpha_2)^{1/2}]^2 & \text{for } \alpha_2\leq z\leq\alpha_3\\ [(\alpha_1-z)^{1/2}\mp(\alpha_2-z)^{1/2}]^2 & \text{for } 0\leq z\leq\alpha_1 ,\end{cases} \quad (18)$$

where $D_\pm$ corresponds to $P_1(Q)$ and $P_2(Q)$, respectively.

In the following we have two subcases to analyze: (a) $0\leq z\leq\alpha_1$ and (b) $0\leq\alpha_2\leq z\leq\alpha_3$ or $\alpha_2\leq 0\leq z\leq\alpha_3$.

(a) The solutions of Eq. (16) on the interval $0\leq z\leq\alpha_1$ can be expressed in terms of Jacobi elliptic functions [36–38]:

$$z(t)=\frac{\alpha_1\alpha_3\mathrm{sn}^2(\mu t,k)}{\alpha_3-\alpha_1\mathrm{cn}^2(\mu t,k)} , \quad (19)$$

where $\mu=4[\alpha_2(\alpha_3-\alpha_1)]^{1/2}$ and $k^2=[\alpha_1(\alpha_3-\alpha_2)]/[\alpha_2(\alpha_3-\alpha_1)]$ if the modulus of the elliptic functions.

The roots of the polynomial on the rhs of Eq. (15) are (see [39], pp. 24)

$$\begin{aligned} &Q_1=(\alpha_1-z)^{1/2}+(\alpha_2-z)^{1/2}+(\alpha_3-z)^{1/2} ,\\ &Q_2=-(\alpha_1-z)^{1/2}-(\alpha_2-z)^{1/2}+(\alpha_3-z)^{1/2} ,\\ &Q_3=-(\alpha_1-z)^{1/2}+(\alpha_2-z)^{1/2}-(\alpha_3-z)^{1/2} ,\\ &Q_4=(\alpha_1-z)^{1/2}-(\alpha_2-z)^{1/2}-(\alpha_3-z)^{1/2} . \end{aligned} \quad (20)$$

With the help of (20) the solutions of Eq. (15) are

$$Q=\begin{cases}\dfrac{(Q_1(Q_2-Q_4)-Q_2(Q_1-Q_4)\mathrm{sn}^2(px,m)}{(Q_2-Q_4)-(Q_1-Q_4)\mathrm{sn}^2(px,m)} , & Q\geq Q_1 \text{or} Q\leq Q_4\\ \dfrac{Q_3(Q_2-Q_4)-Q_4(Q_2-Q_3)\mathrm{sn}^2(px,m)}{(Q_2-Q_4)-(Q_2-Q_3)\mathrm{sn}^2(px,m)} , & Q_3\leq Q\leq Q_2 ,\end{cases} \quad (21)$$

where $p=(\alpha_2-\alpha_1)^{1/2}$ and $m^2=(\alpha_3-\alpha_2)/(\alpha_3-\alpha_1)$ is the modulus of the elliptic function.

From Eqs. (10) and (19) we obtain the expression for $\varphi(t)$

$$\varphi(t)=2(\alpha_1+\alpha_2-\alpha_3)t+\frac{4\alpha_3}{\mu}\Pi(n;\mu t,k) , \quad (22)$$

where $n=\alpha_1/(\alpha_1-\alpha_3)$ and $\Pi(n;\mu t,k)$ is the incomplete elliptic integral of the third kind [36–38]. Now comparing the solutions for $z(t)$, $Q(x,t)$, and $\varphi(t)$ for $\alpha=1$ obtained in [35] with the corresponding ones for $\alpha=-1$ given by relations (19), (21), and (22), we arrive at the conclusion that we have different analytical solutions for $Q(x,t)$ but exactly the same explicit forms for the functions $z(t)$ and $\varphi(t)$. The explicit analytical solutions (19), (21), and (22) are periodic in x and t so we obtain the result that the solution $\psi(x,t)$ is double periodic with respect to the "time" variable t and periodic with respect to the "spatial" variable x.

(b) The solution of Eq. (16) on the interval $\alpha_2 \le z \le \alpha_3$ with $z \ge 0$ is

$$z(t)=\frac{\alpha_2(\alpha_3-\alpha_1)-\alpha_1(\alpha_3-\alpha_2)\mathrm{sn}^2(\mu t,k)}{(\alpha_3-\alpha_1)-(\alpha_3-\alpha_2)\mathrm{sn}^2(\mu t,k)} . \tag{23}$$

From (17) and (18) we obtain the following roots $Q_i(t)$:

$$\begin{aligned} Q_{1,2}&=(\alpha_3-z)^{1/2}\pm i[(z-\alpha_1)^{1/2}-(z-\alpha_2)^{1/2}] ,\\ Q_{3,4}&=-(\alpha_3-z)^{1/2}\pm i[(z-\alpha_1)^{1/2}-(z-\alpha_2)^{1/2}] . \end{aligned} \tag{24}$$

With the help of (24) the solution of Eq. (15) is

$$\begin{aligned} Q=&(\alpha_3-z)^{1/2}+[(z-\alpha_1)^{1/2}-(z-\alpha_2)^{1/2}]\\ &\times \tan(\varphi+\theta_1/2+\theta_2/2) \quad \text{for } \alpha_3\neq 0 \end{aligned} \tag{25}$$

where

$$\tan\theta_1=\left[\frac{z-\alpha_1}{\alpha_3-z}\right]^{1/2}, \quad \tan\theta_2=-\left[\frac{z-\alpha_2}{\alpha_3-z}\right]^{1/2},$$

$$\sin\varphi=\mathrm{sn}(px,m)$$

with

$$m^2=\frac{4(\alpha_3-\alpha_1)^{1/2}(\alpha_3-\alpha_2)^{1/2}}{[(\alpha_3-\alpha_1)^{1/2}-(\alpha_3-\alpha_2)^{1/2}]^2} ,$$

$$p=\left[\frac{(\alpha_3-\alpha_1)^{1/2}-(\alpha_3-\alpha_2)^{1/2}}{(\alpha_2-\alpha_1)[(\alpha_3-\alpha_1)^{1/2}+(\alpha_3-\alpha_2)^{1/2}]}\right]^{1/2} .$$

From Eqs. (10) and (23) we obtain for $\varphi(t)$

$$\begin{aligned} \varphi(t)=&2(\alpha_1+\alpha_2+\alpha_3)t-4\mu\alpha_1 t\\ &+4(\alpha_1-\alpha_2)\Pi(n;\mu t,k) , \end{aligned} \tag{26}$$

where $n=(\alpha_3-\alpha_2)/(\alpha_3-\alpha_1)$.

In the case $z=\alpha_3$ we have

$$Q=[(\alpha_3-\alpha_2)^{1/2}-(\alpha_3-\alpha_1)^{1/2}]\frac{\mathrm{cn}(px,m)}{\mathrm{sn}(px,m)} , \tag{27}$$

where $p=[(\alpha_3-\alpha_1)^{1/2}-(\alpha_3-\alpha_2)^{1/2}]^{-1}$,

$$m^2=1-\left[\frac{(\alpha_3-\alpha_1)^{1/2}+(\alpha_3-\alpha_2)^{1/2}}{(\alpha_3-\alpha_1)^{1/2}-(\alpha_3-\alpha_2)^{1/2}}\right]^{1/2} ,$$

and for the function $\varphi(t)$ we obtain

$$\varphi(t)=2(\alpha_1+\alpha_2-\alpha_3)t . \tag{28}$$

(ii) Case B. Let α_1,α_2 be complex conjugate numbers ($\alpha_1=\alpha_2^*=\rho+i\eta$) and $\alpha_3 \ge 0$. Then the solution of Eq. (16) can be written as

$$z(t)=\frac{\alpha_3(1-\nu)[1+\mathrm{cn}(2\mu t,k)]}{2[1-\nu\,\mathrm{cn}(2\mu t,k)]} , \tag{29}$$

where

$$\nu=(f-g)/(f+g), \quad \mu=4(fg)^{1/2} ,$$

$$f=[(\alpha_3-\rho)^2+\eta^2]^{1/2}, \quad g=(\rho^2+\eta^2)^{1/2} ,$$

and

$$k^2=\frac{1}{2}\left[1-\frac{\eta^2+\rho(\rho-\alpha_3)}{fg}\right]$$

is the modulus of the Jacobi elliptic function.

We mention that the expression for $z(t)$ in [35] [see Eq. (26) in [35]] and those obtained with the help of this formula, e.g., Eqs. (30), (55), (56), (58), and (59) in [35], contain errors. The correct form of the function $z(t)$ for $\alpha=1$, i.e., the case treated in [35], is given by the relation (29). Thus the function $z(t)$ has the same expression for both the normal-dispersion regime ($\alpha=-1$) and the anomalous-dispersion regime ($\alpha=1$).

From (20) and (29) we obtain the roots of the polynomial on the rhs of Eq. (15):

$$Q_{1,2}=-b\pm d, \quad Q_{3,4}=b\pm ic ,$$

where $b=(\alpha_3-z)^{1/2}$; $d,c=\{2[(\rho-z)^2+\eta^2]^{1/2}\pm 2(\rho-z)\}^{1/2}$.

The function $Q(x,t)$ can be written as

$$Q=-b-d\frac{[r-\mathrm{cn}(2px,m)]}{[1-r\,\mathrm{cn}(2px,m)]} \tag{30}$$

for $Q \le Q_2$ or $Q \ge Q_1$, where

$$p=[(\rho-\alpha_3)^2+\eta^2]^{1/4} ,$$

$$r=\frac{(p^2+b-bd)^{1/2}+(p^2+b+bd)^{1/2}}{(p^2+b-bd)^{1/2}-(p^2+b+bd)^{1/2}} ,$$

and

$$m^2=\frac{1}{2}\left[1+\frac{\alpha_3-\rho}{p^2}\right]$$

is the modulus of the Jacobi elliptic function.

We note that the function $Q(x,t)$ has the same expression for both situations ($\alpha=\pm 1$) but the solution (30) is valid for $\alpha=1$ only in the interval $Q_2 \le Q \le Q_1$. For $\alpha=-1$, the explicit solution (30) is valid outside this interval, i.e., for $Q \le Q_2$ or $Q \ge Q_1$.

From (10) and (29) we finally find the correct expression for $\varphi(t)$ which is valid for both $\alpha=1$ and $\alpha=-1$ [the expression (30) for $\varphi(t)$ given in [35] being wrong]:

$$\begin{aligned} \varphi(t)=&2(2\rho+\alpha_3)t\\ &+\frac{4g}{\mu}[(1-n_1)\Pi(n_1;\mu t,k)\\ &+(n_2-1)\Pi(n_2;\mu t,k)] , \end{aligned} \tag{31}$$

where $n_1=2fk^2/(f-g+\alpha_3)$ and $n_2=2fk^2/(f-g-\alpha_3)$.

In the following we will find from the general solutions obtained above the dark-soliton solution. This particular solution is obtained for $\alpha_1=\alpha_2=0$ and $z=0$. In this case we have the following expressions for functions Q and φ:

$$Q(x)=\alpha_3^{1/2}\tanh(\alpha_3^{1/2}x)\ , \tag{32}$$

$$\varphi(t)=2\alpha_3 t\ . \tag{33}$$

Thus we obtained the standard form for the fundamental dark-soliton solution:

$$\psi_d(x,t)=q\tanh(qx)\exp(2iq^2t)\ , \tag{34}$$

where $q=\alpha_3^{1/2}$ is a form factor that determines the pulse amplitude and width.

In order to obtain other classes of solutions of NLSE (1) we can impose more generally that the functions $u(x,t)$ and $v(x,t)$ obey the following relationship:

$$P_n[u(x,t),v(x,t)]=0\ ,$$

where P_n is a polynomial of degree n in the variables u and v with coefficients which depend only on the "time" variable t. By using the method described above we can construct in principle the dynamical system corresponding to Eq. (1). In what follows we give an example for the case $n=2$ and for $P_2[u(x,t),v(x,t)]=u^2(x,t)+v^2(x,t)-R^2(t)$. We observe that the solution $\psi(x,t)$ can be put in the form

$$\psi(x,t)=R(t)\exp[i\Phi(x,t)]\ , \tag{35}$$

where the relationships between $u(x,t),v(x,t)$ and $R(t),\Phi(x,t)$ are the following:

$$\begin{aligned} u(x,t)&=R(t)\cos\Phi(x,t),\\ v(x,t)&=R(t)\sin\Phi(x,t)\ . \end{aligned} \tag{36}$$

The Jacobian of the transformation (36) is given by

$$\frac{\partial(u,v)}{\partial(R,\Phi)}=R(t)\ . \tag{37}$$

After rather simple calculations we are left with the following solution of NLSE (1) with $\alpha=-1$:

$$\psi(x,t)=\frac{C}{t^{1/2}}\exp\left[i\left[-\frac{x^2}{4t}+2C^2\ln t\right]\right]\ . \tag{38}$$

This solution is singular in $t=0$ and ∞ because the Jacobian (37) is singular in $t=0$ and ∞. We note that the corresponding solution of the NLSE (1) for $\alpha=1$ was given in [34] by the Darboux-transformation method.

In conclusion the method developed in this paper allows us to obtain a class of solutions of the NLSE describing the propagation of picosecond light pulses in optical fibers in the normal group-velocity dispersion region. The class of general solutions contains as particular cases, important from the physical point of view, the dark-soliton and stationary solutions. Finally we mention that this method may be applied to the study of solitons formed in the femtosecond region, a research field of recent interest due to possible applications to ultrafast optical switching and optical computing.

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