Degenerate and nondegenerate two-mode normal squeezing in a two-level atom and two-mode system

A. M. Abdel-Hafez
Faculty of Science, El-Minia University, El-Minia, Egypt
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We investigate the phenomena of degenerate and nondegenerate two-mode squeezing for a generalized Jaynes-Cummings model with a two-level atom and two modes. The effect of the relative phases between the atomic superposition state and the coherent field on normal squeezing is studied. Different values for the parameters of the atomic coherent state are taken. It is found that the nondegenerate two-mode squeezing is more effective than the one-mode degenerate type, and it recurs at later times.

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Recently, the definition of one-mode squeezing has been extended so as to investigate squeezing by two-mode interactions [1,2]. The formalism given in Ref. [1] has been used to investigate normal squeezing produced by the interaction of a two-level atom with two modes [3,4].

As for the theoretical work, Alsing and Zubairy [5] have investigated the collapse and revival of Rabi oscillation of an effective two-level atom that is in interaction with a quantized single-mode field through an intermediate state. A generalized Jaynes-Cummings (JC) model is investigated where the transitions are mediated by photons from two different modes [3,1]. In a recent study, the atom is taken to be prepared initially in a coherent superposition of its upper and lower levels, and it interacts with a single coherent mode [6,7]. It has been shown recently that the population inversion and field spectrum show dramatic changes with the change in the relative phase between the atomic dipole and the coherent field [6]. Also, squeezing is effected by atomic coherence [7].

In the present article, we find the wave function for the system of two modes in interaction with one atom. Then we calculate degenerate and nondegenerate two-mode squeezing for different values of the parameters in the atomic coherent state $|\Psi_{\text{atom}}(0)\rangle$ where the modes are initially in coherent states. The effects of the change in the relative phases on the one-mode and two-mode squeezing are studied.

The system we consider here is an effective two-level atom with upper and lower states denoted by $|e\rangle$, and $|g\rangle$, respectively. We start now by introducing the total Hamiltonian for the JC model of a two-level atom and two modes of the electromagnetic field in an interaction which is given in the rotating-wave approximation [3,4], by

$$\hat{H}=\tfrac{1}{2}\omega_0 S_z+\sum_{j=1}^{2}\omega_j\hat{a}_j^\dagger\hat{a}_j+\lambda(\hat{a}_1^\dagger\hat{a}_2^\dagger S_-+\hat{a}_1\hat{a}_2S_+)\ , \qquad (1)$$

where $\hat{a}_j$ ($\hat{a}_j^\dagger$) is annihilation (creation) operator for a photon of the field jth mode, ω_1 and ω_2 are the field frequencies for the two modes, λ is the coupling constant between the atom and the field, ω_0 is the transition frequency of the atom, and

$$S_z=|e\rangle\langle e|-|g\rangle\langle g|\ ,$$
$$S_-=|g\rangle\langle e|,\quad S_+=|e\rangle\langle g|\ .$$

By considering the case of two-photon resonance $\omega_1+\omega_2=\omega_0$, then the Hamiltonian in the interaction picture is

$$\hat{H}_{\text{int}}=\lambda(\hat{a}_1^\dagger\hat{a}_2^\dagger S_-+\hat{a}_1\hat{a}_2S_+)\ . \qquad (2)$$

Let us consider the atom prior to the interaction to be prepared in a coherent superposition of its excited and ground states [6,7]

$$|\Psi_{\text{atom}}(t=0)\rangle=\cos(\theta/2)|e\rangle+e^{-i\phi}\sin(\theta/2)|g\rangle\ . \qquad (3)$$

The initial atom-field state is then a product of the atomic superposition state and the field in a photon coherent state or the squeezed field

$$|\Psi(0)\rangle=\sum_{m,n}q_{mn}|m,n\rangle|\Psi_{\text{atom}}\ \ (t=0)\rangle\ , \qquad (4)$$

where

$$q_{lm}=e^{-(N_1+N_2)/2}\frac{\alpha_1^l\alpha_2^m}{\sqrt{l!m!}}\ , \qquad (5)$$

with $\alpha_j=\sqrt{N_j}\exp(i\psi_j)$, and $N_j=|\alpha_j|^2$ ($j=1,2$) is the mean number of photons in the coherent state. It must be emphasized that the first subscript is related to the first mode and the second is related to the second mode.

At any time $t>0$ the wave function of the total system in the interaction picture with the initial condition (4) is found from the Hamiltonian interaction (2) to be

$$|\Psi(t)\rangle = \exp(-it\hat{H}_{\text{int}})|\Psi(0)\rangle$$
$$= \sum_{m,n} q_{mn}\{\cos(\theta/2)[\cos(\lambda t\mu_{m+1,n+1})|m,n;e\rangle - i\sin(\lambda t\mu_{m+1,n+1})|m+1,n+1;g\rangle]$$
$$+ e^{-i\phi}\sin(\theta/2)[\cos(\lambda t\mu_{m,n})|m,n;g\rangle - i\sin(\lambda t\mu_{m,n})|m-1,n-1;e\rangle]\}\ , \qquad (6)$$

where $\mu^2_{m,n} = mn$. It must be noted that the state $|m,n;e\rangle$ means that the first mode is in the mth Fock state, the second mode is in the nth Fock state, while the third subscript stands for the atomic state. Thus the expectation values of any operator and its dependence on time can be obtained through the formula (6)

$$\langle Q(t)\rangle = \langle\Psi(t)|Q|\Psi(t)\rangle\ . \qquad (7)$$

We now study the normal squeezing in the JC model. We first define the single-mode normal squeezing when the quadrature operators X_j and Y_j given by [1]

$$X_j(t) = \tfrac{1}{2}(\hat{A}_j + \hat{A}_j^\dagger),\quad Y_j(t) = \frac{1}{2i}(\hat{A}_j - \hat{A}_j^\dagger) \quad (j=1,2)\ , \qquad (8)$$

where $\hat{A}_j = \hat{a}_j\exp[i(\omega_j t - \psi_j)]$ is a slowly varying operator, and ψ_j is the phase of the field coherent state. These operators satisfy the commutation relation

$$[X_j, Y_j] = \frac{i}{2}\ . \qquad (9)$$

The commutation relation of Eq. (9) implies the uncertainty

$$(\Delta X_j)^2(\Delta Y_j)^2 \geq (\tfrac{1}{4})^2,\quad j=1,2\ . \qquad (10)$$

A state is normally squeezed in the variables if

$$(\Delta X_j)^2 < \tfrac{1}{4}\ \text{ or }\ (\Delta Y_j)^2 < \tfrac{1}{4},\quad j=1,2\ . \qquad (11)$$

These are the conditions for single-mode normal squeezing.

However, these quantities are generalized for the two-mode case to [1]

$$Z_1(t) = \frac{1}{2\sqrt{2}}(\hat{A}_1 + \hat{A}_1^\dagger + \hat{A}_2 + \hat{A}_2^\dagger) = \frac{1}{\sqrt{2}}(X_1 + X_2)\ , \qquad (12a)$$

$$Z_2(t) = \frac{1}{2i\sqrt{2}}(\hat{A}_1 - \hat{A}_1^\dagger + \hat{A}_2 - \hat{A}_2^\dagger) = \frac{1}{\sqrt{2}}(Y_1 + Y_2)\ . \qquad (12b)$$

These operators satisfy the commutation relation (9), and the uncertainty relation (10) holds. By definition, the two-mode normal squeezing occurs when

$$(\Delta Z_1)^2 < \tfrac{1}{4}\ \text{ or }\ (\Delta Z_2)^2 < \tfrac{1}{4}\ . \qquad (13)$$

In terms of creation and annihilation operators of first mode and second mode these become

$$(\Delta Z_1)^2 = \tfrac{1}{2}[(\Delta X_1)^2 + (\Delta X_2)^2 + \tfrac{1}{2}(\langle\hat{A}_1\hat{A}_2\rangle + \langle\hat{A}_1^\dagger\hat{A}_2\rangle - \langle\hat{A}_1\rangle\langle\hat{A}_2\rangle - \langle\hat{A}_1^\dagger\rangle\langle\hat{A}_2\rangle + \text{c.c.})]\ , \qquad (14)$$

$$(\Delta Z_2)^2 = \tfrac{1}{2}[(\Delta Y_1)^2 + (\Delta Y_2)^2 - \tfrac{1}{2}(\langle\hat{A}_1\hat{A}_2\rangle - \langle\hat{A}_1^\dagger\hat{A}_2\rangle - \langle\hat{A}_1\rangle\langle\hat{A}_2\rangle + \langle\hat{A}_1^\dagger\rangle\langle\hat{A}_2\rangle + \text{c.c.})]\ , \qquad (15)$$

where the variances in the first mode are given by

$$(\Delta X_1)^2 = \tfrac{1}{4}[1 + 2\langle\hat{A}_1^\dagger\hat{A}_1\rangle + \langle\hat{A}_1^2\rangle + \langle\hat{A}_1^{\dagger 2}\rangle - (\langle\hat{A}_1\rangle + \langle\hat{A}_1^\dagger\rangle)^2]\ , \qquad (16)$$

$$(\Delta Y_1)^2 = \tfrac{1}{4}[1 + 2\langle\hat{A}_1^\dagger\hat{A}_1\rangle - \langle\hat{A}_1^2\rangle - \langle\hat{A}_1^{\dagger 2}\rangle + (\langle\hat{A}_1\rangle - \langle\hat{A}_1^\dagger\rangle)^2]\ , \qquad (17)$$

and the variances in the second mode can be obtained by replacing the subscript 1 with 2 in Eqs. (16) and (17). We find

$$(\Delta X_1)^2 \rightarrow (\Delta X_2)^2,\quad (\Delta Y_1)^2 \rightarrow (\Delta Y_2)^2\ . \qquad (18)$$

By using Eq. (7), we obtain the expectation value in the general form for the field operators $\hat{A}_1^{+k}\hat{A}_1^r\hat{A}_2^s$,

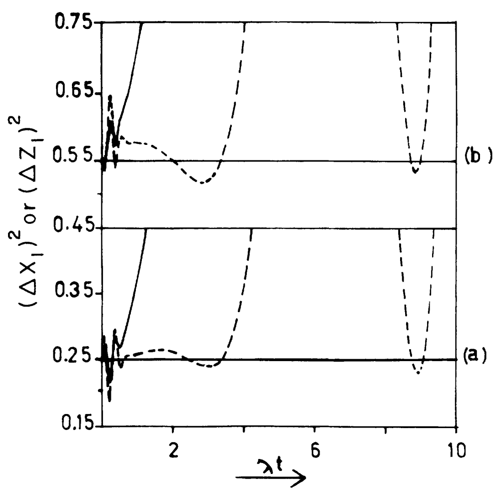

FIG. 1. (a) The time evolution of $(\Delta X_1)^2$ (degenerate mode) (solid line) and $(\Delta Z_1)^2$ (nondegenerate mode) (dashed line) for $N_1 = N_2 = 10$ and $\theta = 0$ (the atom initially in the excited state). (b) $(\Delta X_1)^2 + 0.30$ and $(\Delta Z_1)^2 + 0.30$. The same as in Fig. 1(a) but for $\theta = \pi$ (the atom initially in the ground state).

$$\langle \hat{A}_1^{+k}\hat{A}_1^r\hat{A}_2^s\rangle=|\alpha_1|^{k+r}|\alpha_2|^s\sum_{m,n}p_m^{(1)}p_n^{(2)}\left[\sin^2(\theta/2)\left[\cos(\lambda t\mu_{m+r,n+s})\cos(\lambda t\mu_{m+k,n})+\mu_m^2\mu_n\sin(\lambda t\mu_{m+k,n})\frac{\sin(\lambda t\mu_{m+r,n+s})}{\mu_{m+k}\mu_{m+r,n+s}}\right]\right.$$
$$+\cos^2(\theta/2)\left[\cos(\lambda t\mu_{m+1+k,n+1})\cos(\lambda t\mu_{m+1+r,n+1+s})+\mu_{m+1+k}\mu_{m+1+r,n+1+s}\sin(\lambda t\mu_{m+1+r,n+1+s})\times\frac{\sin(\lambda t\mu_{m+1+k,n+1})}{\mu_{m+1}^2\mu_{n+1}}\right]$$
$$+\frac{i}{2}\frac{\sin(\theta)}{\sqrt{N_1N_2}}e^{i\beta}\left[\mu_m^2\mu_n\cos(\lambda t\mu_{m+r,n+s})\frac{\sin(\lambda t\mu_{m+k,n})}{\mu_{m+k}}-\mu_{m+r,n+s}\sin(\lambda t\mu_{m+r,n+s})\cos(\lambda t\mu_{m+k,n})\right]$$
$$-\frac{i}{2}\sqrt{N_1N_2}\sin(\theta)e^{-i\beta}\left[\cos(\lambda t\mu_{m+1+k,n+1})\times\frac{\sin(\lambda t\mu_{m+1+r,n+1+s})}{\mu_{m+1+r,n+1+s}}-\mu_{m+1+k}\cos(\lambda t\mu_{m+1+r,n+1+s})\left.\times\frac{\sin(\lambda t\mu_{m+1+k,n+1})}{\mu_{m+1}^2\mu_{n+1}}\right]\right],\tag{19}$$

where $P_s^{(j)}=\exp(-N_j)(N_j^s/s!)$, and $\beta=\phi-(\psi_1+\psi_2)$ is the relative phase between the atomic state (phase ϕ) and the field coherent state [phase $(\psi_1+\psi_2)$]. It is easy to see that when we exchange the field modes we have

$$\langle \hat{A}_1^{+k}\hat{A}_1^r\hat{A}_2^s\rangle\to\langle \hat{A}_2^{+k}\hat{A}_2^r\hat{A}_1^s\rangle \text{ with } 1\leftrightarrow 2\ . \tag{20}$$

We will now discuss the temporal behavior of the variances $(\Delta X_1)^2$ and $(\Delta Z_1)^2$, which give information on degenerate and nondegenerate two-mode squeezing, respectively, when we take $N_1=N_2=10$ and the different values of the angle θ and β. We shall study the effect of the relative phase β on two different modes of squeezing for the nonlinear JC model.

Numerical results for Eqs. (14) and (16) are presented in Figs. 1–4. Here we plotted the one- and two-mode squeezing $(\Delta X_1)^2$ and $(\Delta Z_1)^2$, respectively, against λt in the interval [0,10] for $N_1=N_2=10$ and different values of θ (namely 0, $\pi/4$, $\pi/2$, $3\pi/4$, and π) and also the relative phase β (namely 0, $+\pi/2$, and $-\pi/2$).

In Fig. 1(a) we display the results of the JC model with $\theta=0$ (with the atom initially in excited state), and in Fig. 1(b) we have the case for $\theta=\pi$ (with the atom initially in the ground state). We observe that the squeezing parameter of two-mode case oscillates, and one-mode squeezing is lost for $\theta=\pi$ [see Fig. 1(b)], while for $\theta=0$ [see Fig. 1(a)] we note that the two-mode squeezing starts to appear a little later than the degenerate mode squeezing. Furthermore, its value is larger than that of one-mode squeezing. It recurs later on for larger values of λt, in contrast to the degenerate mode squeezing which is revoked.

Hence we find that initially, the amount of degenerate and nondegenerate two-mode squeezing is greater for the atom initially in the excited state $\theta=0$; however, it occurs after a time lapse. It is also apparent from the calculations that for these special cases, phases do not affect squeezing. Normal squeezing for the standard JC model (see Ref. [7]) shows a recurrence of squeezing at a later time for $\theta=0$, which does not occur for the degenerate case in the present model. The nondegenerate two-mode squeezing, however, shows similar behavior, but with stronger squeezing and more frequencies for both values of $\theta=0$ and π.

The characteristics of normal squeezing are shown in Figs. 2(a)–2(c) at the relative phase $\beta=0$, and for the values of θ ($=\pi/4$, $\pi/2$, and $3\pi/4$), respectively. We see that in Fig. 2(a) the one-mode and two-mode squeezing occur for $\theta=\pi/4$ for a short time. But for $\theta=\pi/2$ and $3\pi/4$, we note that degenerate squeezing disappears all together in these cases [see Figs. 2(b) and 2(c)]. While in the nondegenerate case, squeezing recurs later on as time increases in all the cases under consideration.

The amounts of squeezing are studied through curves in Fig. 3, when the relative phase $\beta=\pi/2$, and for all values of θ considered before. We notice that for the

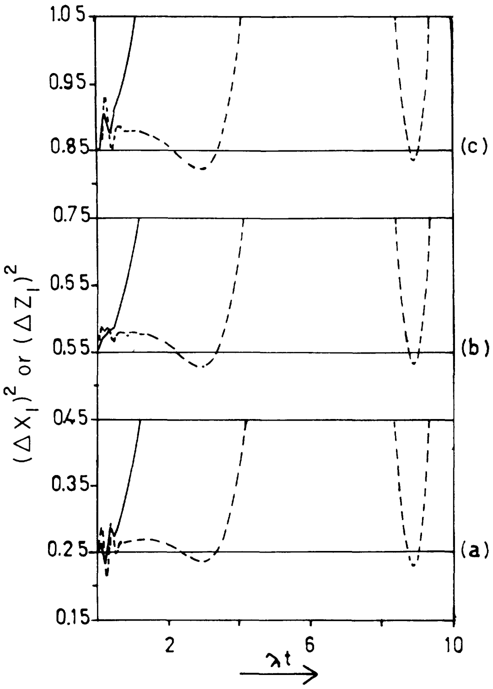

FIG. 2. The time evolution of $(\Delta X_1)^2$ (solid line) and $(\Delta Z_1)^2$ (dashed line) for the one-photon JC model. The relative phase is $\beta=0$, and $N_1=N_2=10$. (a) $(\Delta X_1)^2$ and $(\Delta Z_1)^2$ for $\theta=\pi/4$. (b) $(\Delta X_1)^2+0.30$ and $(\Delta Z_1)^2+0.30$ for $\theta=\pi/2$. (c) $(\Delta X_1)^2+0.60$ and $(\Delta Z_1)^2+0.60$ for $\theta=3\pi/4$.

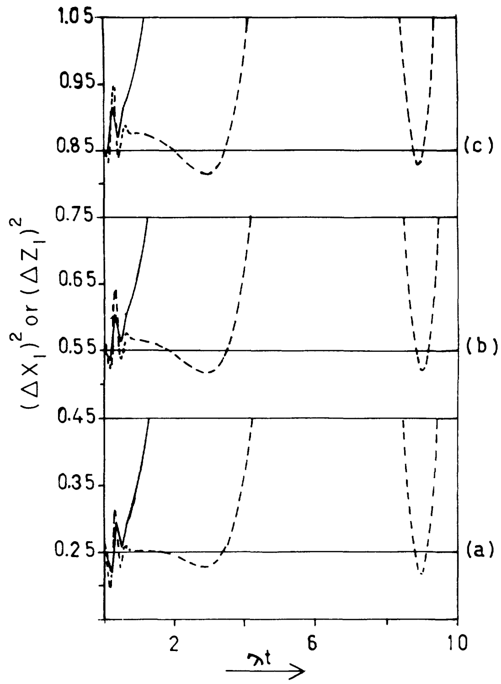

FIG. 3. Same as in Fig. 2, but with $\beta=\pi/2$.

short time $0\le\lambda t<1$ the observed maximum amount squeezing for degenerate and nondegenerate modes is enhanced for $\theta=\pi/4$ [see Fig. 3(a)], and the curve for nondegenerate two-mode squeezing oscillates for all values of θ. We also observe that the degenerate and nondegenerate two-mode squeezing (Fig. 3) ($\beta=\pi/2$) is larger than those in Figs. 1(b) and 2. Furthermore, we notice that in the nondegenerate two-mode case squeezing occurs at later times for $2<\lambda t<4$ and $8.5<\lambda t<9.5$. For the first interval, by increasing θ we find that the largest amount of nondegenerate two-mode squeezing increases, while the order is reversed for the latter interval.

From Figs. 4(a)–4(c), at $\beta=-\pi/2$, we note that the nondegenerate two-mode squeezing parameter oscillates as θ increases, and squeezing is attained at the beginning, in contrast to the degenerate mode squeezing for the short interval $0\le\lambda t<1$. The degenerate mode squeezing is lost after that, while the behavior of the nondegenerate two-mode squeezing is the same as that observed in Fig. 3.

Thus we conclude that the effect of the relative phases on normal squeezing is the strongest in the nondegenerate two-mode JC model for the different values of θ when the relative phase $\beta=\pi/2$. We have shown that when the relative phase (β) is fixed, the amount of squeezing for both degenerate and nondegenerate cases increases as the value of θ decreases for the very short interval $0\le\lambda t<1$. However, for the interval $2<\lambda t<4$ the nondegenerate two-mode squeezing increases as θ does, in contrast to the normal degenerate squeezing for the present model and the standard JC model. Also we notice that the observed amount of nondegenerate two-mode squeezing is larger than the degenerate mode squeezing corresponding to the same values of θ. Furthermore, the degenerate mode squeezing is lost after a short time. Finally, we note that the level of the noise is lower for the nondegenerate two-mode squeezing.

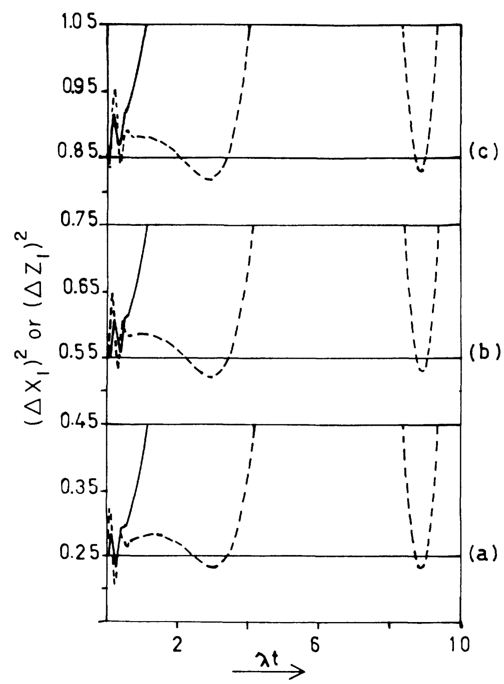

FIG. 4. Same as in Fig. 2, but with $\beta=-\pi/2$.

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