Pulsating laser oscillations depend on extremely-small-amplitude noise

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Pulsating laser oscillations consist of short and intense pulses separated by long regimes with nearly zero intensities. These oscillations appear if the dimensionless relaxation rate of the inversion of population γ is a small quantity (class-B lasers). During the low-intensity regimes, the laser is particularly sensitive to small-amplitude noise. We investigate the effect of noise by studying the limit-cycle oscillations of the laser with a saturable absorber. Specifically, we analyze how the size of the limit cycle is modified as the amplitude of the noise d is progressively increased from zero. We show that if $\gamma \ll 1$ and $d > d_c = O(\exp(-1/\gamma))$, the size of the limit cycle decreases by an $O(1)$ quantity and depends on d. The mathematical problem is a singular perturbation problem that involves two small parameters (d and γ). We first show that the slow evolution of the limit-cycle solution can be studied as a slow passage through a steady bifurcation point. We then analyze the case of a constant imperfection (small injected signal) and the case of additive noise (small-amplitude Gaussian white noise). Both cases lead to the same conclusions.

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I. INTRODUCTION

Pulsating oscillations in the output of lasers have been observed since the first studies of lasers [1]. They are quite different from oscillatory responses observed in mechanical, electronic, or chemical systems. Typically, the intensity of the laser output is a sequence of short pulses. Between each pulses, the intensity is almost zero. Consequently, lasers are highly sensitive to noise during these intervals of time. Most theoretical and numerical studies on the effects of noise due to spontaneous emission have concentrated on this small intensity regime.

In the absence of noise, approximate solutions of the laser equations can be obtained by using asymptotic methods appropriate for the description of relaxation oscillations. The leading problem describing the small intensity regime is usually a linear first- or second-order equation with slowly varying coefficients. This approximation is best explained by a simple example. Consider the case of the laser rate equations for class-B lasers [1]. They consist of two coupled nonlinear ordinary differential equations for the intensity I and the inversion of population D and are given by

$$\frac{dI}{dt} = 2I(-1 + AD) \tag{1.1}$$

and

$$\frac{dD}{dt} = \gamma[1 - D(1+I)] \ . \tag{1.2}$$

The pump parameter $A > 1$ is the bifurcation parameter. The atomic parameter γ is small for class-B lasers that include CO_2 and semiconductor lasers. For small γ, Eqs. (1.1) and (1.2) admit a slowly decaying oscillatory solution. Each oscillation corresponds to a small intensity regime where D is *slowly* increasing followed by a large amplitude intensity pulse where D is *quickly* decreasing. In the phase plane (I,D), the small intensity regime of the nth oscillation starts at $D = D_n < 1$ and $I = I_n \ll 1$ and finishes as soon as I becomes larger than I_n. Since I is small, we set $I=0$ in Eq. (1.2) and find that the solution of this equation with $D(0) = D_n$ is

$$D(\gamma t) = 1 + (D_n - 1)e^{-\gamma t} \ . \tag{1.3}$$

Substituting then (1.3) into (1.1) leads to a linear first-order equation for I

$$\frac{dI}{dt} = 2f(\gamma t)I, \quad I(0) = I_n \ . \tag{1.4}$$

In (1.4), $f(\gamma t)$ is a slowly varying function defined by

$$f(\gamma t) = -1 + A + A(D_n - 1)e^{-\gamma t} \ . \tag{1.5}$$

Note that $I=0$ is an exact solution of (1.4). Assuming $AD_n < 1$ and $A > 1$, the function $f(\gamma t)$ is negative at $t=0$ and positive as $t \to \infty$. Equation (1.4) describes a *slow passage through a bifurcation point:* the "bifurcation point" $t = t^*$ is defined by the condition $f(\gamma t^*) = 0$ and the "growth rate" $f(\gamma t)$ is slowly increasing from negative to positive values. First- or second-order equations with slowly varying coefficients can be obtained for more complex laser systems, such as the laser with a saturable absorber [2] or the laser-Lorentz equations modeling a cryogenic maser [3]. The approximation is again based on the observation of a slowly varying regime corresponding to small-amplitude intensities. For the laser rate equations (1.1) and (1.2), the effect of noise has been modeled by adding a noise-source term in the right-hand side of Eq. (1.4). It is expected that the new equation is a physically valid approximation of quantum noise if the amplitude is sufficiently small. In the case of the laser rate equations, this approximation was first proposed by Morozov [4] [in his paper, the function $f(\gamma t)$ has been further simplified by expanding in Taylor series the exponential function and keeping the linear term in γt]. The problem has been reexamined recently by Balle, Colet, and San Miguel [5] and Valle, Pesquera, and Rodriguez [6]. For more complicated laser systems, the effects of noise have been investigated numerically [3,7]. These computations have underlined the mathematical (as well as numerical) difficulties to obtain accurate quantitative information.

On the other hand, slow passage through bifurcation or limit points have been studied independently to model experimental studies of bifurcation transitions (see Refs. [8–14] for studies in the area of lasers). Simple linear and nonlinear first- and second-order equations have been investigated in detail. In the case of a steady bifurcation described by an equation of the form of Eq. (1.4), detailed results have been obtained with additive and multiplicative noise [15–18]. The main purpose of this paper is to show that these bifurcation studies are useful for the asymptotic description of the laser pulsating oscillations. Specifically, we shall consider a laser problem exhibiting limit-cycle oscillations and investigate the effect of small-amplitude noise both analytically and numerically. In particular, we are interested to determine the critical amplitude of the noise d_c above which the oscillations depend dramatically on noise. As we shall demonstrate, d_c is an exponentially small quantity compared to γ, defined as the rate of change of the small intensity regime.

The plan of the paper is as follows. Section II is devoted to the formulation of the laser equations. Section III gives the asymptotic description of the limit-cycle solution. Sections IV and V consider the case of a constant imperfection and the case of additive noise, respectively. Section VI compares the results obtained for the case of noise and the results of a numerical study of the complete laser equations with Gaussian white noise.

II. FORMULATION

The laser containing an intracavity saturable absorber has been studied since the early days of laser theory and is known to produce sustained oscillations of the laser output intensity [1] called passive Q switch (PQS). The physical mechanism leading to these oscillations can be described by simple rate equations and the bifurcation diagram of the periodic solutions has been analyzed both analytically and numerically [19]. More recently, combined numerical and experimental studies of the laser with a saturable absorber (LSA) have led to the description of other forms of PQS as well as irregular pulsating regimes and have contributed to the formulation of better models (see Zambon [20] for recent references).

The simplest model of the LSA considers the active and passive media of the LSA as two homogeneously broadened (two-level) atomic systems. The transition frequencies are assumed equal and perfectly tuned with one cavity mode of frequency ω. In the rate-equation approximation and in terms of dimensionless quantities, the response of the LSA is described by the following equations for the normalized field intensity I and the atomic inversion densities D ($\overline{D}$) of the active (passive) atoms,

$$\frac{dI}{dt}=2I(-1+AD+\overline{A}\overline{D})\ , \tag{2.1}$$

$$\frac{dD}{dt}=\gamma[1-D(1+I)]\ , \tag{2.2}$$

$$\frac{d\overline{D}}{dt}=\overline{\gamma}[1-\overline{D}(1+aI)]\ . \tag{2.3}$$

All quantities with overbars refer to the absorber. γ and $\overline{\gamma}$ are the longitudinal decay constants normalized by the field decay rate in the empty cavity. The parameter a is the relative saturability of the absorber with respect to the active medium. We are interested in the case $a>1$ in which the absorber can be saturated more easily than the amplifier. The quantities $A>0$ and $\overline{A}<0$ represent the pump parameters for the lasing medium and the absorber. In general $\overline{A}$ is fixed and A is used as the control parameter. In Ref. [21], CO_2 lasers were studied with different gaseous absorbers. It appears that γ and $\overline{\gamma}$ are $O(10^{-3})$ small quantities and motivates an investigation of the LSA equations in the limit $\gamma=O(\overline{\gamma})\rightarrow 0$. In this limit, limit-cycle oscillations are possible if [2]

$$a>1,\quad \overline{A}<-(a-1)^{-1}\ . \tag{2.4}$$

Moreover, it is known from detailed bifurcation studies that the oscillations become pulsating as

$$\alpha=A-A^* \tag{2.5}$$

approaches zero. A^* is defined by

$$A^*=1+\overline{A}\ . \tag{2.6}$$

As $\alpha\rightarrow 0^+$, the period of the oscillations increases and is infinite at $\alpha=0$. The closed orbit in the phase plane is then homoclinic and connects a saddle point located at

$$(I,D,\overline{D})=(0,1,1)\ . \tag{2.7}$$

As $\gamma=O(\overline{\gamma})\rightarrow 0$, we may construct the limit cycle by using singular perturbation techniques [2]. The analysis given in [2] considers the case $\gamma=\overline{\gamma}$ but the method can be applied to the case $\gamma\neq\overline{\gamma}$. Since we need part of this analysis, we briefly summarize the main results.

III. RELAXATION OSCILLATIONS

We propose to construct the periodic solution of the LSA equations by using an asymptotic method valid in the limit

$$\gamma\rightarrow 0\quad\text{and}\quad \overline{\gamma}=\gamma\rightarrow 0\ . \tag{3.1}$$

Figure 1 shows the limit cycle in the phase plane (I,D). The numerical solution indicates that the oscillations correspond to a periodic sequence of slow "relaxation" phases (depending on γt) where $I\ll 1$ alternating with fast, almost discontinuous, pulses (depending on t) where $I\gg 1$. This suggests to use singular perturbation techniques which are appropriate to describe relaxation oscillations. The solution is well approximated by combining two distinct solutions: (1) the slowly varying regime and (2) the quick and intense pulse.

A. The slowly varying regime

This solution corresponds to $I \ll 1$ and $D,\bar{D}$ slowly increasing. Because the differential equations are autonomous, we may choose the time origin wherever it is convenient. We define $t=0$ as the time where the solution jumps down from $I \gg 1$ to $I=O(\gamma)$ and introduce the following initial conditions:

$$D(0)=D_-,\quad \bar{D}(0)=\bar{D}_-,\quad I(0)=I_- , \tag{3.2}$$

where $D_-(\gamma)=O(1)$, $\bar{D}_-(\gamma)=O(1)$, and $I_-(\gamma)=O(\gamma)$ are unknown and must be determined from our analysis. By using a multiscale perturbation method, we have found that the small intensity and slowly varying solution is given in first approximation by

$$D=1+Be^{-\tau},\quad \bar{D}=1+\bar{B}e^{-\tau},\quad I=I_-\exp[2\gamma^{-1}F(\tau)] , \tag{3.3}$$

where the slow time τ nd the function $F(\tau)$ are defined by

$$\tau=\gamma t,\quad F(\tau)=\alpha\tau-(AB+\bar{A}\bar{B})(e^{-\tau}-1) . \tag{3.4}$$

$B,\bar{B}$ are two constants of integration. Using (3.2), we can relate B and $\bar{B}$ to the initial values D_- and $\bar{D}_-$,

$$D_-=1+B \quad\text{and}\quad \bar{D}_-=1+\bar{B} . \tag{3.5}$$

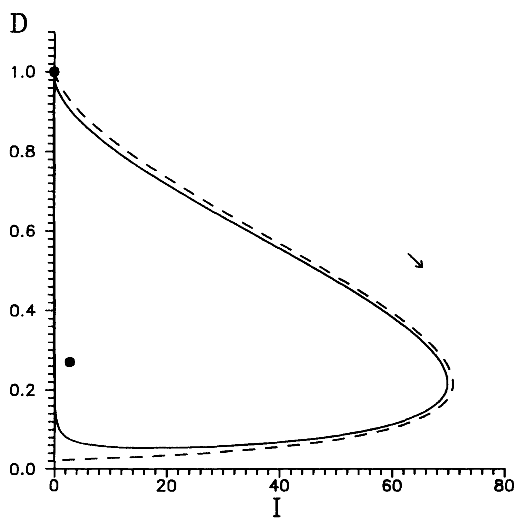

FIG. 1. The limit cycle of the LSA equations. The limit cycle is shown in the phase plane (I,D) and corresponds to a small intensity regime from $D=D_-$ to $D=D_+$ followed by a rapid pulse with D changing from $D=D_+$ to $D=D_-$. The values of the parameters are $\bar{A}=-3.4375$, $A=1-\bar{A}+\alpha$, $\alpha=0.2$, $a=5$, and $\gamma=0.04$. The solid line represents the exact numerical solution. The dashed line corresponds to the asymptotic approximation given in Sec. III. This approximation is given by (3.9) and (3.10) with $D_+=\bar{D}_+=1$ and W changing from 0 to W_0. W_0 is the root of Eq. (3.13) with $D_+=\bar{D}_+=1$. Note that this approximation does not correspond to the leading asymptotic expression as $\alpha\to 0$: we use the exact value of A for both (3.9) and (3.13) instead of $A=1-\bar{A}$. This approximation leads to a better agreement with the exact numerical solution. The two dots in the figure corresponds to the two singular points.

As τ increases from zero, we note from (3.3) and (3.4) that I is an $O(\exp(-\gamma^{-1}))$ small quantity until $\tau\to\tau_c$, where τ_c is defined as the nonzero root of $F(\tau)=0$ or, equivalently,

$$\alpha\tau_c-(AB+\bar{A}\bar{B})[\exp(-\tau_c)-1]=0 . \tag{3.6}$$

At $\tau=\tau_c$, D and $\bar{D}$ take the following values:

$$D\approx D_+=1+B\exp(-\tau_c),\quad \bar{D}\approx\bar{D}_+=1+\bar{B}\exp(-\tau_c) . \tag{3.7}$$

B. The quick pulse

The small intensity regime corresponds to the slow increase from $D=D_-$ to D_+ $(0\le\tau\le\tau_c)$. It is followed by a rapid change of both D and I and is characterized by the quick decrease from $D=D_+$ to $D=D_-$. We introduce a new time variable T defined by

$$T=t-\gamma^{-1}\tau_c-t_0 , \tag{3.8}$$

where $t_0=O(1)$ has been introduced to allow for the possibility of a shift of the transition layer. We then seek a solution of the form $I=\gamma^{-1}J=O(\gamma^{-1})$, $D=O(1)$), $\bar{D}=O(1)$. We find that the leading approximation which satisfies the matching conditions as $T\to-\infty$ (i.e., $J\to 0$, $D\to D_+$, and $D\to\bar{D}_+$) is given by

$$I=\gamma^{-1}J =\gamma^{-1}2\left[\ln(W)-AD_+(W-1)-\frac{\bar{A}}{a}\bar{D}_+(W^a-1)\right] , \tag{3.9}$$

$$D=D_+W , \tag{3.10}$$

and

$$\bar{D}=\bar{D}_+W^a , \tag{3.11}$$

where W is defined as the integral

$$W=\exp\left[-\int_{-\infty}^{T} J(s)ds\right] . \tag{3.12}$$

The orbit is completed as $T\to\infty$. In this limit, $J\to 0$ and $W\to W_0$. From (3.9) we obtain $W=W_0$ as the nonzero root of

$$\ln(W_0)-AD_+(W_0-1)-\frac{\bar{A}}{a}\bar{D}_+(W_0^a-1)=0 . \tag{3.13}$$

Knowing W_0, we may then determine D and $\bar{D}$ using (3.10) and (3.11)

$$D=D_+W_0 \quad\text{and}\quad \bar{D}=\bar{D}_+W_0^a . \tag{3.14}$$

We have completed the periodic orbit if the values (3.14) are matching the initial values $D=D_-$ and $D=\bar{D}_-$. We now analyze these conditions in the limit $\alpha\to 0$.

C. Matching as $\alpha\to 0$

The limit $\alpha\to 0$ corresponds to periodic solutions close to the homoclinic solution or infinite period solution. We first note from (3.6) and (3.7) that

$$\tau_c \to -\alpha^{-1}[(1-\overline{A})B+\overline{A}\,\overline{B}] , \quad (3.15)$$

$$D_+ \to 1 \text{ and } \overline{D}_+ \to 1 \text{ as } \alpha \to 0 . \quad (3.16)$$

We then determine W_0 from (3.13) as the nonzero root of

$$\ln(W_0)-(1-\overline{A})(W_0-1)-\frac{\overline{A}}{a}(W_0^a-1)=0 . \quad (3.17)$$

Using (3.14), (3.16), and the matching condition $D=D_-$ and $D=\overline{D}_-$ as $T\to\infty$, we obtain

$$D_- = W_0 \text{ and } \overline{D}_- = W_0^a . \quad (3.18)$$

The remaining unknowns are B and $\overline{B}$ that can now be obtained using (3.5) and (3.18). We have

$$B=W_0-1 \text{ and } \overline{B}=W_0^a-1 . \quad (3.19)$$

With these values of $\overline{B}$ and $\overline{B}$, we may determine the value of τ_c using (3.6).

In summary, we have shown that the matching conditions are satisfied as $\alpha\to 0$. Using the values of the parameters given in Fig. 1, we obtain

$$W_0 \approx 0.026, \quad B\approx -0.974, \quad \overline{B}\approx -1, \quad \tau_c \approx 4.4 . \quad (3.20)$$

Using (3.5), (3.7), and (3.20), we also determine the minimum and maximum values of D

$$D_- \approx 0.026 \text{ and } D_+ \approx 0.99 . \quad (3.21)$$

IV. CONSTANT IMPERFECTION

In this section we investigate the effect of a constant and (extremely) small imperfection. We model this imperfection by introducing an additive term in the equation of the electrical field E. Instead of Eqs. (2.1)–(2.3), we now consider

$$\frac{dE}{dt}=E(-1+AD+\overline{A}\overline{D})+d , \quad (4.1)$$

$$\frac{dD}{dt}=\gamma[1-D(1+E^2)] , \quad (4.2)$$

$$\frac{d\overline{D}}{dt}=\gamma[1-\overline{D}(1+aE^2)] . \quad (4.3)$$

The additional term in (4.1) corresponds, for example, to a small-amplitude injected signal. Figure 2 shows the main effect. The solid and dashed lines correspond to $d=0$ and 0.001, respectively. We note that a small value of d is sufficient to decrease the size of the limit cycle by an almost $O(1)$ quantity. This effect results from the fact that the small-d perturbation is particularly important in the small intensity regime.

A. Asymptotic analysis $\gamma\to 0$

Applying the asymptotic analysis given in Sec. III, we find that the slowly varying regime of the oscillations is now given by

$$D=1+Be^{-\tau}, \quad \overline{D}=1+\overline{B}e^{-\tau} , \quad (4.4)$$

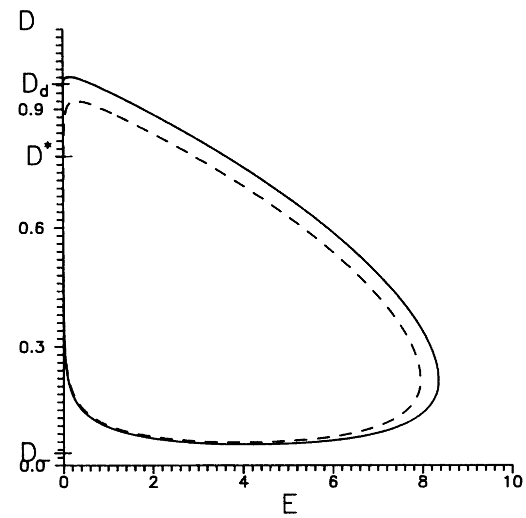

FIG. 2. Effect of a constant imperfection. The solid and dashed lines correspond to the solution of the LSA equations with $d=0$ and 0.001, respectively. The values of the other parameters are the same as in Fig. 1. The points D_-, D^*, and D_d correspond to the starting point, the "bifurcation point," and the final point of the small intensity regime, respectively. They have been computed using the limit α small and are given in Secs. III and IV.

and

$$E=E_-e^{\gamma^{-1}F(\tau)}+d\gamma^{-1}e^{\gamma^{-1}F(\tau)}\int_0^{\tau}e^{-\gamma^{-1}F(s)}ds , \quad (4.5)$$

where $E_-=I_-^{1/2}$ is unknown and $F(\tau)$ is defined by (3.4). We now analyze the integral in (4.5) in the limit $\gamma\to 0$. The function $F(\tau)$ has a local maximum at $\tau=\tau^*$ that satisfies the condition $F'(\tau)=0$ or

$$\alpha+(AB+\overline{A}\overline{B})\exp(-\tau^*)=0 . \quad (4.6)$$

From (4.6), we find that τ^* is given by

$$\tau^*=-\ln(\alpha)+\ln[-(AB+\overline{A}\overline{B})] . \quad (4.7)$$

We then approximate the integral in (4.5) using Laplace's method and obtain

$$E\approx E_-e^{\gamma^{-1}F(\tau)} + d\gamma^{-1}\left[\frac{2\gamma\pi}{\alpha}\right]^{1/2} e^{\gamma^{-1}[F(\tau)-F(\tau^*)]} \quad (\tau>\tau^*) . \quad (4.8)$$

The expression (4.7) is a linear combination of two exponentials. The first exponential becomes large as soon as $\tau>\tau_c$ [$\tau=\tau_c$ satisfies $F(\tau_c)=0$ or Eq. (3.6)]. However, the second exponential may become large as $\tau>\tau_d$, where $\tau_d\geq\tau^*$ must satisfy the condition $F(\tau_d)-F(\tau^*)=0$. Using (3.4) and (4.6), this condition implies

$$\alpha(\tau_d-\tau^*-1)-(AB+\overline{A}\overline{B})\exp(-\tau_d)=0 . \quad (4.9)$$

We introduce u defined as

$$u=\tau_d-\tau^* \quad (4.10)$$

into (4.9), use (4.6) and find the following equation for u:

$$u-1+e^{-u}=0 . \quad (4.11)$$

An analysis of Eq. (4.11) shows that the only root is $u=0$. Thus, we conclude that

$$\tau_d=\tau^* . \quad (4.12)$$

Thus, τ_d is independent of d and the size of the limit cycle has been reduced dramatically because $\tau^*<\tau_c$. However, if d is *exponentially small* [i.e., $d=0(\exp(-\gamma^{-1}))$], τ_d becomes a function of d. We investigate this possibility by first introducing the new parameter $k=O(1)$ defined as

$$d=\gamma^{1/2}\exp(-k\gamma^{-1}) . \quad (4.13)$$

After inserting (4.13) into (4.8), we find that the second exponential becomes exponentially large as $\tau>\tau_d(k)$, where $\tau_d(k)$ now satisfies the condition $-k+F(\tau_d)-F(\tau^*)=0$ or, equivalently,

$$\alpha(\tau_d-\tau^*-1)-(AB+\overline{A}\,\overline{B})\exp(-\tau_d)=k . \quad (4.14)$$

Using (4.10), we now find that u satisfies the following equation:

$$u-1+e^{-u}=k\alpha^{-1} . \quad (4.15)$$

This equation admits two roots but only the positive root is valid because of the condition $\tau\geq\tau^*$ in (4.8).

B. Behavior as $\alpha\to0$

We now analyze our results in the limit $\alpha\to0$. From (4.7), we note that

$$\tau^*\to-\ln(\alpha)+\ln\{-[(1-\overline{A})B+\overline{A}\,\overline{B}]\} \quad \text{as } \alpha\to0 , \quad (4.16)$$

where B and $\overline{B}$ are constants given by (3.19). Note that $\tau^*=O(-\ln(\alpha))$ is clearly smaller than $\tau_c=O(\alpha^{-1})$. Solving (4.15) and using (4.10), we find that $\tau_d(k)$ has the following limit:

$$\begin{aligned}\tau_d(k)&\to\tau^*+k\alpha^{-1}+1\\ &\to k\alpha^{-1}-\ln(\alpha)+1\\ &\quad+\ln\{-[(1-\overline{A})B+\overline{A}\,\overline{B}]\}\\ &\text{as } \alpha\to0 . \end{aligned}\quad (4.17)$$

We can now predict when the intensity of the laser field will grow exponentially. There are two possibilities depending on the values of τ_c and $\tau_d(k)$. *Case (1):* if $\tau_d(k)<\tau_c$, the large intensity pulse appears as soon as $\tau>\tau_d(k)$. *Case (2):* if $\tau_d(k)>\tau_c$, the intensity grows exponentially as soon as $\tau>\tau_c$. Note that τ_c is independent of k (and thus independent of the imperfection d). If $d\to0$, $k\to\infty$, and $\tau_d(k)\to\infty$, then case (2) will dominate. On the other hand, if $d=O[\exp(-\gamma^{-1})]$ is progressively increased, k and $\tau_d(k)$ will decrease and case (1) will dominate. Because the period and the amplitude of the oscillations are in first approximation proportional to the time interval of the slowly varying regime [i.e., either τ_c or $\tau_d(k)$], the size of the limit cycle will be a function of the imperfection in case (1). Experiments with progressively larger values of d should show a dramatic transition from case (2) to case (1) in the vicinity of $d=d_c$. The critical value $d=d_c$, or equivalently, $k=k_c$, is defined by the condition

$$\tau_d(k_c)=\tau_c . \quad (4.18)$$

If $\alpha\to0$, we use (3.15) and (4.17) and obtain

$$k_c=-[(1-\overline{A})B+\overline{A}\,\overline{B}]+\alpha\ln(\alpha)+O(\alpha) . \quad (4.19)$$

In (4.19), B and $\overline{B}$ are functions of the constant W_0 and are given by (3.19).

We have evaluated numerically the approximations of τ^*, $D^*=D(\tau^*)$, τ_d, $D_d=D(\tau_d)$, and k using the values of the parameters of Fig. 2. We obtain

$$\begin{aligned}&\tau^*=1.49, \quad D^*=0.78, \quad \tau_d=3.54 ,\\ &D_d=0.97, \quad k=0.21 .\end{aligned}\quad (4.20)$$

The points D^* and D_d as well as $D_-\approx0.026$ are shown in Fig. 2. Finally, we have determined the value of k_c given by (4.19) and found

$$k_c\approx0.56 . \quad (4.21)$$

V. NOISE

We now consider the effect of noise and analyze the following Langevin equation with an additive noise term:

$$\frac{dE}{dt}=E(-1+AD+\overline{A}\overline{D})+\xi(t) , \quad (5.1)$$

where $\xi(t)$ is a Gaussian white-noise source satisfying the properties

$$\langle\xi(t)\rangle=0 \quad (5.2)$$

and

$$\langle\xi(t)\xi(t')\rangle=2\epsilon\delta(t-t') . \quad (5.3)$$

Solving the differential equation (5.1) with the initial condition $E(0)=E_-$ leads to

$$\begin{aligned}E(t)=&E_-\exp[\gamma^{-1}F(\gamma t)]\\ &+\exp[\gamma^{-1}F(\gamma t)]\int_0^t\exp[-\gamma^{-1}F(\gamma s)]\xi(s)ds ,\end{aligned}\quad (5.4)$$

where $F(\gamma t)$ is defined in (3.4). E_- is a random number which we do not know but, as in Secs. III and IV, its value is not required to predict when E will increase exponentially. Because of (5.2), $\langle E(t)\rangle$ does not depend on $\xi(t)$. We determine $\langle E^2(t)\rangle$ by taking the square of (5.4) and averaging. After using (5.2), we obtain a sum of two terms

$$\langle E^2(t)\rangle = E_-^2 \exp[2\gamma^{-1}F(\gamma t)] + \exp[2\gamma^{-1}F(\gamma t)]\int_0^t\int_0^t \exp[-\gamma^{-1}F(\gamma t')]\exp[-\gamma^{-1}F(\gamma t'')]\langle \xi(t')\xi(t'')\rangle dt'dt'' \ . \tag{5.5}$$

The second term has been written as a double integral. We now use (5.3) and introduce the preexponential factor in the integral

$$\langle E^2(t)\rangle = E_-^2 \exp[2\gamma^{-1}f(\gamma t)] + \int_0^t\int_0^t \exp\{\gamma^{-1}[F(\gamma t)-F(\gamma t')]\}\exp\{-\gamma^{-1}[F(\gamma t)-F(\gamma t'')]\}2\epsilon\delta(t'-t'')dt'dt'' \ . \tag{5.6}$$

Using the property of the δ function, we obtain

$$\langle E^2(t)\rangle = E_-^2 \exp[2\gamma^{-1}F(\gamma t)] + 2\epsilon \exp[2\gamma^{-1}F(\gamma t)] \times \int_0^t \exp[-2\gamma^{-1}F(\gamma t')]dt' \ . \tag{5.7}$$

Note that (5.7) has the same form as (4.5), the solution of the problem with constant imperfection. The asymptotic analysis of Eq. (5.7) for $\gamma \to 0$ is thus similar and leads to the approximation

$$\langle E^2(\tau)\rangle = E_-^2 \exp[2\gamma^{-1}F(\tau)] + 2\epsilon \left[\frac{\pi}{\gamma\alpha}\right]^{1/2} \exp\{2\gamma^{-1}[F(\tau)-F(\tau^*)]\} \quad (\tau > \tau^*) \ , \tag{5.8}$$

which is equivalent to (4.8). Introducing the parameter $k = O(1)$ now redefined as

$$2\epsilon = \gamma^{1/2}\exp(-2\gamma^{-1}k) \ , \tag{5.9}$$

we find from (5.8) that $\langle E^2(\tau)\rangle$ grows exponentially as $\tau > \tau_d$, where τ_d satisfies the condition (4.14). We now assume that the large intensity pulse is not modified by the exponentially small noise. The analysis given in Sec. III remains valid and gives $D_+(n)$ and $D_-(n+1)$ using (3.7) and (3.14) (n denotes the nth oscillation).

VI. NUMERICAL RESULTS

In Sec. V we have shown that the effect of small-amplitude noise can be analyzed by studying a specific slow-passage problem. This problem is described by a linear first-order equation with a noise-source term. In addition, we have obtained a specific relation between the amplitude of the noise and the size of the limit cycle [i.e., condition (5.9)]. The purpose of this section is to verify these two predictions. Specifically, we have considered the full LSA equations with an additive noise-source term in the equation of the electrical field written as

$$\xi(t) = 10^{-\kappa}g(t) \ . \tag{6.1}$$

Two random variables uniformly distributed over the interval (0,1) and the Box-Muller method have been used to generate $g(t)$, a random Gaussian distribution with zero mean and variance equal to one. The amplitude of $\xi(t)$ is controlled by changing κ. Using (5.3), we find that

$$2\epsilon = 10^{-2\kappa} \ . \tag{6.2}$$

Using (5.9), we then obtain a relation between k and κ,

$$-2\gamma^{-1}k = \ln(2\epsilon\gamma^{-1/2}) = \ln(\gamma^{-1/2}10^{-2\kappa}) \tag{6.3}$$

or, equivalently,

$$k \to \gamma\kappa \ln(10) \quad \text{as } \kappa \to \infty \quad (\gamma \text{ fixed}) \ . \tag{6.4}$$

We have analyzed the validity of the asymptotic result (4.14) which relates k and τ_d, defined as the critical time for the pulse. We analyzed numerically the quantity

$$L = \alpha(\tau_j - \tau^* - 1) - (AB + \overline{A}\,\overline{B})\exp(-\tau_j) \ , \tag{6.5}$$

where τ_j is defined as the time for the limit cycle to follow the slowly varying intensity regime from $E \le 0.01$ to $E \ge 0.01$. This quantity has been evaluated numerically for many oscillations. $\tau = \tau^*$ is given by (4.7). B and $\overline{B}$ are computed for each oscillation: from (3.5) we have

$$B = D_- - 1 \quad \text{and} \quad \overline{B} = \overline{D}_- - 1 \ , \tag{6.6}$$

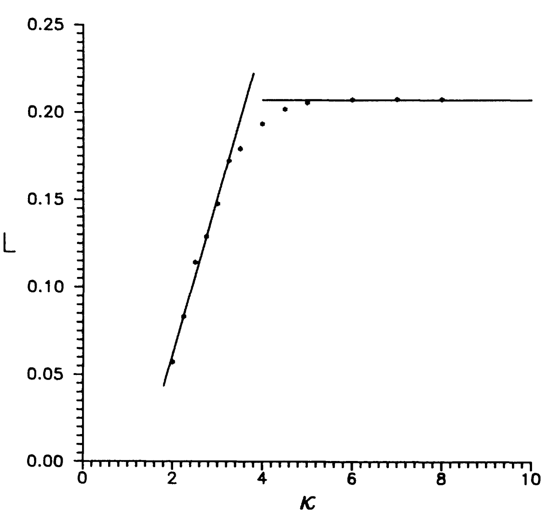

FIG. 3. Effect of exponentially small-amplitude noise. This figure summarizes the numerical study of the three LSA equations with additive noise. κ is related to the amplitude of the noise, namely $10^{-\kappa}$. The function L allows us to find the critical value of κ below which noise has an important effect on the size of the limit cycle. If $\kappa > 5$, L becomes independent of κ as expected since the amplitude of the noise decreases as κ increases. If $\kappa < 5$, exponentially small-amplitude noise starts to have an effect on the size of the limit cycle. The slope of the best-fit line agrees with the value predicted by the theory.

and D_- and $\bar{D}_-$ are the values of $D \ll 1$ and $\bar{D} \ll 1$ corresponding to $E = 0.01$.

If $\tau_j \approx \tau_d$, we use (4.14) and conclude that $L \approx k$. Consequently, there must be a linear relation between L and κ given by

$$L \approx \gamma \ln(10)\kappa \quad \text{as } \kappa \to \infty \ . \tag{6.7}$$

Our numerical results are summarized in Fig. 3. We note that the function $L(\kappa)$ follows two regimes: (i) a linear regime for $\kappa < 5$ and (ii) a constant regime for $\kappa \geq 5$. The first regime follows the predicted relation (6.7). The slope of the best fit line is 0.089 and compares well with the predicted slope $\gamma \ln(10) = 0.092$ ($\gamma = 0.04$). The second regime noted in the figure corresponds to L independent of κ and is also expected from the theory because, as $\kappa \to \infty$, the amplitude of the noise becomes too small and $\tau_d \approx \tau_c$ (τ_c is the value predicted in the absence of noise and is independent of κ).

VII. DISCUSSION

We have analyzed the equations for a LSA and show that the limit-cycle solution depends on a slow passage through a steady bifurcation point. Motivated by recent studies of slow-passage problems, we have shown that the properties of the limit cycle can be modified dramatically if the small amplitude of the imperfection or the noise surpasses a critical value. This critical value is an exponentially small quantity, and consequently, noise effects should be important in experimental studies of pulsating laser oscillations. Our results are in agreement with a detailed numerical study of the LSA equations.

Our analysis of the LSA suggests to investigate other laser problems exhibiting pulsating oscillations with long periods of almost zero intensities. The effects of small-amplitude noise may be analyzed by first studying a simplified equation describing a slow passage through a bifurcation point.

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