Number-phase uncertainty products and minimizing states

A. Lukš and V. Peřinová

Laboratory of Quantum Optics, Natural Science Faculty, Palacký University, Třída Svobody 26, 771 46 Olomouc, Czechoslovakia

(Received 27 August 1991)

The similarity between the system of the plane rotator and that of the optical oscillator (mode) has been exposed. In both the systems special states have been examined, which minimize the usual and unusual uncertainty relations. Number-sine-cosine uncertainty relations for these systems have been derived and the number-sine minimum-uncertainty states have been investigated in greater detail than in the literature.

PACS number(s): 42.50.Dv

I. INTRODUCTION

The experimental proof of the squeezed states of light has initiated an enhanced interest in special states of the electromagnetic field. Since the dawn of quantum optics, Glauber's coherent states [1] are known to be minimum-uncertainty states. The experimentally proved squeezed states are realizations of the two-photon coherent states [2], which result as a natural generalization of the coherent states minimizing the same uncertainty relation. The squeezed states are known to possess decreased fluctuations in one quadrature component at the cost of the increased noise of the other quadrature component. Other uncertainty relations are associated with the definitions of other minimizing states. The uncertainty relations and the minimizing states illustrate well the phase problem in the physical system of the optical oscillator and the easy study of the rotation angle. In the past, the phase problem tended to the consideration of some uncertainty relations and led to so many definitions of new states that the authors themselves doubted whether these states were more than mathematical constructions. We approve this attitude, but we confess that the beauty of the mathematical considerations is connected with their inconceivable efficiency.

II. APPLICATIONS OF THE SCHWARTZ INEQUALITY IN QUANTUM THEORY

In quantum theory we encounter physical quantities that are named after the classical ones but have a different meaning. This reinterpretation is underscored by Heisenberg in his early paper [3]. The new meaning of the quantities manifests itself in the fact that not all of them are compatible, i.e., the measurement of one observable may affect the precision of the other. This phenomenon is described by the uncertainty relations. It is marvelous that, in spite of the prominent physical role of the uncertainty relations, their mathematical proof rests upon the mere Schwartz inequality.

The phase problem is formulated for the phase of the optical oscillator. Both past and recent work on this problem comprise also the study of the rotation angle of the plane rotator. Of many quantities, which can be considered in the physical systems, some are basic, for instance the quadrature components in the harmonic and anharmonic oscillator in quantum optics and the angular momentum and the rotation angle in the plane rotator. The quantum theory of these systems arises in the quantum interpretation of these quantities as operators and of the Poisson brackets as the commutators. Usually we can discern whether an examined observable is basic or not. In the second case it must be possible to express the investigated observable in terms of basic ones. This possibility is yielded also by the quantum theory if we accept the necessity of ordering quantum operators in some cases.

The best known instance, perhaps, is the physical system of an electron in the Coulombic field of a nucleus and the bound motion of the electron. Considering only one degree of freedom, we arrive at the linear (anharmonic) oscillator. Here the basic observables are the position $\hat{x}$ and the conjugate momentum $\hat{p}_x$ and it holds that

$$[\hat{x},\hat{p}_x]=i\hbar\hat{1} . \quad (1)$$

Familiar are the formulas introducing the annihilation operator $\hat{a}$ describing the harmonic motion in this physical system. The matter-field interaction has led to the idea of the physical system of the optical oscillator, which comprises also the annihilation operator $\hat{a}$. In the system of a plane rotator we can arrive at the basic quantities also when reducing the number of basic quantities for a known motion of a particle. If $\hat{x},\hat{y}$ are the plane-position operators and $\hat{p}_x,\hat{p}_y$ their respective conjugate momenta, this reduction is expressed by the formulas

$$\hat{J}=\hat{y}\hat{p}_x-\hat{x}\hat{p}_y , \quad (2)$$

$$\cos\hat{\phi}_\theta=(\hat{x}^2+\hat{y}^2)^{-1/2}\hat{x} ,$$

$$\sin\hat{\phi}_\theta=(\hat{x}^2+\hat{y}^2)^{-1/2}\hat{y} , \quad (3)$$

where $\hat{J}$ is the angular-momentum operator, $\cos\hat{\phi}_\theta$ and $\sin\hat{\phi}_\theta$ are the cosine and sine operators, and θ is the minimum value of the measured angle.

At quantum applications of the Schwartz inequality we arrive as follows. Let us consider a concrete physical system and two observables $\hat{A}$, $\hat{B}$ therein. On the assumption that the system is in the pure state $|\psi\rangle$, $\langle\psi|\psi\rangle=1$, the observables $\hat{A},\hat{B}$ have the expectations,

$$\langle \hat{A}\rangle=\langle\psi|\hat{A}|\psi\rangle\ ,\quad \langle\hat{B}\rangle=\langle\psi|\hat{B}|\psi\rangle\ , \tag{4}$$

respectively. The physical interpretation of the following mathematical derivation is based on the mean squares of the operators:

$$\Delta\hat{A}=\hat{A}-\langle\hat{A}\rangle\ ,\quad \Delta\hat{B}=\hat{B}-\langle\hat{B}\rangle\ . \tag{5}$$

Introducing the vectors

$$|\varphi\rangle=\Delta\hat{A}|\psi\rangle\ ,\quad |\chi\rangle=\Delta\hat{B}|\psi\rangle \tag{6}$$

and considering the Schwartz inequality in the form

$$|\langle\varphi|\chi\rangle|^2\le\langle\varphi|\varphi\rangle\langle\chi|\chi\rangle\ , \tag{7}$$

we obtain the inequality

$$|\langle\Delta\hat{A}\Delta\hat{B}\rangle|^2\le\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle\ . \tag{8}$$

On the right-hand side there is a product of variances. To interpret the left-hand side more easily, we resolve the constituent product

$$\begin{aligned}\Delta\hat{A}\Delta\hat{B}&=\tfrac{1}{2}(\{\Delta\hat{A},\Delta\hat{B}\}+[\Delta\hat{A},\Delta\hat{B}])\\ &=\tfrac{1}{2}(\{\Delta\hat{A},\Delta\hat{B}\}+i\hat{C})\ ,\end{aligned} \tag{9}$$

where

$$\hat{C}=-i[\Delta\hat{A},\Delta\hat{B}]=-i[\hat{A},\hat{B}]\ . \tag{10}$$

Then relation (8) becomes

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\ge\text{cov}^2(\hat{A},\hat{B})+\tfrac{1}{4}\langle\hat{C}\rangle^2\ , \tag{11}$$

where

$$\text{cov}(\hat{A},\hat{B})=\tfrac{1}{2}\langle\{\Delta\hat{A},\Delta\hat{B}\}\rangle\ . \tag{12}$$

Regardless of the above modifications, the equality in relation (11) is attained on the assumption that either

$$\Delta\hat{B}|\psi\rangle=0\ , \tag{13}$$

or a suitable complex number κ can be found that

$$(\Delta\hat{A}+\kappa\Delta\hat{B})|\psi\rangle=0\ . \tag{14}$$

We rewrite this relation in the form

$$(\hat{A}+\kappa\hat{B})|\psi\rangle=\lambda|\psi\rangle\ , \tag{15}$$

where

$$\lambda=\langle\hat{A}\rangle+\kappa\langle\hat{B}\rangle\ . \tag{16}$$

If the equality sign in relation (11) applies, then

$$\kappa=-\frac{\langle\Delta\hat{B}\Delta\hat{A}\rangle}{\langle(\Delta\hat{B})^2\rangle}=\frac{-\text{cov}(\hat{A},\hat{B})+\frac{i}{2}\langle\hat{C}\rangle}{\langle(\Delta\hat{B})^2\rangle}\ . \tag{17}$$

For the states $|\psi\rangle$ fulfilling

$$\langle\hat{C}\rangle=0\ , \tag{18}$$

relation (11) and formula (17) simplify. In the optical oscillator system and for the case of basic observables

$$\text{(i)}\quad \hat{A}=\hat{Q}\ ,\quad \hat{B}=\hat{P}\ ,\quad [\hat{Q},\hat{P}]=2i\hat{1}\ , \tag{19}$$

where $\hat{Q},\hat{P}$ are the quadrature operators,

$$\hat{Q}=\hat{a}+\hat{a}^\dagger\ ,\quad \hat{P}=-i(\hat{a}-\hat{a}^\dagger)\ , \tag{20}$$

no states have the property (18). For the couples of operators like (19), the problem (15) could not be derived from the Schwartz inequality. Nevertheless, the choice

$$\text{(ii)}\quad \hat{A}=\hat{n}\ ,\quad \hat{B}=-\hat{Q}\ ,\quad [\hat{n},\hat{Q}]=-i\hat{P}\ , \tag{21}$$

where $\hat{n}=\hat{a}^\dagger\hat{a}$ is the number operator

$$\hat{n}=\tfrac{1}{4}(\hat{Q}^2+\hat{P}^2-2)\ , \tag{22}$$

ensures that the relation $\langle\hat{C}\rangle=\langle\hat{P}\rangle=0$ is fulfilled by some states.

Before we present case (iii) concerning the plane rotator, we give relation (11) simplified using (18),

$$\text{cov}^2(\hat{A},\hat{B})\le\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle\ , \tag{23}$$

which means only that the variances of operators and the covariance between the operators obey the same relations as the quantities in the classical theory of statistics. This proves that they have been consistently defined. Equation (17) simplifies to the form

$$\kappa=\gamma\ ,\quad \gamma=-\frac{\text{cov}(\hat{A},\hat{B})}{\langle(\Delta\hat{B})^2\rangle}=-\text{sgn}[\text{cov}(\hat{A},\hat{B})]\frac{\Delta A}{\Delta B}\ , \tag{24}$$

where

$$\Delta A=[\langle(\Delta\hat{A})^2\rangle]^{1/2}\ ,\quad \Delta B=[\langle(\Delta\hat{B})^2\rangle]^{1/2}\ . \tag{25}$$

From the properties of inequalities it follows that relation (23) holds also regardless of condition (18). The equality sign in relation (23) takes place when the correlation between the observables $\hat{A},\hat{B}$ is as maximal as possible. We easily find that in case (ii) this occurs for the displaced number state [4] and the shift in the phase space is realized in the direction of the $\hat{Q}$ (real) axis. Quite perspicuously we get Glauber's coherent state $|\gamma\rangle$ as a solution of Eq. (15) for the state $|\psi\rangle$ with $\lambda=-\gamma^2$. The correlation between the number of photons and the quadrature $\hat{Q}$ is maximal, i.e., in relation (23) the equality sign holds, for the coherent state $|\gamma\rangle$, contrary to the inequality for any state derived by a small perturbation of the coherent state. Relation (24) takes on the form in case (ii),

$$\gamma=\frac{\text{cov}(\hat{n},\hat{Q})}{\langle(\Delta\hat{Q})^2\rangle}\ . \tag{26}$$

The plane rotator provides the case

$$\text{(iii)}\quad \hat{A}=\hat{N}\ ,\quad \hat{B}=-\cos\hat{\phi}_\theta\ ,\quad [\hat{N},\cos\hat{\phi}_\theta]=-i\sin\hat{\phi}_\theta\ . \tag{27}$$

The operator $\hat{N}$ is related to the angular momentum $\hat{J}$ defined in (2),

$$\hat{N}=\frac{1}{\hbar}\hat{J}\ , \tag{28}$$

where $\hbar$ is the reduced Planck constant.

So far we assumed condition (18), which unsatisfactorily emphasizes the quantum nature of the relation for the product of uncertainties. Now we restrict ourselves to the states $|\psi\rangle$ fulfilling the condition

$$\mathrm{cov}(\hat{A},\hat{B})=0\ . \tag{29}$$

Then relation (11) reads

$$\langle(\Delta\hat{A})^2\rangle\langle(\Delta\hat{B})^2\rangle\geq\tfrac{1}{4}\langle\hat{C}\rangle^2\ , \tag{30}$$

or equivalently

$$\Delta A\,\Delta B\geq\tfrac{1}{2}|\langle\hat{C}\rangle|\ , \tag{31}$$

i.e., the Heisenberg uncertainty relation. But the pioneers of the quantum theory restricted themselves first to the case $\hat{C}=\hbar\hat{1}$. Relation (17) becomes, under condition (29),

$$\kappa=i\gamma\ ,\quad \gamma=\frac{\langle\hat{C}\rangle}{2\langle(\Delta\hat{B})^2\rangle}=\mathrm{sgn}\langle\hat{C}\rangle)\frac{\Delta A}{\Delta B}\ . \tag{32}$$

The uncertainty relations (30) and (31) hold generally regardless of condition (29).

In the system of the optical oscillator in case (i), assumption (29) can be fulfilled and the equality sign in relation (30) is achieved for the standard squeezed states [5], if the axis of squeezing is parallel with the $\hat{Q}$ (real) or $\hat{P}$ (imaginary) axis in the phase space. With respect to the equality attained in relation (30), we speak of the minimum-uncertainty states.

The solution of the eigenproblem (15) for the case

$$\text{(iv)}\quad \hat{A}=\hat{n}\ ,\quad \hat{B}=\hat{P}\ ,\quad [\hat{n},\hat{P}]=i\hat{Q}\ , \tag{33}$$

is of slightly different character [6]. Here, $\langle\hat{P}\rangle=0$. No similar constraint occurs in case (i). Nevertheless, this constraint is identical with the assumption $\langle\hat{C}\rangle=\langle\hat{P}\rangle=0$ in case (ii), which has been deliberately adjusted to case (iv). The problem (iv) issued in the study of the optical phase as a linearization result.

A similar parallel occurs between cases (iii) and (v),

$$\text{(v)}\quad \hat{A}=\hat{N}\ ,\quad \hat{B}=\sin\hat{\phi}_\theta\ ,\quad [\hat{N},\sin\hat{\phi}_\theta]=i\cos\hat{\phi}_\theta\ . \tag{34}$$

This case will be studied in greater detail in Sec. III.

Cases (iv) and (v) illustrate the possibility that $\langle\hat{C}\rangle$ may depend on which state realizing the equality in (30) is considered. The uncertainty product $\Delta A\,\Delta B=\frac{1}{2}|\langle\hat{C}\rangle|$ is state dependent, and so some authors speak of intelligent states in place of minimum-uncertainty ones [7].

III. UNCERTAINTY RELATIONS IN THE PLANE ROTATOR

The free plane rotator is described by the Hamiltonian

$$\hat{H}=\frac{1}{2I}\hat{J}^{\,2}\ , \tag{35}$$

where I is the moment of inertia. The angular-momentum operator and the rotation angle of the body, $\hat{\phi}_\theta$, are canonically conjugate variables. The angle operator $\hat{\phi}_\theta$ does exist and its eigenkets fulfill the relations

$$\hat{\phi}_\theta|\varphi\rangle_e=\varphi|\varphi\rangle_e\ ,\quad \theta\leq\varphi<\theta+2\pi\ , \tag{36}$$

$$_e\langle\varphi|\varphi'\rangle_e=2\pi\delta(\varphi-\varphi')\ . \tag{37}$$

These angular kets cannot be normalized. It holds that

$$[\hat{J},\hat{\phi}_\theta]=i\hbar(\hat{1}-|\theta\rangle_{e\,e}\langle\theta|)\ , \tag{38}$$

which unfortunately differs by the projector $|\theta\rangle_{e\,e}\langle\theta|$ from the established picture of this commutator, but

$$\begin{aligned}[][\hat{J},\cos\hat{\phi}_\theta]&=-i\hbar\sin\hat{\phi}_\theta\ ,\\ [\hat{J},\sin\hat{\phi}_\theta]&=-i\hbar\cos\hat{\phi}_\theta\ ,\end{aligned} \tag{39}$$

as can be expected. If the angle operator $\hat{\phi}_\theta$ plays the role of the argument of cosine and sine functions, the index θ can be omitted. The stationary states of the free rotator, $|N\rangle$, are simultaneously the eigenstates of the operator $\hat{J}$,

$$\hat{J}|N\rangle=\hbar N|N\rangle\ ,\quad \langle N|N'\rangle=\delta_{NN'}\ ,\qquad N,N'=0,\pm1,\pm2,\ldots, \tag{40}$$

and it is valid that

$$\langle N|\varphi\rangle_e=\exp(iN\varphi)\ . \tag{41}$$

If we restrict ourselves to the states $|\psi\rangle_e$, which are the superposition of only the states of the positive angular momentum, then the free plane rotator identifies, from the mathematical viewpoint, with the anharmonic oscillator in quantum optics. Therefore, the index e is affixed because the Hilbert space of the plane rotator represents an extension of the Hilbert space of the optical oscillator.

In terms of classical variables, the phase space of the optical oscillator is a plane, whereas the phase space of the plane rotator is a cylinder extending to infinity in both directions. Restricting ourselves to the positive angular momentum, we manage with half this cylinder, and using the polar coordinates we can prove the canonical equivalence with the phase space of the optical oscillator.

Respecting the relationship of the free plane rotator and the anharmonic oscillator, we treat cases (iii) and (v) in greater detail. In case (iii) the eigenvalue problem (15) reads

$$(\hat{N}-\gamma\cos\hat{\phi})|\psi\rangle_e=\lambda|\psi\rangle_e\ . \tag{42}$$

We easily find that

$$|\psi\rangle_e=\sum_{N=-\infty}^{\infty}J_{N-\lambda}(\gamma)|N\rangle\ ,\quad {}_e\langle\psi|\psi\rangle_e=1\ , \tag{43}$$

where λ is an integer expressed by the formula

$$\lambda=\langle\hat{N}\rangle-\gamma\langle\cos\hat{\phi}\rangle \tag{44}$$

and $J_{N-\lambda}$ is the Bessel function of the integral order. From the expression for the wave function $(2\pi)^{-1/2}{}_e\langle\varphi|\psi\rangle_e$ of the state (43) it follows that the phase distribution is uniform as if it were in any eigenstate of the angular-momentum operator. The distribution of the angular momentum exhibits oscillations.

Case (v) is more familiar. The eigenvalue problem (15) can be written as follows:

$$(\hat{N}+i\gamma \sin\hat{\phi})|\psi\rangle_e=\lambda|\psi\rangle_e \ . \quad (45)$$

In [8] the eigenvalue problem (45) is solved in terms of the modified Bessel functions of the first kind of order $(N-\lambda)$,

$$|\psi\rangle_e=[I_0(2\gamma)]^{-1/2}\sum_{N=-\infty}^{\infty} I_{N-\lambda}(\gamma)|N\rangle \ , \quad {}_e\langle\psi|\psi\rangle_e=1 \ , \quad (46)$$

where $\lambda=\langle\hat{N}\rangle$ is an integer. The appropriate wave function is easily found in the angular representation and the angular distribution can be identified with the von Mises distribution [9]. The uncertainty relation (30) reads in this case

$$\langle(\Delta\hat{N})^2\rangle\langle(\Delta \sin\hat{\phi})^2\rangle \ge \tfrac{1}{4}\langle\cos\hat{\phi}\rangle^2 \ . \quad (47)$$

Following [8], we have considered a symmetric form of the uncertainty relation and—unlike [8]—we have arrived at relation [10],

$$M_e \ge \tfrac{1}{4} \ , \quad (48)$$

where

$$M_e=[\langle(\Delta\hat{N})^2\rangle+\tfrac{1}{4}][1-|\langle\exp(i\hat{\phi})\rangle|^2] \ . \quad (49)$$

Because of this symmetry, the hope of attaining the equality in (48) is lost, since it would require the fulfillment of both the conditions

$$\mathrm{cov}(\hat{N},\cos\hat{\phi})=\mathrm{cov}(\hat{N},\sin\hat{\phi})=0 \ , \quad (50)$$

formally rewritten

$$\mathrm{cov}(\hat{N},\exp(i\hat{\phi}))=0 \ . \quad (51)$$

Compared with the ordinary uncertainty relation, the $\frac{1}{4}$ in the first factor is striking, a correction for the discreteness of the variable $\hat{N}$. The second factor is the angular dispersion, free of variances as recommended in the statistical studies of directional data [9]. The uncertainty product (49) is useful [10,11] and open for further numerical analysis since no minimizing states of relation (48) are likely to exist.

Combining problems (42) and (45), we arrive at the eigenvalue problem

$$[\hat{N}-\gamma \exp(-i\hat{\phi})]|\psi\rangle_e=\lambda|\psi\rangle_e \ , \quad (52)$$

which is not connected with any uncertainty relation, but is expected on the basis of symmetry. This problem has the solution

$$|\psi\rangle_e=[I_0(2\gamma)]^{-1/2}\sum_{N=\lambda}^{\infty}\frac{\gamma^{N-\lambda}}{(N-\lambda)!}|N\rangle \ , \quad (53)$$

where λ is an integer. The wave function of this eigenstate is easily expressed in the angular representation. It determines the angular probability distribution to be that of von Mises [9].

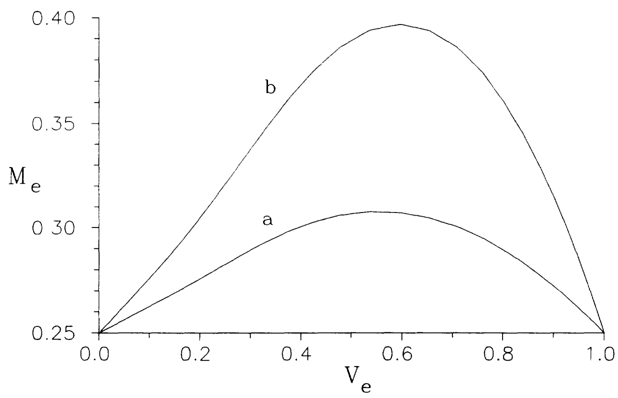

FIG. 1. The uncertainty product $M_e \equiv M_e(\gamma)$ vs the phase dispersion $V_e(\gamma)$ for the $(\hat{N}, i\sin\hat{\phi})$ state (curve *a*) and the $(\hat{N}, -\exp(-i\hat{\phi}))$ state (curve *b*).

The uncertainty product (49) for state (46) and state (53) reads

$$M_e(\gamma)=\left[\frac{\gamma}{2}\frac{I_1(2\gamma)}{I_0(2\gamma)}+\frac{1}{4}\right]\left[1-\left[\frac{I_1(2\gamma)}{I_0(2\gamma)}\right]^2\right] \quad (54)$$

and

$$M_e(\gamma)=\left\{\gamma^2\left[1-\left[\frac{I_1(2\gamma)}{I_0(2\gamma)}\right]^2\right]+\frac{1}{4}\right\}\left[1-\left[\frac{I_1(2\gamma)}{I_0(2\gamma)}\right]^2\right] , \quad (55)$$

respectively. The absence of λ in these formulas is not only convenient for drawing graphs but also a result of an interplay between ΔN in formula (49) and the λ shift in formulas (46) and (53). The product $M_e(\gamma)$ is plotted versus the quantity V_e,

$$V_e \equiv V_e(\gamma)=1-|\langle\exp(i\hat{\phi})\rangle|^2 \ , \quad (56)$$

which is, for both states,

$$V_e(\gamma)=1-\left[\frac{I_1(2\gamma)}{I_0(2\gamma)}\right]^2 , \quad (57)$$

in Fig. 1. In this picture we can see that the former state (46) is of better quality than the latter state (53), not only for the uncertainty relation (47) but also for (48).

IV. UNCERTAINTY RELATIONS IN THE OPTICAL OSCILLATOR

The optical oscillator is described by the Hamiltonian either in terms of the quadrature operators

$$\hat{H}=\tfrac{1}{4}\hbar\omega(\hat{Q}^2+\hat{P}^2) \quad (58)$$

or in terms of the number operator

$$\hat{H}=\hbar\omega(\hat{n}+\tfrac{1}{2}) \ . \quad (59)$$

The steady states of the harmonic oscillator $|n\rangle$ are eigenstates of the operator $\hat{n}$,

$$\hat{n}|n\rangle = n|n\rangle, \langle n|n'\rangle = \delta_{nn'}, n,n'=0,1,2,\ldots . \quad (60)$$

To explain our remark on the analogy between the free plane rotator and the optical anharmonic oscillator, we present the Hamiltonian of the latter system,

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} , \quad (61)$$

where

$$\hat{H}_0 = \hbar\omega(\hat{n} + \tfrac{1}{2}) , \quad \hat{H}_{\text{int}} = \hbar\chi\hat{n}^2 . \quad (62)$$

The interaction representation depends only on $\hat{H}_{\text{int}}$, which clears up the analogy.

In classical physics the mathematics of the system of the optical oscillator is very similar to that of the plane rotator, which leads us nearly to the concept of physical analogy. But in the quantum theory the mathematics of the former system is rather different from the mathematics of the latter system. Although the operator $\hat{n}$ corresponds well to the operator $\hat{N}$ of the plane rotator, the phase of the optical oscillator does not resemble the rotation angle of the plane rotator very well. In fact, the definition of the phase states $|\varphi\rangle$,

$$|\varphi\rangle = \sum_{n=0}^{\infty} \exp(in\varphi)|n\rangle , \quad (63)$$

is as good as possible, but there is no simple relation analogous to relation (37). In [11] it is shown that the requirement of working with a unitary phase operator can be fulfilled algebraically by adopting a type of antinormal ordering of the operators $(\hat{n}+\hat{1})^{-1/2}\hat{a}$, $\hat{a}^{\dagger}(\hat{n}+\hat{1})^{-1/2}$. This leads to the classical quantum correspondence defined in the following way. To each phase function $M(\varphi)$ there corresponds a phase operator,

$$\hat{M} = \frac{1}{2\pi}\int_{\theta}^{\theta+2\pi} M(\varphi)|\varphi\rangle\langle\varphi|d\varphi , \quad (64)$$

which does not depend on θ provided that $M(\varphi)$ is 2π periodic. The choice $M(\varphi)\equiv 1$ results in $\hat{M}=\hat{1}$ and turns (64) into a resolution of the identity [12]. Particularly, using relation (64), we obtain the operators

$$\widehat{\cos}\varphi = \frac{1}{2\pi}\int_0^{2\pi}\cos\varphi|\varphi\rangle\langle\varphi|d\varphi ,$$
$$\widehat{\sin}\varphi = \frac{1}{2\pi}\int_0^{2\pi}\sin\varphi|\varphi\rangle\langle\varphi|d\varphi , \quad (65)$$

which were brought out in [13], and

$$\widehat{\exp}(i\varphi) = \frac{1}{2\pi}\int_0^{2\pi}\exp(i\varphi)|\varphi\rangle\langle\varphi|d\varphi . \quad (66)$$

A part of the mathematics is shared with the plane rotator

$$[\hat{n}, \widehat{\cos}\varphi] = -i\,\widehat{\sin}\varphi , \quad (67)$$

$$[\hat{n}, \widehat{\sin}\varphi] = i\,\widehat{\cos}\varphi , \quad (68)$$

compared to (27) and (34), respectively. The rest of the mathematics is more complicated; the operators $\widehat{\cos}\varphi$ and $\widehat{\sin}\varphi$ do not commute, as they are incompatible. Although the operator $\hat{\phi}_\theta$, fulfilling the relation

$$[\hat{n}, \hat{\phi}_\theta] = i(\hat{1} - |\theta\rangle\langle\theta|) , \quad (69)$$

does exist [11], it only joins the set of mutually incompatible operators defined by relation (64). Starting from the correspondence (65) and regarding the theory of operator ordering, we have derived the quantum analog of the measure of the phase dispersion in the classical theory of statistics. Consistent with the Barnett-Pegg procedure [14], we have found that the quantity V,

$$V = 1 - |\langle\widehat{\exp}(i\varphi)\rangle|^2 , \quad (70)$$

represents a suitable phase dispersion.

In analogy to (48) we can derive the uncertainty relation

$$M \geq \tfrac{1}{4} , \quad (71)$$

where

$$M = [\langle(\Delta\hat{n})^2\rangle + \tfrac{1}{4}][1 - |\langle\widehat{\exp}(i\varphi)\rangle|^2] . \quad (72)$$

We tested this inequality with the aid of the Jackiw state, (see [8] and references therein), which we investigated more thoroughly than it is usual in the literature [15,8]. Jackiw formulated and solved the eigenvalue problem [15]

$$(\hat{n} + i\gamma\,\widehat{\sin}\varphi)|\psi\rangle = \lambda|\psi\rangle . \quad (73)$$

Because the sine operator can be expressed simply as

$$\widehat{\sin}\varphi = \frac{1}{2i}[\widehat{\exp}(i\varphi) - \text{H.c.}] ,$$
$$\widehat{\exp}(i\varphi) = \sum_{n=0}^{\infty}|n\rangle\langle n+1| , \quad (74)$$

we get the solution in the form

$$|\psi\rangle = Z\sum_{n=0}^{\infty} I_{n-\bar{n}}(\gamma)|n\rangle , \quad (75)$$

where $\bar{n}$ is a real number from the interval $(\bar{N}, \bar{N}+1)$ for $\bar{N}$ even or from the interval $(\bar{N}-1, \bar{N})$ for $\bar{N}$ odd and $Z>0$ is the normalization constant; let us compare formula (46) for an analogy. The eigenvalue $\bar{n}$ is to be considered a multivalued function of the parameter γ,

$$\bar{n} = \bar{n}(\gamma) . \quad (76)$$

At the cost of a more complicated denotation, we will consider a set of single-valued functions

$$\bar{n} = \bar{n}(\bar{N}, \gamma) , \quad (77)$$

where $\bar{N}$ is the number of the single-valued branch of the multivalued function (76). This index was chosen so that it is valid,

$$\bar{n}(\bar{N}, \gamma)_{\gamma=0} = \bar{N} , \quad (78)$$

i.e., the eigenvalues of the number operator, namely $0, 1, 2, \ldots, \infty$. The graph of (77) versus γ in Fig. 2 comprises pairs of nearly horizontal line segments, which are on the right-hand side closed by nearly parabolic arcs, starting from the ordinates $\bar{N}$.

Jackiw [15] considered the constraint $\langle\widehat{\sin}\varphi\rangle = 0$ when

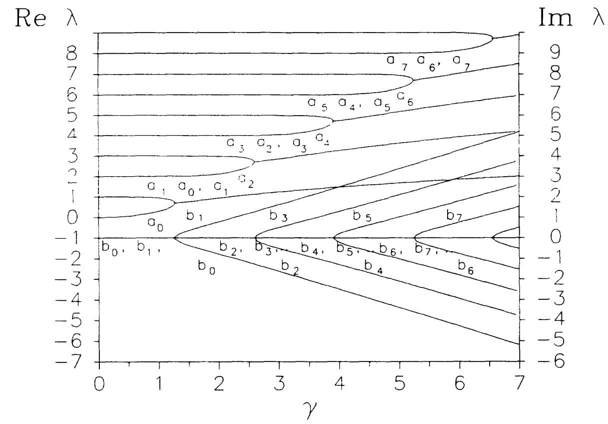

FIG. 2. The eigenvalue of the Jackiw eigenvalue problem in the dependence on γ, $\lambda_j = \mathrm{Re}\lambda_j + i\,\mathrm{Im}\lambda_j$, $j=0,1,2,\ldots$, (real parts, curves a_j; imaginary parts, curves b_j).

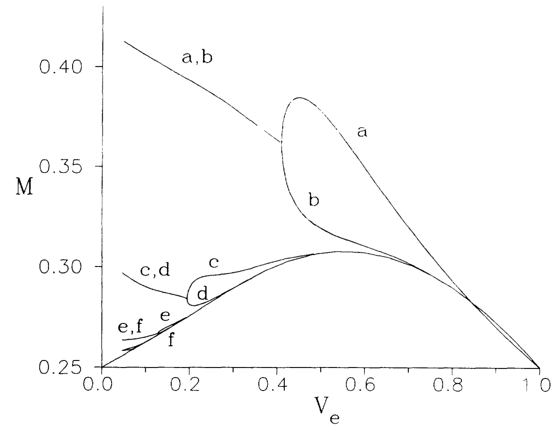

FIG. 3. The uncertainty product $M \equiv M(\overline{N},\gamma)$ vs the phase dispersion $V_e(\gamma)$ for $\overline{N}=0,1,2,3,4,5$ (curves a, b, c, d, e, and f, respectively).

solving (73). Having withdrawn this condition, we tried to specify a complex function [11]

$$\lambda = \lambda(\gamma) , \tag{79}$$

appropriate to a general solution

$$|\psi\rangle = Z \sum_{n=0}^{\infty} I_{n-\lambda}(\gamma)|n\rangle . \tag{80}$$

Introducing the index $\overline{N}$ analogously as in (77), we remove the infinite valuedness of $\lambda(\gamma)$, retaining the paired branches with the same real part. The numerical representation of the function

$$\lambda = \lambda(\overline{N},\gamma) \tag{81}$$

is given in Fig. 2; the dependence of real parts of eigenvalues λ on γ, $\mathrm{Re}\lambda = \overline{n}$, is completed with the dependence of imaginary parts of λ on γ as described above. The neighboring pairs of real eigenvalues converge with an increase of γ and particularly the even eigenvalues approach faster and the nearly parabolic arc appears sooner with the lowest pair. After the eigenvalues meet, complex solutions are considered. For large γ a larger number of pairs of real eigenvalues are replaced by pairs of complex eigenvalues, whereas real large-enough eigenvalues do not differ significantly from the integers, which are the eigenvalues of the number operator.

Substituting state (80) into (72), we get the uncertainty product $M = M(\overline{N},\gamma)$. An interplay between Δn in formula (72) and the λ shift in formula (80) results in a weak dependence of $M(\overline{N},\gamma)$ on $\overline{N}$, but since λ takes on nonintegral values, the dependence on $\overline{N}$ is present. For $\overline{N}$ tending to infinity, $M(\overline{N},\gamma)$ (the optical oscillator) converges to the uncertainty product $M_e(\gamma)$ according to (55) calculated for the state defined by the eigenvalue problem (45) (the plane rotator). The graphical illustration of $M(\overline{N},\gamma)$ versus $V_e(\gamma)$ in Fig. 3, is a bundle of curves with its center at the point (1,0.25) corresponding to the number states ($\gamma=0$). The presence of complex eigenvalues for smaller values of $V_e(\gamma)$ makes itself evident by an identification of the pairs of branches appropriate to $(\overline{N},\overline{N}+1)$, for $\overline{N}$ even and small. The occurrence of real eigenvalues for larger values of $V_e(\gamma)$ manifests itself as a bifurcation and the branching curves approach a limit arc for $\overline{N}$ large, which is the curve a in Fig. 1.

V. CONCLUSION

In this paper we have tried to show that the formally related definitions of states connected with the uncertainty relations, the states about which one may doubt whether they are more than mathematical constructions in the physical system of the optical oscillator, have real physical significance in the system of the plane rotator. The comparison of two types of states in the plane rotator has been realized with the aid of the number-sine-cosine uncertainty product. In the physical system of the optical oscillator, the Jackiw state, the mathematical origin of which was underlined, has been studied in the full extent. Its properties known from the literature have been completed with further convincing details. This paper includes, moreover, some illustrative examples. At least about one of them [$(\hat{n}, i\hat{P})$ state] it is known that the mathematical definition provides the state generated by the reduction of the state, i.e., by a physically interesting method, which kindles a hope for a more frequent occurrence of this connection in the future.

ACKNOWLEDGMENT

One of us (A.L.) thanks Dr. Z. Hradil for numerous discussions concerning the topic of the paper.

[1] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[2] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).
[3] W. Heisenberg, Z. Phys. **33**, 879 (1925).
[4] M. S. Kim, F. A. M. de Oliveira, and P. L. Knight, Opt. Commun. **72**, 99 (1989).
[5] R. Loudon and P. L. Knight, J. Mod. Opt. **34**, 709 (1987).
[6] Z. Hradil, Phys. Rev. A **44**, 792 (1991).
[7] C. Aragone, E. Chalbaud, and S. Salamo, J. Math. Phys.

17, 1963 (1976).
[8] P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).
[9] R. C. Rao, *Statistical Inference and its Applications* (Wiley, New York, 1973), p. 175.
[10] A. Lukš and V. Peřinová, Czech. J. Phys. **41**, 1205 (1991).
[11] A. Lukš, V. Peřinová, and J. Křepelka, Czech, J. Phys. **42**, 59 (1992).
[12] The definition of an operator with the aid of the identity resolution is known from the measurement theory, C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976), p. 57. This prescription is analogous to the definition of an operator for the antinormal ordering by means of functions, G. S. Agarwal and E. Wolf, Phys. Rev. D **2**, 2161 (1970); **2**, 2187 (1970).
[13] L. Susskind and J. Glogower, Physics **1**, 49 (1964).
[14] S. M. Barnett and D. T. Pegg, J. Mod. Opt. **36**, 7 (1989).
[15] R. Jackiw, J. Math. Phys. **9**, 339 (1968).