Phase from Q function via linear amplification

W. Schleich
Abteilung für Quantenphysik, Universität Ulm, Albert-Einstein-Allee 11, D-7900 Ulm, Germany and Max-Planck-Institut für Quantenoptik, D-8046 Garching bei München, Germany

A. Bandilla and H. Paul
Zentralinstitut für Optik und Spektroskopie, Rudower Chaussee 6, O-1199 Berlin-Adlershof, Germany
(Received 26 December 1991)

We relate the phase distribution of a quantum state obtained by integrating its corresponding Q function over radius (1) to a specific measurement scheme using a linear amplifier and (2) to a particular phase operator. A simple relation between the P distribution of the amplified macroscopic state and the Q function of the initial unamplified microscopic state allows us to prove the identity of the phase distributions obtained by integrating the corresponding distributions.

PACS number(s): 42.50.Dv, 42.50.Ar

To find the phase distribution of a quantum state is a nontrivial task. The reason for this is that Hermitian phase operators are rare [1]. However, one approach that is free of any such problems immediately offers itself: Express the Wigner function of this state that is ordinarily given in the (dimensionless) variables, coordinate x and momentum p, in polar coordinates, radius r and angle φ, and integrate over the radius [2]. The resulting distribution $W_\varphi^{(W)}$ is periodic in the "phase angle" φ and, for various examples of states, it satisfies all properties required by a proper phase distribution. What prevents us from subjecting the Glauber P distribution [3,4] or the Q function [5] to this very procedure? Is it the fact that the resulting phase distributions $W_\varphi^{(P)}$ or $W_\varphi^{(Q)}$ are different from the Wigner-function–borne curve $W_\varphi^{(W)}$? Which of the three curves, if any, is the true phase distribution?

In the present article we address these questions and show that the curve $W_\varphi^{(Q)}$ obtained from the Q function by integration over the radius corresponds to a specific experimental scheme [3, 4] for measuring the phase: We amplify the original quantum state depicted in Fig. 1 schematically by a "blob" located in the neighborhood of the origin of phase space by a phase-insensitive linear amplifier to a state whose average number of photons is so large that our conventional notions of phase apply. Hence the Glauber P distribution $P(t;\alpha)$ of the final, that is, the amplified state, is that of a classical state, and integration over the radius shown in the lower right magnifying glass yields the phase distribution $W_\varphi^{(P)}$. We show that this distribution is identical to that obtained by integrating the Q function of the original state over the radius. This corresponds to probing the internal structure of the Glauber P distribution of the original state by coherent states of identical phase aligned along a ray, as indicated by the lower left magnifying glass of Fig. 1. This example also demonstrates most clearly the influence of the measuring device—the amplifier—on the investigated quantum system. This is reminiscent of the idea of Refs. [6] and [7] of probing a quantum state described by the Wigner function with a measuring device in a coherent state. Here we carry this concept over to the Glauber P distribution. We conclude by relating the calculated phase distribution $W_\varphi^{(Q)}$ to a phase operator [4].

Our goal is to connect the phase distribution

$$W_\varphi^{(P)} \equiv \int_0^\infty d|\alpha|\,|\alpha| P(t;|\alpha|e^{i\varphi}) \tag{1}$$

obtained by integrating the *P distribution of the amplified state* over radius $|\alpha|$, to

$$W_\varphi^{(Q)} \equiv \int_0^\infty d|\alpha|\,|\alpha| Q(t_0=0;|\alpha|e^{i\varphi}) , \tag{2}$$

that is, to the curve obtained by integrating the *Q function of the initial, unamplified state.*

The amplification process forces the density matrix $\rho(0)$ of this state, expressed in the coherent-state representation [8]

$$\rho(0)= \int d^2\alpha_0 P(t_0=0;\alpha_0)|\alpha_0\rangle\langle\alpha_0| , \tag{3}$$

to evolve to

$$\rho(t)= \int d^2\alpha\, P(t;\alpha)|\alpha\rangle\langle\alpha| . \tag{4}$$

Here $P(t_0=0;\alpha_0)$ denotes the Glauber P distribution indicated schematically by the blob in Fig. 1 and

$$P(t;\alpha)= \int d^2\alpha_0 P(t;\alpha|t_0=0;\alpha_0)P(t_0=0;\alpha_0) , \tag{5}$$

where $P(t;\alpha|t_0=0;\alpha_0)$ is the conditional quasiprobability.

The relation between $W_\varphi^{(P)}$ and $W_\varphi^{(Q)}$ becomes immediately apparent when we model the linear amplification process by an inverted harmonic oscillator coupled to a heat bath consisting of ordinary harmonic oscillators [9]. In this case the conditioned quasiprobability in an appropriately rotating frame reduces [9,10] to

$$P(t;\alpha|t_0=0;\alpha_0)=\pi^{-1}N(t)^{-1}\exp(-N^{-1}|\alpha-\alpha_0 e^{\kappa t}|^2)\ , \tag{6a}$$

where

$$N(t)=(1+\langle n\rangle)(e^{2\kappa t}-1)\ . \tag{6b}$$

Here κ and $\langle n\rangle$ denote the amplification constant and the mean number of photons of the heat bath, respectively. We now focus on the limit of strong as well as most-quiet amplification, that is, $\kappa t \gg 1$ and $\langle n\rangle=0$. These conditions simplify Eq. (6) to

$$P(t;\alpha|t_0=0;\alpha_0)\cong(\pi e^{2\kappa t})^{-1}\exp(-e^{-2\kappa t}|\alpha-\alpha_0 e^{\kappa t}|^2)\ . \tag{7}$$

When we substitute Eq. (7) into Eq. (5) the Glauber P distribution of the *amplified* state reads

$$P(t;\alpha)\cong(\pi e^{2\kappa t})^{-1}\int d^2\alpha_0 \exp(-|\alpha e^{-\kappa t}-\alpha_0|^2)\times P(t_0=0;\alpha_0)\ . \tag{8}$$

This expression we now compare with and contrast to the

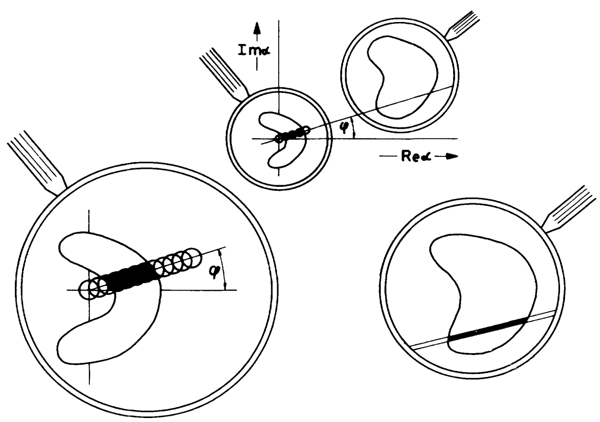

FIG. 1. We amplify a quantum state, depicted here by the "blob"—a pictorial representation of its P distribution. Initially this state rests close to the origin of the Reα-Imα phase space. The quanta of energy fed into the state from the amplifier displace the blob from the origin. Moreover, the amplification process introduces quantum noise, which causes the blob to grow in size, as if to bury all singularities and negativities; that is, all abnormalities the initial P distribution might have had. The P distribution of the amplified state is always positive and classical. Integration of this distribution over the radius, that is, the overlap of this state with the divergent but infinitesimal thin beam, as indicated in the lower right magnifying glass, provides the phase distribution $W_\varphi^{(P)}(\rho(t))$ of the amplified state. To probe the original, unamplified state $\rho(0)$ with coherent states aligned along the ray at angle φ, that is, to calculate the overlap of this P distribution with this finite-sized "one-way street" shown in the lower left magnifying glass, corresponds to the phase distribution $W_\varphi^{(Q)}(\rho(0))$. The latter results from integrating the Q function of the original state over the radius. In the limit of strong but quiet amplification the two areas of overlap in the lower magnifying glasses are identical, that is, $W_\varphi^{(P)}(\rho(t))\cong W_\varphi^{(Q)}(\rho(0))$.

Q function of the initial, *unamplified* state following from Eq. (3),

$$Q(t_0=0;\alpha)\equiv\pi^{-1}\langle\alpha|\rho(0)|\alpha\rangle =\pi^{-1}\int d^2\alpha_0|\langle\alpha|\alpha_0\rangle|^2 P(t_0=0;\alpha_0)\ .$$

With the help of the nonorthogonality relation of the coherent states

$$|\langle\alpha|\alpha_0\rangle|^2=\exp(-|\alpha-\alpha_0|^2)\ ,$$

we arrive at the familiar expression

$$Q(t_0=0;\alpha)=\pi^{-1}\int d^2\alpha_0 \exp(-|\alpha-\alpha_0|^2)P(t_0=0;\alpha_0)\ . \tag{9}$$

Comparison between Eqs. (8) and (9) yields the relation

$$P(t;\alpha)\cong e^{-2\kappa t}Q(t_0=0;\alpha e^{-\kappa t})\ , \tag{10}$$

which is valid in the limit of strong amplification, that is, $\kappa t\gg 1$ and $\langle n\rangle=0$.

We have thus found a rather simple description of the amplification process: We obtain the Glauber P function for the amplified field by first calculating the Q function for the initial field and then displacing this distribution toward higher photon numbers. Hence we can formally divide the amplification process into two steps. The first step accounts, once and for all, for the amplifier noise (note that the Q function is the most strongly smeared-out quasiprobability distribution, compared to Glauber's P function and Wigner's W function), while the second step describes noiseless amplification [11].

When we substitute Eq. (10) into Eq. (1), that is,

$$W_\varphi^{(P)}=\int_0^\infty d|\alpha|\ |\alpha|e^{-2\kappa t}Q(t_0=0;|\alpha|e^{i\varphi}e^{-\kappa t})\ ,$$

and introduce the new integration variable

$$\tilde{\alpha}\equiv|\alpha|e^{-\kappa t}\ ,$$

we arrive at

$$W_\varphi^{(P)}=\int_0^\infty d|\tilde{\alpha}|\ |\tilde{\alpha}|Q(t_0=0,|\tilde{\alpha}|e^{i\varphi})\ .$$

According to Eq. (2), we then find the interesting property

$$W_\varphi^{(P)}=W_\varphi^{(Q)}\ , \tag{11}$$

that is, the phase distribution obtained by integrating the *Q function of the initial state* is identical to that found by integrating the *P distribution of the final state.*

Hence we have connected the abstract idea of obtaining a phase distribution of a quantum state by integrating its Q function over the radius with a specific experiment: We first amplify the quantum state of interest to a classical state and then find its phase-probability distribution. We may also interpret this result in the spirit of the left magnifying glass of Fig. 1: Our measuring device—the linear amplifier—is not an infinitely thin divergent beam, but it has a finite width corresponding to the aligned coherent states.

We are now able to relate the phase distribution $W_\varphi^{(Q)}$

of Eq. (11) to the distribution associated with the operator [4,12]

$$(e^{in\varphi})_{\rm op}\equiv\frac{1}{\pi}\int_{-\pi}^{\pi}d\varphi\, e^{in\varphi}\times\int_0^\infty d|\alpha|\,|\alpha|\,\big|\,|\alpha|e^{i\varphi}\rangle\langle\,|\alpha|e^{i\varphi}|\ . \quad (12)$$

Its expectation value in a quantum state described by a density matrix $\hat\rho=\hat\rho(t)$ reads

$$\langle(e^{in\varphi})_{\rm op}\rangle=\int_{-\pi}^{\pi}d\varphi\, e^{in\varphi}\pi^{-1}\times\int_0^\infty d|\alpha|\,|\alpha|\langle\,|\alpha|e^{i\varphi}|\hat\rho(t)|\,|\alpha|e^{i\varphi}\rangle$$

or

$$\langle(e^{in\varphi})_{\rm op}\rangle=\int_{-\pi}^{\pi}d\varphi\, e^{in\varphi}W_\varphi^{(Q)}\ .$$

Hence

$$W_\varphi^{(Q)}=\pi^{-1}\int_0^\infty d|\alpha|\,|\alpha|\langle\,|\alpha|e^{i\varphi}|\hat\rho(t)|\,|\alpha|e^{i\varphi}\rangle=\int_0^\infty d|\alpha|\,|\alpha|Q(t;|\alpha|e^{i\varphi})\ ,$$

that is, the phase distribution obtained by integrating the Q function of this state over the radius does indeed allow us to calculate the expectation value of the phase operator equation (12).

We conclude by summarizing our main results, shown schematically in Fig. 1. In the limit of strong but quiet amplification, the P distribution of the final amplified state is identical to the Q function of the original unamplified state displaced by the numbers of photons fed in by the amplifier, as expressed by Eq. (10). Consequently, the phase distributions $W_\varphi^{(P)}(\rho(t))$ and $W_\varphi^{(Q)}(\rho(0))$, obtained by integrating the two distributions over the radius, are identical. When we first amplify a quantum state to a macroscopic state and then homodyne it with a classical field, we measure the phase distribution [13], which corresponds to measuring the integrated Q function of the initial quantum state. The phase operator equation (12) describes such a phase measurement.

We can also state these results slightly differently: An experiment that reads out the phase distribution of a quantum state with a coherent-state–phase-space strip of finite width such as a laser state rather than with an infinitely thin phase-space beam, measures the integrated Q function rather than the integrated Wigner distribution.

Two of us (A.B. and H.P.) want to thank H. Walther and the Max-Planck-Institut für Quantenoptik for the hospitality extended to us during a visit that helped to stimulate this work.

[1] See the classic review by P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968); for the suggestion of a novel Hermitian phase operator, see S. M. Barnett and D. T. Pegg, J. Mod. Opt. **36**, 7 (1989); D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665 (1989); for a recent discussion, see J. H. Shapiro and S. R. Shepard, *ibid.* **43**, 3795 (1991).

[2] W. Schleich, R. J. Horowicz, and S. Varro, Phys. Rev. A **40**, 7405 (1989); in *Quantum Optics V*, edited by J. D. Harvey and D. F. Walls (Springer, Heidelberg, 1989); W. Schleich, J. P. Dowling, R. J. Horowicz, and S. Varro, in *New Frontiers in QED and Quantum Optics*, edited by A. Barut (Plenum, New York, 1990).

[3] A. Bandilla and H. Paul, Ann. Phys. (Leipzig) **23**, 323 (1969).

[4] H. Paul, Fortschr. Phys. **22**, 657 (1974).

[5] Phase distribution obtained by integrating the Q distribution of a quantum state has been proposed and most prominently pursued by J. H. Shapiro and S. S. Wagner, IEEE J. Quantum Electron. **QE-20**, 803 (1984) and N. G. Walker and J. E. Carroll, Opt. Quantum Electron. **18**, 355 (1986). For an interesting application to generalized squeezed states, see S. L. Braunstein and C. M. Caves, Phys. Rev. A **42**, 4115 (1990).

[6] S. Stenholm, Eur. J. Phys. **1**, 244 (1980).

[7] K. Wodkiewicz, Phys. Rev. Lett. **52**, 1064 (1984); in *Quantum Electrodynamics and Quantum Optics*, edited by A. Barut (Plenum, New York, 1984); see also S. Braunstein, C. Caves, and G. Milburn, Phys. Rev. A **43**, 1153 (1991).

[8] R. Glauber, in *Quantum Optics and Electronics*, edited by C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, New York, 1965).

[9] R. Glauber, New Techniques and Ideas in Quantum Measurement Theory, edited by D. Greenberger [Ann. N.Y. Acad. Sci. **480**, 115 (1987)].

[10] R. J. Glauber, in *Frontiers in Quantum Optics*, edited by E. R. Pike and S. Sarkar (Hilger, London, 1986).

[11] In the context of the decay of unstable states, F. Haake [Phys. Rev. Lett. **41**, 1685 (1978)] already noted that a linear amplification process is accompanied by a Gaussian white noise, which, using the P distribution, looks like a *deterministic* linear amplification process when the Q function is used. The effects of fluctuations then show up in the initial probability density. However, the above reference does not contain Eq. (10), the relation so central to the discussion of phase distributions.

[12] We emphasize that here we have defined the operator of the nth moment of phase, that is, $(e^{in\varphi})_{\rm op}$ and *not* the nth moment $(e^{i\varphi})^n_{\rm op}$ of the operator $(e^{i\varphi})_{\rm op}=\pi^{-1}\int_{-\pi}^{\pi}d\varphi\, e^{i\varphi}\int_0^\infty d|\alpha|\,|\alpha|\,\big|\,|\alpha|e^{i\varphi}\rangle\langle\,|\alpha|e^{i\varphi}|$. Since coherent states are not orthogonal, the two expressions do not agree, that is, $(e^{in\varphi})_{\rm op}\neq(e^{i\varphi})^n_{\rm op}$.

[13] For a comparison between an infinitely narrow phase state $|\varphi\rangle=(2\pi)^{-1/2}\sum_{m=0}^{\infty}e^{im\varphi}|m\rangle$ and a homodyne detection state

$$|\mathcal{O}_{E(\varphi)}\rangle=(2\pi)^{-1/4}\sum_{m=0}^{\infty}\frac{[\Gamma(2m+1)]^{1/2}}{2^m\Gamma(m+1)}e^{2im\varphi}|2m\rangle\ ,$$

see, for example, W. Vogel and W. Schleich, Phys. Rev. A **44**, 7642 (1991).