Inertia and break of self-organized criticality in sandpile cellular-automata models

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We present a sandpile cellular-automata model that includes the effects of inertia. The model is studied in both one and two dimensions. We find that the model changes the normal self-organized critical behavior, creating a dominance of big events in the system and leading to very large fluctuations in the mass of the system. We show that those changes of behavior can only be noticed in large sandpiles, which is in accord with previous experimental results.

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I. INTRODUCTION

In a series of papers Bak, Tang, and Wiesenfeld [1] introduced the concept of self-organized criticality (SOC). Some extended dynamical systems can evolve into a statistically stationary state where events, i.e., avalanches of all sizes are observed with no spatial and temporal correlations between them. The distribution functions of those events display a power-law behavior. This so-called critical state is an attractor of the dynamics of the system. This discovery suggests a rather general dynamical mechanism for the emergence of scaling and fractal behavior in nature.

The prototypic example of SOC has been the sandpile cellular-automata model [1]. Consider a sandpile, in which grains of sand are added one by one. Topplings will happen when the local slope reaches a critical value. If we begin with a flat surface, initially the grains of sand will simply accumulate on the surface. However, after some time, the sandpile will evolve to a steady state in which its slope fluctuates around an average value. In this statistically stationary state the amount of sand added to the system is balanced, on average, by the amount of sand that falls out of the edge, and the fluctuations observed in the total mass of the sandpile are extremely small relative to its total mass. Once the sandpile is in this state, the addition of a single grain of sand can trigger an avalanche of any size. The distribution functions of the size and duration of those avalanches obey a power law.

In the real world this theory has been successful in explaining the behavior of earthquakes [2]. It provides a simple explanation for the observed Gutenberg-Richter power-law distribution of earthquake sizes. However, in experiments on real sandpiles this behavior was not fully observed [3,4]. At least one experiment tried to reproduce the features of the sandpile model in detail [4]. Single grains of sand were dropped in the center of the pile only after a full relaxation of previous avalanches. The drop-number distribution function was measured. For small sandpiles the drop-distribution functions obeyed the scaling laws observed in the cellular-automata sandpile model. However, as the size of the system grew larger, there was a transition in the behavior of the pile. Big events became dominant and the relaxation of the system was done through those big events, leading to a fluctuation pattern of the total mass which was quite similar to the time histograms observed by Jaeger, Liu, and Nagel [3]. They also observed that the profile of the sandpile, after those big avalanches, became concave.

We argue that the reason for this is probably the effect of inertia and energy dissipation present in real sandpiles. Real sandpiles are driven by gravitational forces. Because of inertia, large avalanches cannot be stopped. If they become large enough, they can only be stopped at the boundary. This would lead to avalanches that include a significant fraction of the size of the sandpiles and would lead to the introduction of a system-size-independent length scale which is the minimal size of those large avalanches.

In this paper we make an attempt to introduce a model to simulate this behavior. We developed a sandpile cellular-automata model in which those effects are taken into account. We observe that when modeling small sandpile models those effects can hardly be seen, leading to the excepted-scaling SOC behavior. However, as the size of the systems grows larger, the behavior changes drastically. The global features of our model as well as its drop-number distribution functions present features that are similar to the observed experimental results.

This paper is organized as follows: in Sec. II we present our model in both one and two dimensions; in Sec. III we organize the results of our simulations and compare them with previous experimental observations. In Sec. IV we summarize the conclusions.

II. THE INERTIA VERSION OF THE SANDPILE MODEL

When we look at a stone rolling down a mountain, it becomes obvious that there is a strong effect of inertia. We can argue that in a sequence of topplings of sand grains something similar also happens. As an avalanche evolves, potential energy is transformed into kinetic energy, the avalanching grains of sand gain momentum, and the whole mass of sand is accelerated. If the process lasts long enough it eventually can only be stopped at the boundary of the system. We introduce this "snowball effect" by making the critical slope Z_c (the threshold value to trigger a single event) a decreasing function of the energy or the momentum, accumulated by a grain of sand during a sequence of topples.

Models in both one and two dimensions driven randomly or deterministically were studied. In one dimension we considered the simplest model that does not present a trivial dynamics. This model was first suggested by Kadanoff *et al.* [5]. In each site of a one-dimensional lattice of size n the height of the pile of sand is given by h_i. We define the slope at site i as

$$z_i = h_i - h_{i-1} \ .$$

The system is perturbed by adding one grain of sand at a randomly chosen site i. Whenever the slope of the sandpile at this site exceeds a threshold value Z_c, an avalanche is triggered. Two grains of sand would topple from site i into its two forward nearest neighbors, more precisely, if

$$z_i \geq Z_c$$

then

$$h_i \rightarrow h_i - 2 \ ,$$

$$h_{i+1} \rightarrow h_{i+1} + 1 \ ,$$

$$h_{i+2} \rightarrow h_{i+2} + 1 \ .$$

The grains of sand that reach the sites $i+1$ and $i+2$ have already toppled once, therefore, it will be a little harder to stop them. We make the critical slope at those sites a little smaller than Z_c, that is

$$Z_{c,i} = Z_c - \alpha n_i$$

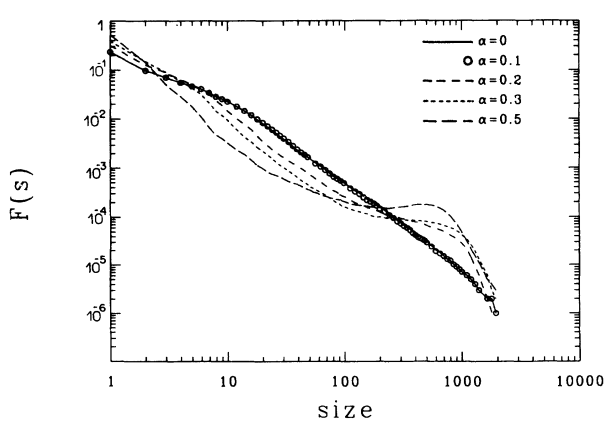

FIG. 1. The size distribution function for different values of α. The size of the system was held constant ($L=100$). All distribution functions were smoothed according to the procedure described in Ref. [6]. The values were averaged over intervals of 10%.

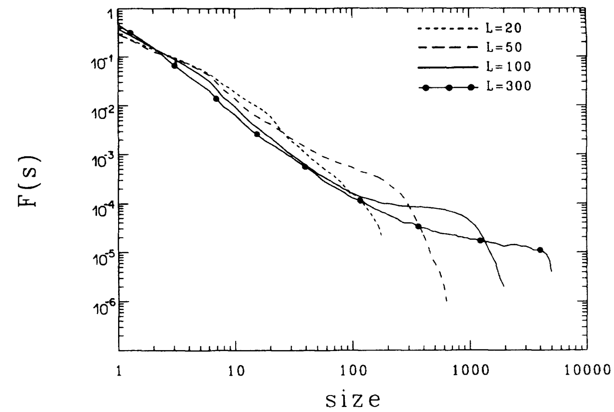

FIG. 2. The size distribution function for different lattice sizes. The value of α was held constant ($\alpha=0.3$). The distribution functions were smoothed as described in Fig. 1.

where n_i is the number of topplings a given grain of sand has done before reaching the site i and $Z_{c,i}$ is the critical value at this site. If new backward falls start we would begin to count the number of topples for those particles again. In any real process the velocity of the falling sand cannot grow forever. So a minimum value Z_0 for Z_c was introduced.

In two dimensions we used the original model of Bak, Tang, and Wiesenfeld [1], with directional flow of the energy. At any toppling the energy is transmitted in equal parts to the fall positions and so on. The critical slope will depend on the energy which is accumulated at a site. We transfer the energy only in the forward direction since that is the direction of the flow in this system.

The energy is conserved; however, if there is no avalanche in a given site all the energy in this position is dissipated. We define an energy matrix $e_{i,j}$ which is updated together with a matrix Z_{ij} that defines the sandpile slopes at each step of the avalanching process. Those rules can be summarized as follows: if

$$Z_{ij} < Z_{c,ij}(e_{ij}) \ ,$$

then $e_{ij}=0$; if

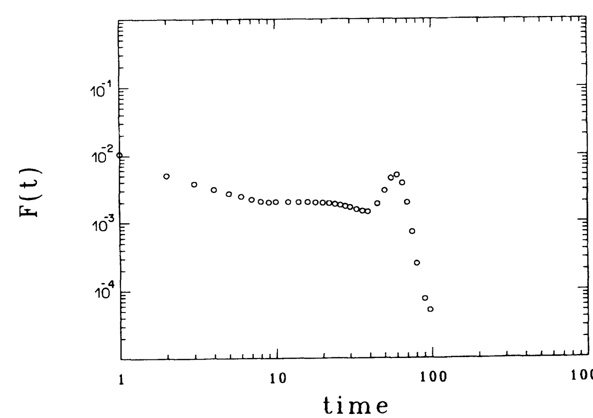

FIG. 3. The number of events as a function of time for the one-dimensional system, $L=50$ and $\alpha=0.5$.

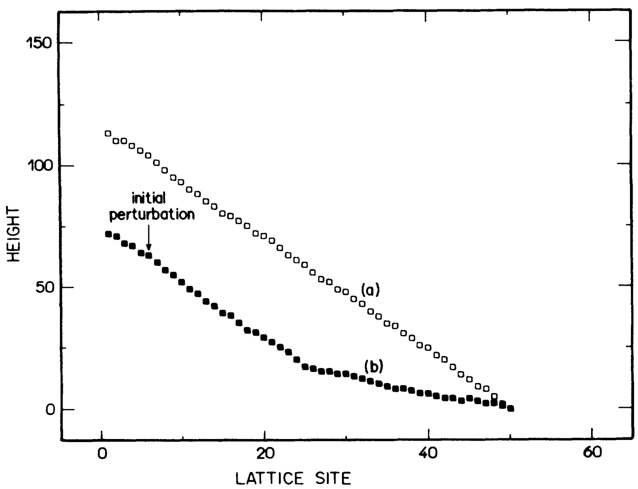

FIG. 4. The profile of the one-dimensional pile with $L=50$ with $\alpha=0.5$ (a) before and (b) after a big avalanche. Notice that after the big avalanche the system is concave.

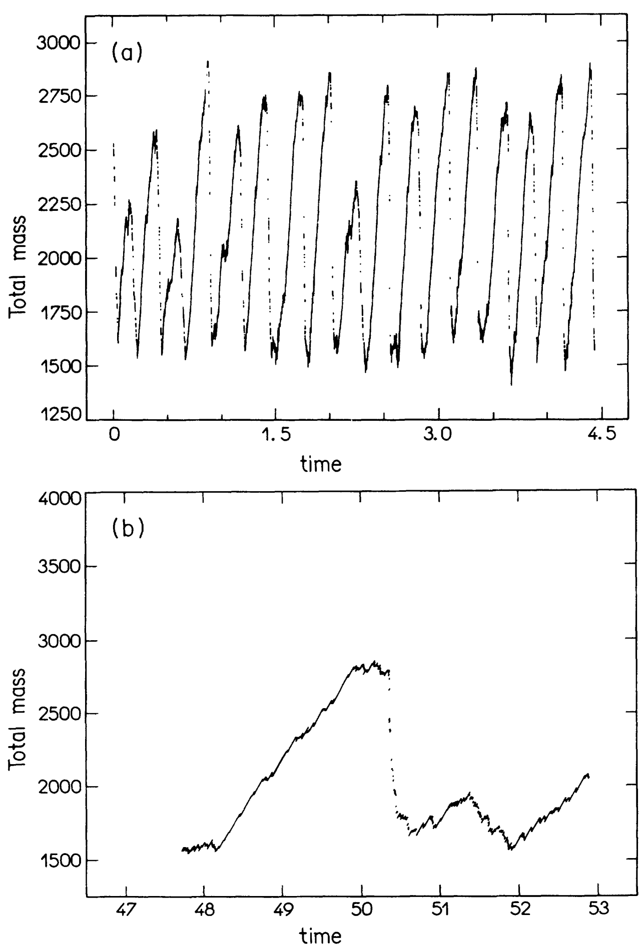

FIG. 5. The fluctuations of the mass of a one-dimensional system as a function of time. Notice the separation between the buildup and the devastation cycles. (b) shows an enlargement of (a). The time is measured according to the number of avalanches. The time parameter in (a) is the number of avalanches, divided by 10^4 in (a) and by 10^3 in (b).

$$Z_{ij} \ge Z_{c,ij}(e_{ij}) ,$$

then

$$Z_{ij}=Z_{ij}-4 ,$$

$$Z_{i\pm 1j}=Z_{i\pm 1j}+1 ,$$

$$Z_{ij\pm 1}=Z_{ij\pm 1}+1 ,$$

$$e_{ij}=0 ,$$

$$e_{i+1j}=e_{ij}/2+1 ,$$

$$e_{ij+1}=e_{ij}/2+1 .$$

To produce the conus morphology of sandpiles, we connected the left boundary of the system to the upper boundary and left the other two sides of the system open. To reproduce the experimental situation we perturbed the system only at the upper left corner. We perturbed the system randomly every 200 steps so that we would not get any periodic trajectories in phase space.

We define

$$Z_{c,ij}(e_{ij})=Z_c-\alpha e_{ij} .$$

Notice that in the two-dimensional case we do not have to choose a limiting value for Z_c. There will always be a limiting value for it. Suppose that during the toppling process a particle gains an energy E each time it topples and that this energy is transferred in equal parts to c neighboring positions. Assuming an infinite sequence of topplings, we see that the accumulated energy will be

$$m_0=E/(c-1)$$

where E is the energy added in each toppling and c is the connectivity of the lattice. The minimum value of the critical slope will be

$$Z_0=z_c-\alpha m_0 .$$

III. RESULTS

We probed the behavior of the system in both one and two dimensions for different lattice sizes and values of the parameters. The main results can be seen in Figs. 1, 2, and 6–8.

In the one-dimensional case we let the system reach a stationary state using the original Kadanoff rules with no inertia. We then analyze the distribution function of avalanche sizes, $F(s)$, for the next 300 000 events. Figure 1 shows $F(s)$ for different values of α. Figure 2 shows how $F(s)$ is affected by changes in the size of the lattice. We see that for small values of α there is no change in the distribution function. $F(s)$ has a clear power-law behavior that lasts for almost two decades. However, as we increase the value of α, we observe an increasing number of large avalanches, and a corresponding decrease in the number of intermediate events. Notice, however, that in all cases we still can see a region of linear behavior with the same scaling exponent. In all cases there is still a (decreasing) region of linear behavior, and a slight decrease in the number of avalanches of intermediate sizes. It is also clear from Fig. 2 that no significant change in behavior can be seen in small lattices. If we hold α constant and vary the size of the lattice there will be a critical size L_c below which there is a power-law behavior. After that point, however, we see a continuous change in $F(s)$, as observed experimentally by Held *et al.* [4] and in exactly the same way we have already described. The critical size L_c depends on α according to the simple law $L_c = A/\alpha$. Notice that this size does not scale with the size of the system as is the usual case in the simple SOC cases. The inertia creates a size-independent scale above which it is very hard to stop an avalanche. This behavior is even clearer if we look at the graph of $F(t)$, the distribution function of the duration of the avalanches (see Fig. 3). Those results are in agreement with experimental results found in Ref. [3], where the absence of short-duration avalanches is probably due to the fact that they were able to register only avalanches that were big enough to cause some particles to fall off the edge of the system.

The interpretation is simple: In small avalanches there is not enough accumulation of inertia (i.e., energy) and there is no visible effect of the inertia. But eventually, if the system is big enough, there will be a big avalanche with the long sequence of topplings. This avalanche can only be stopped at the boundary. It will take more sand out of the system than it otherwise would, causing, as is observed experimentally, a flow of sand in deeper layers of the sandpile. After a set of such big avalanches, the profile of the sandpile looks quite concave, as shown in Fig. 4, where in most of the sandpile the slope is well below the critical value. The system then starts to rebuild. Most of the sand added to it will not cause an avalanche. The mass of the system grown almost constantly in a linear manner. Once rebuilt, it will experience again a sequence of a few big devastating avalanches.

The separation between the buildup and devastation of the pile has profound effects on the global statistics of avalanches. The average slope fluctuates in a very strong manner unlike the average slope in simple SOC systems which is almost uniform. This has a very profound effect on the correlations between events. In such a model the big avalanches are quite correlated. Those effects can be clearly seen in Fig. 5, which describes the mass of the system as a function of time.

For the two-dimensional models, we analyzed the distribution functions for the drop number $F(d)$ and the size of avalanches for 50 000 events. The results are presented in Figs. 6–8.

As α is increased there is a dramatic change in $F(d)$. Even when there is no visible change in the size distribution function (see Fig. 5), $F(d)$ develops a large tail of large drops (see Fig. 7). As α increases further the larger drops become more and more dominant as can also be seen in Fig. 6. At a value $\alpha = 0.8$ a continuous change in $F(s)$ begins. The bigger avalanches become more dominant. However, as in the one-dimensional case one can see that the initial power law is still present and corresponds to the buildup process as we have already stated.

As can be seen in Fig. 8, when the system becomes larger a transition occurs in the form of $F(d)$. As we enlarge the system, $F(d)$ develops a peaked structure at a

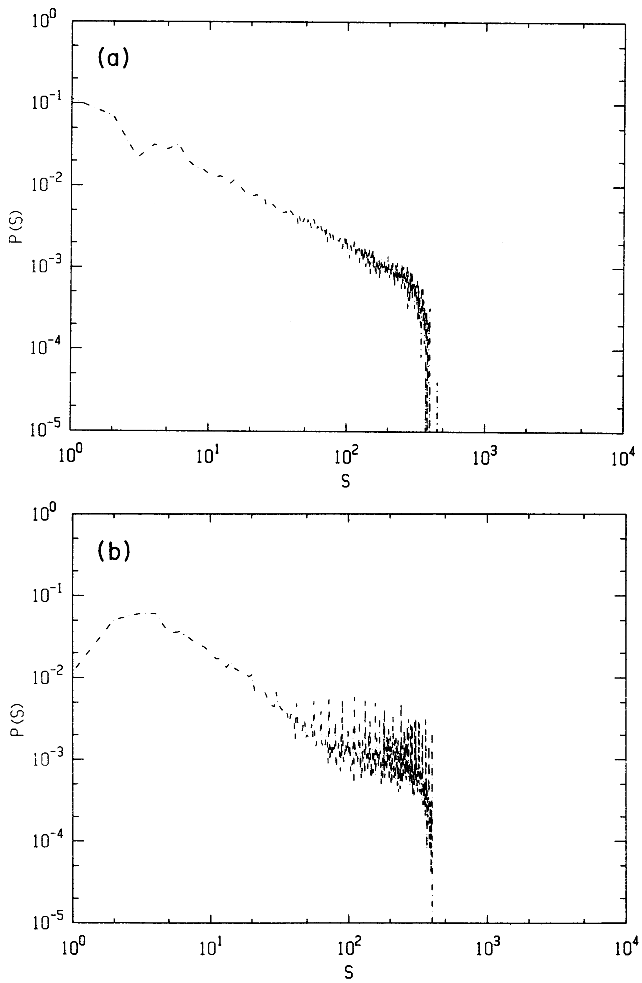

FIG. 6. The size distribution function for the two-dimensional model for a size $L = 20$ and for (a) $\alpha = 0.5$ and (b) 1.0. Notice that for $\alpha = 0.5$ the original power law is still seen.

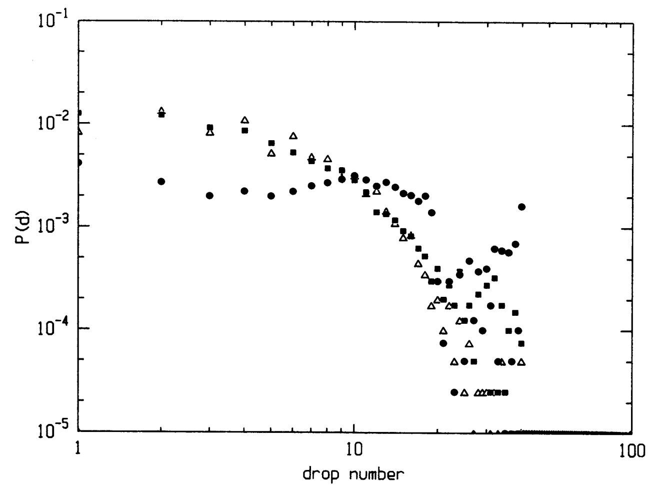

FIG. 7. The drop-distribution function for $\alpha = 0.0$ (open triangles), $\alpha = 0.5$ (closed squares), and $\alpha = 1.0$ (closed circles), where $L = 20$ in all cases.

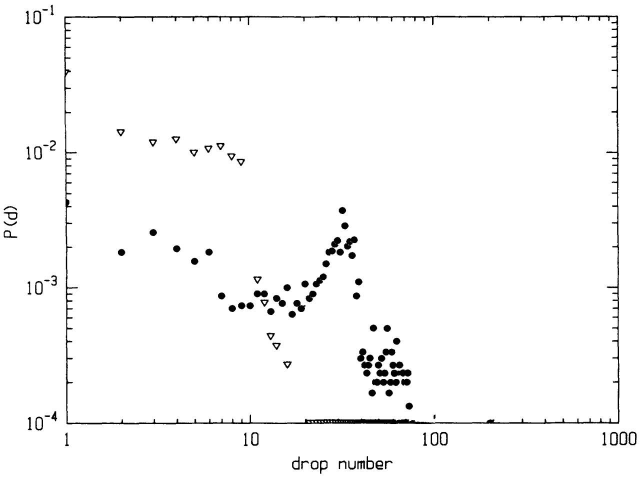

FIG. 8. The drop-distribution functions for different system sizes $L=10$ (open triangles) and 40 (closed triangles). The parameter α was held constant: $\alpha=0.5$.

high drop number (Fig. 8). Notice that a size of about 20 seems to be a critical size for this case ($\alpha=0.5$).

Above this size the effect is well seen in the size distribution function. Actually it is well seen in the drop-distribution function even before it. Our algorithm affects the drop-distribution function even before it destroys the criticality in $F(s)$. Notice that, like in the one-dimensional case, there is a separation between the build-up and the collapse of the system.

All those basic features were experimentally observed in Ref. [4]. When the system becomes larger, events with big drop numbers become dominant. For intermediate sizes a tail of large events is seen. However, we notice that the maximal drop number is the size of the boundary as it is in a simple SOC model, unlike what was observed in the experiments.

IV. CONCLUSIONS

We have shown that most of the features of experiments dealing with sandpiles are reproduced by our model. The size-dependent behavior observed in [4] can be understood as an effect of the accumulation of energy (or momentum) in big avalanches, which results in the introduction of a system-size-independent scale length. In our model the system's mass fluctuates very strongly unlike what will happen in a SOC model, but like what is seen in experiments. This has a very strong effect on the statistics of such systems and on the size of the avalanches. Such models might be proper in describing the behavior of snow, earth, or rock avalanches where after some critical stage the avalanches will go on until the boundary is reached.

The results of our simulations should be seen in a more general context. The scaling results observed for the SOC models are a result of the precise balance between branching and killing probabilities inherent in the cellular-automata model. If this balance is destroyed either by introducing inertia or by some kind of nonconservation, the power-law behavior will be destroyed above some scale which is associated with the relevant laws.

In other words, the power-law behavior observed in the SOC systems is a result of the constant branching probabilities that are present in those models. Our laws change dramatically the probabilities associated with this process. In our model the average number of topplings increases with the size of the avalanche. As the event grows larger its probability to continue will grow larger. So, if the system is big enough, the avalanching process will finally explode and big avalanches will become dominant. Since the system is finite, this process will destroy the system at least partially and this description will not be valid any longer.

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