# Moments of $P$ functions and nonclassical depths of quantum states

Ching Tsung Lee
*Department of Physics, Alabama A&M University, Normal, Alabama 35762*
(Received 18 November 1991)

The practical aspect of the moment problem of the $P$ function is developed; i.e., a simple explicit expression for the $P$ function in terms of its moments and the derivatives of the Dirac $\delta$ function in the complex domain is derived. Its connection with the concept of the nonclassical depth of a quantum state introduced recently is explored. The basic formulas derived are used in the calculations of several specific examples that are popular in the quantum-optics community, such as photon-number state, binomial state, two-photon thermal state, squeezed-vacuum state, etc.

PACS number(s): 42.50.Dv, 42.50.Ar

## I. INTRODUCTION

To determine the probability distribution function from a knowledge of its moments is generally referred to as the problem of moments [1]. Historically, only normalizable positive distribution functions have been considered in such a problem. However, in the phase-space description of quantum mechanics, we are dealing with quasidistributions that are not always positive, such as the distribution function introduced by Wigner [2], the $P$ functions introduced by Glauber [3] and by Sudarshan [4], etc. In these cases, we are confronted with a new kind of moment problem having to do with quasidistributions. This quantum-mechanical-moment problem was first mentioned by Moyal in his seminal paper [5] on Wigner distributions, which was also discussed more recently by Narcowich [6].

In this paper we shall focus our attention on the moment problem of the $P$ function. We shall not address ourselves to the fundamental issues, such as the problem of uniqueness or the problem of necessary and/or sufficient conditions for the existence of distribution functions, etc. Instead, we shall confine ourselves to the more practical problem of determining the explicit expression for the $P$ function from its known moments.

The $P$ function is widely used to represent quantum states in discussing fundamental problems of quantum optics. However, it seems to be not as widely used in practical calculations. In some cases this is because it is more expedient to bypass it, but in other cases this is simply because it is not easy to obtain an explicit expression for the $P$ function. An explicit expression for the $P$ function will be very useful, if not necessary, for determining the nonclassicality of a quantum state according to the definition introduced recently by Lee [7].

Inspired by the work of Morten and Krall [8], we derive an explicit expression for the $P$ function in terms of Dirac's $\delta$ function in the complex domain and its derivatives; the moments appear in the coefficients of such an expansion. Our approach will be very simple if the general expression for the moments can be easily obtained. It is well known within the community of quantum optics that the $P$ function of a nonclassical state [9] is typically a singular function; for such a state, our approach is most direct and natural. Ironically, our approach is not as direct for a classical state having a regular function as its $P$ function; we need to convert the seemingly singular function into a regular function through the Fourier transformation followed by its inverse. Fortunately, we are, of course, more interested in nonclassical states than classical ones.

In Sec. II, we first reproduce the essential basics of generalized functions of a complex variable and their Fourier transforms, we then derive the explicit expression for the $P$ function in terms of its moments, and we also discuss its connection with the concept of the nonclassical depth of a quantum state. The basic formulation of Sec. II is illustrated by a few specific examples presented in Sec. III. Section IV summarizes the essential points of this paper.

## II. BASIC FORMULATION

### A. Generalized functions of a complex variable

The domain of the $P$ function is the complex plane and the function is not necessarily analytic. Therefore, we should include both the complex variable $z$ and its conjugate $\bar{z}$ as its arguments and treat them as two independent variables. For the sake of completeness and convenience, we reproduce in the following the bare essentials of the theory of generalized functions of a complex variable [10].

A distribution or generalized function $F(z,\bar{z})$ is a continuous linear functional of a test function $\phi(z,\bar{z})$ from an appropriately defined function space with values given by

$$\langle F,\varphi\rangle \equiv \int F(z,\bar{z})\varphi(z,\bar{z})d^2z \ , \tag{2.1}$$

where the integration is over the complex plane.

The most important distributions for our purpose are the Dirac $\delta$ function in the complex domain and its derivatives, defined as follows:

$$\langle \delta,\varphi\rangle \equiv \int \delta(z,\bar{z})\varphi(z,\bar{z})d^2z = \varphi(0,0) \tag{2.2}$$

and

$$\langle \delta^{(k,l)},\varphi\rangle \equiv (-1)^{k+l}\langle \delta,\varphi^{(k,l)}\rangle \ , \tag{2.3}$$

 

where the superscript $(k,l)$ indicates the $k$th and the $l$th derivatives with respect to $z$ and $\bar{z}$, respectively.

The Fourier transformation involving the $P$ function or other related distribution or quasidistribution functions will play a very important role in the subsequent development. To avoid possible confusion, we should be very precise about what we mean by the Fourier transform of a function in the complex domain, since it can be defined in several different ways. Cahill and Glauber [11] defined the complex Fourier transform of a function $\varphi(z,\bar{z})$ in the complex domain as

$$\tilde{\varphi}(w,\bar{w}) \equiv \frac{1}{\pi}\int \exp(w\bar{z}-\bar{w}z)\varphi(z,\bar{z})d^2z \ . \tag{2.4}$$

According to this definition, the processes of Fourier transformation and Fourier inversion are completely symmetric; namely,

$$\varphi(z,\bar{z}) = \frac{1}{\pi}\int \exp(z\bar{w}-\bar{z}w)\tilde{\varphi}(w,\bar{w})d^2w \ , \tag{2.5}$$

which implies

$$\tilde{\tilde{\varphi}}(z,\bar{z}) = \varphi(z,\bar{z}) \ . \tag{2.6}$$

This definition of the Fourier transform of a regular function in the complex domain can be extended to define the Fourier transform of a distribution $F(z,\bar{z})$ in the complex domain by using the Parseval formula

$$\langle \tilde{F},\varphi\rangle \equiv \langle F,\tilde{\varphi}\rangle \ . \tag{2.7}$$

According to this definition, the Fourier transform of the $\delta$ function can be easily established as

$$\tilde{\delta}(w,\bar{w}) = 1/\pi \ ; \tag{2.8}$$

so we also have a representation of the $\delta$ function as

$$\delta(z,\bar{z}) = \frac{1}{\pi^2}\int \exp(z\bar{w}-\bar{z}w)d^2w \ . \tag{2.9}$$

The Fourier transforms of the derivatives of the $\delta$ function in the complex domain can also be established as follows. By definition we have

$$\begin{aligned}\langle (\delta^{(k,l)})_F\varphi\rangle &\equiv \langle \delta^{(k,l)},\tilde{\varphi}\rangle = (-1)^{k+l}\langle \delta,\tilde{\varphi}^{(k,l)}\rangle \\ &= (-1)^{k+l}\tilde{\varphi}^{(k,l)}(0,0)\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\tilde{\varphi}^{(k,l)}(w,\bar{w}) &\equiv \frac{1}{\pi}\frac{\partial^{k+l}}{\partial w^k \partial \bar{w}^l}\int \exp(w\bar{z}-\bar{w}z)\varphi(z,\bar{z})d^2z \\ &= \frac{1}{\pi}\int \bar{z}^k(-z)^l \exp(w\bar{z}-\bar{w}z)\varphi(z,\bar{z})d^2z \ ,\end{aligned} \tag{2.11}$$

where $(\ )_F$ denotes a Fourier transform. The combination of Eqs. (2.10) and (2.11) gives

$$(\delta^{(k,l)})_F(w,\bar{w}) = \frac{1}{\pi}(-\bar{w})^k w^l \ , \quad k,l=0,1,2,\ldots \ . \tag{2.12}$$

### B. *P* function in terms of its moments

Let $\mu_{k,l}$ be the $(k,l)$ moment of the $P$ function defined by

$$\mu_{k,l} \equiv \frac{1}{\pi}\int P(z,\bar{z})z^k\bar{z}^l d^2z = \langle (\hat{a}^\dagger)^l(\hat{a})^k\rangle \ , \tag{2.13}$$

where $a^\dagger$ and $\hat{a}$ are the creation and the annihilation operators, respectively. As an extension of the work of Morten and Krall [8] to the complex domain, we can view $P(z,\bar{z})$ as a linear functional and $z^k\bar{z}^l$ as a test function in the appropriate space. Then Eq. (2.13) can be rewritten as

$$\mu_{k,l} = \frac{1}{\pi}\langle P,z^k\bar{z}^l\rangle \ . \tag{2.14}$$

We now consider an arbitrary function $\varphi(z,\bar{z})$ that has the Maclaurin-type series expansion

$$\varphi(z,\bar{z}) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\varphi^{(k,l)}(0,0)\frac{z^k\bar{z}^l}{k!l!} \tag{2.15}$$

with

$$\varphi^{(k,l)}(0,0) = \left.\frac{\partial^{k+1}}{\partial z^k\partial\bar{z}^l}\varphi(z,\bar{z})\right|_{z=\bar{z}=0} \ . \tag{2.16}$$

The function $\varphi(z,\bar{z})$ can be used as the test function in the following expression:

$$\begin{aligned}\langle P,\varphi\rangle &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\varphi^{(k,l)}(0,0)\frac{\langle P,z^k\bar{z}^l\rangle}{k!l!} \\ &= \pi\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\varphi^{(k,l)}(0,0)\frac{\mu_{k,l}}{k!l!} \ ,\end{aligned} \tag{2.17}$$

where we have used Eq. (2.14). On the other hand, $\varphi^{(k,l)}(0,0)$ can be represented in terms of the derivative of the Dirac $\delta$ function as

$$\varphi^{(k,l)}(0,0) = (-1)^{k+l}\langle \delta^{(k,l)}(z,\bar{z}),\varphi\rangle \ . \tag{2.18}$$

Using Eq. (2.18) in Eq. (2.17) we have

$$\langle P,\varphi\rangle = \pi\left\langle \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l}\frac{\mu_{k,l}}{k!l!}\delta^{(k,l)}(z,\bar{z}),\varphi\right\rangle \ . \tag{2.19}$$

Therefore, we can express the $P$ function in terms of its moments and the derivatives of the $\delta$ function as

$$P(z,\bar{z}) = \pi\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l}\frac{\mu_{k,l}}{k!l!}\delta^{(k,l)}(z,\bar{z}) \ . \tag{2.20}$$

Sudarshan, in his original paper on coherent-state representations of quantum fields [4], gives a formula for the $P$ function also in terms of an infinite series of derivatives of the $\delta$ function. We shall show in Appendix A that Sudarshan's formula is equivalent to our Eq. (2.20). The essential foundation for our representation is that the collections $\{(-1)^{k+l}\delta^{(k,l)}(z,\bar{z})\}_{k,l=0}^{\infty}$ and $\{z^m\bar{z}^n/m!n!\}_{m,n=0}^{\infty}$ form a biorthogonal set; i.e.,

$$\langle (-1)^{k+l}\delta^{(k,l)}(z,\bar{z}),z^m\bar{z}^n/m!n!\rangle = \delta_{k,m}\delta_{l,n} \ . \tag{2.21}$$

Except for some special cases, such as the coherent state to be seen later, we do not known how to carry out the summations in Eq. (20), which involves the derivatives of the δ function, to obtain a closed-form expression directly. An obvious indirect way of doing it is through the Fourier transformation followed by its inverse. Using Eq. (2.12) in Eq. (2.20), the Fourier transform of the $P$ function can be written as

$$\tilde{P}(w,\bar{w})=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\mu_{k,l}\frac{\bar{w}^k(-w)^l}{k!l!} . \tag{2.22}$$

It is obvious that the summations in Eq. (2.22) are easier to carry out than those in Eq. (2.20). The inverse Fourier transform of $\tilde{P}(w,\bar{w})$ will be the $P$ function. But in some cases, the Fourier transform may be more useful than the $P$ function itself.

Another well-known distribution function is called the $Q$ function, corresponding to the antinormal ordering of the creation and annihilation operators. The two distribution, or quasidistribution, functions are related to each other through the following convolution transformation [7]:

$$Q(z,\bar{z})=\frac{1}{\pi}\int e^{-|z-w|^2}P(w,\bar{w})d^2w . \tag{2.23}$$

The $Q$ function is actually the diagonal element of the density operator

$$Q(z,\bar{z})=\langle z|\hat{\rho}|z\rangle ; \tag{2.24}$$

hence, it is always a positive-definite regular function.

Using the theorem that the fourier transform of the convolution of two functions is the product of the Fourier transforms of the two functions, we can express the Fourier transform of the $Q$ function according to Eq. (2.23) as $\tilde{Q}(w,\bar{w})=\exp(-w\bar{w})\tilde{P}(w,\bar{w})$; so we have another expression for the Fourier transform of the $P$ function as

$$\tilde{P}(w,\bar{w})=\exp(w\bar{w})\tilde{Q}(w,\bar{w}) . \tag{2.25}$$

For some cases it may be easier to obtain the Fourier transform of the $Q$ function first; then we can use it in Eq. (2.25), expand the right-hand side of this equation in the power series, and compare the coefficients of the expansion with those in Eq. (2.22) to obtain the expressions for the moments. We can further use Eq. (2.12) to obtain the expression for the $P$ function in terms of the δ function and its derivatives.

### C. Nonclassical depth of a quantum state

Following Cahill and Glauber [11], but in a slightly different way, a continuous parameter $\tau$ was recently introduced [7] into the convolution transformation of Eq. (2.23) to define a general distribution function as

$$R(z,\bar{z},\tau)=\frac{1}{\pi\tau}\int \exp\left[-\frac{1}{\tau}|z-w|^2\right]P(w,\bar{w})d^2w . \tag{2.26}$$

We shall call $R(z,\bar{z},\tau)$ the $R$ function. The original $P$ and $Q$ functions are two limiting cases of the $R$ function with $\tau=0$ and 1, respectively.

Our motivation for introducing this $\tau$ parameter is to define a measure of how nonclassical quantum states are. It is well known that the origin of the nonclassical effects is that the $P$ functions of all pure quantum states are singular and not necessarily positive definite, as shown by Hillery [12]; hence they are called quasidistribution functions. On the other hand, as pointed out earlier, the $Q$ function is always a positive-definite regular function. The reason that the $Q$ function behaves better than the $P$ function is that a convolution transformation can be viewed as a *moving average,* so it has the effect of making the transformed function smoother. The smoothing effect of the convolution transformation of Eq. (2.26) is enhanced as $\tau$ increases. If $\tau$ is large enough so that the $R$ function become acceptable as a classical distribution function, i.e., it is a positive-definite regular function and normalizable, then we say that the smoothing operation is complete. Let $\Omega$ denote the set of all the $\tau$ that will complete the smoothing of the $P$ function of a quantum state and let the greatest lower bound, or infimum, of all the $\tau$ in $\Omega$ be denoted by

$$\tau_m \equiv \inf_{\tau\in\Omega}(\tau) . \tag{2.27}$$

We recently proposed [7] defining $\tau_m$ as the nonclassical depth of the quantum state.

According to this definition, we have $\tau_m=0$ for an arbitrary coherent state $|\alpha\rangle$ because its $P$ function is a δ function; this is very reasonable since the coherent state is known to be on the border line between classical and nonclassical states. On the other hand, for $\tau=1$ we have $R(z,\bar{z},1)=Q(z,\bar{z})$, which is always acceptable as a classical distribution function for any quantum state; hence 1 is an upper bound for $\tau_m$. Therefore, we can specify the range of $\tau_m$ to be

$$0\le\tau_m\le 1 . \tag{2.28}$$

Applying the convolution theorem of Fourier transforms to Eq. (2.26), we can express the Fourier transform of the $R$ function as

$$\tilde{R}(w,\bar{w},\tau)=\exp(-\tau w\bar{w})\tilde{P}(w,\bar{w}) . \tag{2.29}$$

If $R(z,\bar{z},\tau)$ is a non-negative regular function, and since $w\bar{z}-\bar{w}z$ is always a pure imaginary number, we have

$$|\tilde{R}(w,\bar{w},\tau)|\le\frac{1}{\pi}\int|\exp(w\bar{z}-\bar{w}z)|R(z,\bar{z},\tau)d^2z$$
$$=\frac{1}{\pi}\int R(z,\bar{z},\tau)d^2z=\tilde{R}(0,0,\tau)=1 , \tag{2.30}$$

which provides a necessary condition for the $R$ function to be a regular distribution function [13]. In some cases, it may be easier to determine $\tau_m$ by using this condition.

The $Q$ function can also be considered as the convolution transform of the $R$ function according to

$$Q(z,\bar{z})=\frac{1}{\pi(1-\tau)}\int \exp\left[-\frac{|z-w|^2}{1-\tau}\right]R(w,\bar{w},\tau)d^2w . \tag{2.31}$$

Therefore, we can also express the fourier transform of the $R$ function in terms of that of the $Q$ function as

$$\tilde{R}(w,\bar{w},\tau)=\exp[(1-\tau)w\bar{w}]\tilde{Q}(w,\bar{w}) . \quad (2.32)$$

## III. EXAMPLES

To illustrate the applications of the basic formulas derived in Sec. II, we shall consider a few specific examples as follows.

### A. Thermal state

The density matrix of the thermal state can be written as

$$\hat{\rho}_{\rm th}=\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{(\langle n\rangle+1)^{n+1}}|n\rangle\langle n| . \quad (3.1)$$

So, we can obtain the moments as follows:

$$\begin{aligned}\mu_{k,l}&=\mathrm{Tr}[(\hat{a}^\dagger)^l(\hat{a})^k\hat{\rho}_{\rm th}]\\&=\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{(\langle n\rangle+1)^{n+1}}\langle n|(\hat{a}^\dagger)^l(\hat{a})^k|n\rangle\\&=\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{(\langle n\rangle+1)^{n+1}}\frac{n!}{(n-k)!}\delta_{k,l}\\&=k!\langle n\rangle^k\delta_{k,l} .\end{aligned} \quad (3.2)$$

Substitution of Eq. (3.2) into Eq. (2.20) gives

$$P_{\rm th}(z,\bar{z})=\sum_{k=0}^{\infty}\frac{\langle n\rangle^k}{k!}\delta^{(k,k)}(z,\bar{z}) . \quad (3.3)$$

This is not a desirable expression for the $P$ function of the thermal state, since we already know that it is an ordinary function. To recover the ordinary function, we can resort to Fourier transformation. The Fourier transform of Eq. (3.3) is

$$\tilde{P}_{\rm th}(w,\bar{w})=\sum_{k=0}^{\infty}(-1)^k\frac{\langle n\rangle^k}{k!}\bar{w}^k w^k=\exp(-\langle n\rangle|w|^2) , \quad (3.4)$$

where we have sued Eq. (2.12). An inverse Fourier transformation of Eq. (3.4) gives

$$P_{\rm th}(z,\bar{z})=\frac{1}{\langle n\rangle}\exp\left[-\frac{z\bar{z}}{\langle n\rangle}\right] , \quad (3.5)$$

as expected.

This example is included here to illustrate the point that our approach is not very attractive, but still works, when applied to classical states.

### B. Coherent state

Considering a coherent state $|\alpha\rangle$, we have

$$\mu_{k,l}=\langle\alpha|(\hat{a}^\dagger)^l(\hat{a})^k|\alpha\rangle=\bar{\alpha}^l\alpha^k . \quad (3.6)$$

Substitution of Eq. (3.6) into Eq. (2.20) gives

$$P_\alpha(z,\bar{z})=\pi\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{k+l}\frac{\alpha^k}{k!}\frac{\bar{\alpha}^l}{l!}\delta^{(k,l)}(z,\bar{z}) . \quad (3.7)$$

The right-hand side of the above equation is of the form of the Maclaurin expansion, given in Eq. (2.15), of the $\delta$ function, but about the point $(z,\bar{z})$ instead of the origin; so we may conclude that

$$P_\alpha(z,\bar{z})=\delta(z-\alpha,\bar{z}-\bar{\alpha}) , \quad (3.8)$$

as expected. However, the Maclaurin series expansion for a singular function may not be considered well justified by some readers. Therefore, to be on more solid ground, we shall legitimize the relation through the Fourier transformation of Eq. (3.7) as follows:

$$\begin{aligned}\tilde{P}_\alpha(w,\bar{w})&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^k\frac{\alpha^k}{k!}\frac{\bar{\alpha}^l}{l!}\bar{w}^k w^l=\exp(\bar{\alpha}w-\alpha\bar{w})\\&=\int\delta(z-\alpha,\bar{z}-\bar{\alpha})\exp(w\bar{z}-\bar{w}z)d^2z ,\end{aligned} \quad (3.9)$$

where we have again used Eq. (2.12). An inverse Fourier transformation of Eq. (3.9) gives Eq. (3.8).

This example is rather trivial; it is included here because, as mentioned earlier, the coherent state is right on the border line between classical and nonclassical states: For classical states, Fourier transformation is necessary to recover their $P$ functions as regular functions; for nonclassical states, Fourier transformation is not necessary because their $P$ functions must be singular anyway; but for coherent states, whether it is necessary or not depends on the level of rigorousness one demands.

### C. Photon-number state

Considering a photon-number (Fock) state $|n\rangle$ with exactly $n$ photons, we have

$$\mu_{k,l}=\langle n|(\hat{a}^\dagger)^l(\hat{a})^k|n\rangle=\begin{cases}\dfrac{n!}{(n-k)!}\delta_{k,l} & \text{for } k\le n\\ 0 \text{ for } k>n .\end{cases} \quad (3.10)$$

Substitution of Eq. (3.10) into Eq. (2.20) gives

$$P_n(z,\bar{z})=\pi\sum_{k=0}^{n}\frac{1}{k!}\begin{bmatrix}n\\k\end{bmatrix}\delta^{(k,k)}(z,\bar{z}) . \quad (3.11)$$

This is perhaps the best example to illustrate the power of our approach. The result is obtained so easily and it is a useful result since it is of a different form from that derived originally by Sudarshan [4].

The Fourier transform of $P_n(z,\bar{z})$ can be obtained immediately by using Eq. (2.12) as

$$\tilde{P}_n(w,\bar{w})=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\begin{bmatrix}n\\k\end{bmatrix}(w\bar{w})^k=L_n(|w|^2) , \quad (3.12)$$

where $L_n(x)$ is the Laguerre polynomial. Using this expression in Eq. (2.29), we can write the Fourier transform of the $R$ function for the photon-number state as

$$\tilde{R}_n(w,\bar{w},\tau)=\exp(-\tau w\bar{w})L_n(w\bar{w}) . \quad (3.13)$$

However, this simple expression is not very useful for determining $\tau_m$. For this purpose, it is easier to consider the $R$ function itself.

Using Eq. (3.11) in Eq. (2.26) and exchanging the order of integration and summation, we obtain the $R$ function for the photon-number state as

$$R_n(z,\bar{z},\tau)=\sum_{k=0}^{n}\frac{1}{k!}\begin{bmatrix}n\\k\end{bmatrix}\frac{1}{\pi\tau}\int\exp\left[-\frac{1}{\tau}(z-w)(\bar{z}-\bar{w})\right]\times\delta^{(k,k)}(w,\bar{w})d^2w\ . \quad (3.14)$$

Repeating integration by parts, we have

$$\frac{1}{\tau}\int\exp\left[-\frac{1}{\tau}(z-w)(\bar{z}-\bar{w})\right]\delta^{(k,k)}(w,\bar{w})d^2w=\frac{1}{\tau}\exp\left[-\frac{z\bar{z}}{\tau}\right]\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{k!}{j!}\left[-\frac{1}{\tau}\right]^{k-j}\left[\frac{z\bar{z}}{\tau^2}\right]^j . \quad (3.15)$$

Substituting Eq. (3.15) into Eq. (3.14), exchanging the order of summations, and then carrying out the summations, we obtain

$$R_n(z,\bar{z},\tau)=\frac{1}{\tau}\left[-\frac{1-\tau}{\tau}\right]^n\times\exp\left[-\frac{z\bar{z}}{\tau}\right]L_n(z\bar{z}/\tau(1-\tau))\ . \quad (3.16)$$

As shown in Appendix B, this $R$ function can also be obtained as the Fourier transform of Eq. (3.13). Since the Laguerre polynomial can be a positive function only if its argument is restricted to negative values, we conclude that

$$\tau_m=1\ , \quad (3.17)$$

which means that a photon-number state is as nonclassical as can be.

### D. Binomial state

The binomial state has been known to show sub-Poissonian photon statistics [14]. So, we are interested in determining its nonclassical depth. The binomial state is an incoherent superposition of photon-number states according to binomial probability distribution so that its density operator is represented by an $N\times N$ diagonal matrix

$$\hat{\rho}(N,p)=\sum_{n=0}^{N}\begin{bmatrix}N\\n\end{bmatrix}p^n(1-p)^{N-n}|n\rangle\langle n|\ . \quad (3.18)$$

It is obvious that the $P$ function of such a state can be expressed as

$$P_{N,p}(z,\bar{z})=\sum_{n=0}^{N}\begin{bmatrix}N\\n\end{bmatrix}p^n(1-p)^{N-n}P_n(z,\bar{z})\ , \quad (3.19)$$

where $P_n(z,\bar{z})$ is the $P$ function of a photon-number state given in Eq. (3.11). Using Eq. (3.11) in Eq. (3.19) and changing the order of summations, we obtain

$$P_{N,p}(z,\bar{z})=\pi\sum_{k=0}^{N}\frac{1}{k!}\begin{bmatrix}N\\k\end{bmatrix}\delta^{(k,k)}(z,\bar{z})\times\sum_{n=k}^{N}\begin{bmatrix}N-k\\n-k\end{bmatrix}p^n(1-p)^{N-n}=\pi\sum_{k=0}^{N}\frac{p^k}{k!}\begin{bmatrix}N\\k\end{bmatrix}\delta^{(k,k)}(z,\bar{z})\ . \quad (3.20)$$

Similarly, we can use Eq. (3.16) to obtain the $R$ function as

$$R_{N,p}(z,\bar{z},\tau)=\frac{1}{\tau}\exp\left[-\frac{z\bar{z}}{\tau}\right]\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}\frac{1}{j!}\left[\frac{z\bar{z}}{\tau^2}\right]^j\times\left[1-\frac{1}{\tau}\right]^{n-j}=\frac{1}{\tau}\left[1-\frac{p}{\tau}\right]^n\times\exp\left[-\frac{z\bar{z}}{\tau}\right]L_n\left[\frac{pz\bar{z}}{\tau(p-\tau)}\right]\ . \quad (3.21)$$

For this $R$ function to be positive definite, the argument of the Laguerre function must be negative. Therefore, we conclude that

$$\tau_m=p\ . \quad (3.22)$$

The photon statistics of the binomial state is intermediate between that of a coherent state and that of a photon-number state; it reduces to either that of a coherent state (as $p\to 0$ and $N\to\infty$ such that $pN=\langle n\rangle=|\alpha|^2$, a constant) or that of a number state (as $p\to 1$) under the two extreme conditions of the parameter $p$. From Eq. (3.22) we see that the nonclassical depth varies accordingly, as expected.

### E. Two-photon thermal state

Cahill and Glauber [15] have studied the quantum state with the following density operator,

$$\hat{\rho}_{\text{TPT}}=2\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{(\langle n\rangle+2)^{n+1}}|2n\rangle\langle 2n|\ , \quad (3.23)$$

as belonging to the "parametrized family." Such a density operator bears a striking similarity to that of the thermal state, except for the fact that only states with an even number of photons are involved. We shall call such a state a *two-photon thermal state*.

The moments of such a state can be calculated as

$$\mu_{k,l}=2\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{(\langle n\rangle+2)^{n+1}}\langle 2n|(\hat{a}^\dagger)^l(\hat{a})^k|2n\rangle$$
$$=2\sum_{n=0}^{\infty}\frac{\langle n\rangle^n}{(\langle n\rangle+2)^{n+1}}\frac{(2n)!}{(2n-k)!}\delta_{k,l}$$
$$=k!\left[(1+t)\left[\frac{t}{1-t}\right]^k+(1-t)\left[\frac{-t}{1+t}\right]^k\right]\delta_{k,l}\,, \tag{3.24}$$

where

$$t=\sqrt{\langle n\rangle/(\langle n\rangle+2)}\ . \tag{3.25}$$

Substitution of Eq. (3.24) into Eq. (2.20) gives the $P$ function as

$$P_{\mathrm{TPT}}(z,\bar{z})=\pi\sum_{k=0}^{\infty}\frac{1}{k!}\left[(1+t)\left[\frac{t}{1-t}\right]^k+(1-t)\left[-\frac{t}{1+t}\right]^k\right]\times\delta^{(k,k)}(z,\bar{z})\ . \tag{3.26}$$

The Fourier transform of $P_{\mathrm{TPT}}(z,\bar{z})$ can be obtained immediately by using Eq. (2.12) as

$$\tilde{P}_{\mathrm{TPT}}(w,\bar{w})=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left[(1+t)\left[\frac{t}{1-t}\right]^k+(1-t)\left[\frac{-t}{1+t}\right]^k\right](w\bar{w})^k$$
$$=(1+t)\exp\left[-\frac{t}{1-t}w\bar{w}\right]+(1-t)\exp\left[\frac{t}{1+t}w\bar{w}\right]\ . \tag{3.27}$$

Then the Fourier transform of the $R$ function can be easily obtained according to Eq. (2.29) as

$$\tilde{R}_{\mathrm{TPT}}(w,\bar{w},\tau)=(1+t)\exp\left[-\left[\tau+\frac{t}{1-t}\right]w\bar{w}\right]+(1-t)\exp\left[-\left[\tau-\frac{t}{1+t}\right]w\bar{w}\right]\ . \tag{3.28}$$

For this $R$ function to be normalizable, we must have

$$\tau-\frac{t}{1+t}>0\ . \tag{3.29}$$

So, the nonclassical depth of the two-photon thermal state is

$$\tau_m=\frac{t}{1+t}=\frac{\sqrt{\langle n\rangle}}{\sqrt{\langle n\rangle+2}+\sqrt{\langle n\rangle}}\ , \tag{3.30}$$

which varies from 0 to $\frac{1}{2}$ as $\langle n\rangle$ varies from 0 to $\infty$.

### F. Squeezed-vacuum state

The squeezed-vacuum state is generated from the vacuum state by the well-known squeeze operator [16]

$$\hat{S}(\zeta)\equiv\exp(\tfrac{1}{2}\zeta\hat{a}^\dagger\hat{a}^\dagger-\tfrac{1}{2}\bar{\zeta}\hat{a}\hat{a})\ , \tag{3.31}$$

with $\zeta=re^{i\theta}$ being a complex parameter. Let the squeezed-vacuum state be identified by $\zeta$ and denoted by $|\zeta\rangle\equiv\hat{S}(\zeta)|0\rangle$. It is well known that the unitary operator $\hat{S}(\zeta)$ imposes a Bogoliubov transformation [17] on the creation and annihilation operators according to the formulas

$$\hat{b}^\dagger\equiv\hat{S}^\dagger(\zeta)\hat{a}^\dagger\hat{S}(\zeta)=(\cosh r)\hat{a}^\dagger+e^{-i\theta}(\sinh r)\hat{a} \tag{3.32}$$

and

$$\hat{b}\equiv\hat{S}^\dagger(\zeta)\hat{a}\hat{S}(\zeta)=(\cosh r)\hat{a}+e^{i\theta}(\sinh r)\hat{a}^\dagger\ . \tag{3.33}$$

The moments can be expressed in terms of $\hat{b}$ and $\hat{b}^\dagger$ as

$$\mu_{k,l}\equiv\langle\zeta|(\hat{a}^\dagger)^l(\hat{a})^k|\zeta\rangle=\langle 0|(\hat{b}^\dagger)^l(\hat{b})^k|0\rangle\ . \tag{3.34}$$

As shown in Appendix C, we have

$$\hat{b}^{2j}|0\rangle=\sum_{n=0}^{j}\frac{(2j)!}{\sqrt{(2n)!}(j-n)!}(e^{i\theta}\sinh r)^{j+n}\times(\tfrac{1}{2}\cosh r)^{j-n}|2n\rangle \tag{3.35}$$

and

$$\hat{b}^{2j+1}|0\rangle=\sum_{n=0}^{j}\frac{(2j+1)!}{\sqrt{(2n+1)!}(j-n)!}(e^{i\theta}\sinh r)^{j+n+1}\times(\tfrac{1}{2}\cosh r)^{j-n}|2n+1\rangle\ . \tag{3.36}$$

So the moments $\mu_{k,l}$ are nonvanishing only if $k$ and $l$ are either both even numbers or both odd numbers. From Eqs. (3.35) and (3.36) we can easily obtain the expressions for the nonvanishing moments as follows:

$$\mu_{2j,2m}=\sum_{n=0}^{(j,m)}\frac{(2j)!(2m)!}{(2n)!(j-n)!(m-n)!}e^{i(j-m)\theta}\times(\sinh r)^{j+m+2n}(\tfrac{1}{2}\cosh r)^{j+m-2n} \tag{3.37}$$

and

$$\mu_{2j+1,2m+1}=\sum_{n=0}^{(j,m)}\frac{(2j+1)!(2m+1)!}{(2n+1)!(j-n)!(m-n)!}\times e^{i(j-m)\theta}(\sinh r)^{j+m+2n+2}\times(\tfrac{1}{2}\cosh r)^{j+m-2n}\ , \tag{3.38}$$

where $(j,m)\equiv\min(j,m)$ is the smaller of the two integers. Substitution of Eqs. (3.37) and (3.38) into Eq. (2.20) gives the $P$ function as

$$P_\zeta(z,\bar{z})=\pi\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{(j,m)}\frac{e^{i(j-m)\theta}}{(2n)!(j-n)!(m-n)!}(\sinh r)^{j+m+2n}(\tfrac{1}{2}\cosh r)^{j+m-2n}\times\left[\delta^{(2j,2m)}(z,\bar{z})+\frac{\sinh^2 r}{2n+1}\delta^{(2j+1,2m+1)}(z,\bar{z})\right], \quad (3.39)$$

which is a rather complicated expression. To search for a simpler one, let us calculate its Fourier transform. It is important to note that the expression for the summand in the above equation vanishes for $n>(j,m)$, so we can extend the upper limit of the summation over $n$ in the above equation to infinity without changing the result. This step will simplify the calculation tremendously. Using Eq. (2.12) and changing the order of summations, we obtain the Fourier transform as

$$\tilde{P}_\zeta(w,\bar{w})=\sum_{n=0}^{\infty}\sum_{j=n}^{\infty}\sum_{m=n}^{\infty}\frac{e^{i(j-m)\theta}}{(2n)!(j-n)!(m-n)!}\left[1-\frac{\sinh^2 r}{2n+1}\bar{w}w\right]\bar{w}^{2j}w^{2m}(\sinh r)^{j+m+2n}(\tfrac{1}{2}\cosh r)^{j+m-2n}$$

$$=\exp[\tfrac{1}{2}\sinh r\cosh r(e^{i\theta}\bar{w}^2+e^{-i\theta}w^2)-(\sinh r)^2 w\bar{w}]\ . \quad (3.40)$$

Using this expression in Eq. (2.29), we obtain the Fourier transform of the $R$ function for the squeezed-vacuum state to be of the form of a Gaussian function in the complex domain

$$\tilde{R}_\zeta(w,\bar{w},\tau)=\exp(aw^2-bw\bar{w}+c\bar{w}^2) \quad (3.41)$$

with

$$a=\tfrac{1}{2}e^{-i\theta}\sinh r\cosh r\ ,$$

$$b=\sinh^2 r+\tau\ , \quad (3.42)$$

$$c=\tfrac{1}{2}e^{i\theta}\sinh r\cosh r\ .$$

For the $R$ function to be a regular distribution function, the condition of Eq. (2.30) must be satisfied. This condition can be translated into the following three conditions: (i) $a=\bar{c}$, (ii) $b$ is real and positive, and (iii) $b^2-4ac>0$. The first two conditions are obviously satisfied. The third condition leads to the nonclassical depth of a squeezed-vacuum state as

$$\tau_m=(1-e^{-2r})/2\ , \quad (3.43)$$

which varies from 0 to $\frac{1}{2}$ as the squeeze parameter $r$ varies from 0 to $\infty$. There exists some similarity between the nonclassical depth of the squeezed-vacuum state and that of the two-photon thermal state.

## IV. SUMMARY

The moment problem of the $P$ function is to determine the $P$ function from a knowledge of its moments. We have solved the practical aspect of the problem by deriving an explicit expression for the $P$ function in terms of its moments and the derivatives of the Dirac $\delta$ function. From this expression, the complex Fourier transform of the $P$ function can also be obtained immediately. One important application of such an explicit expression for the $P$ function is to determine the nonclassical depth of a quantum state, a concept recently introduced by us [14]. As illustrations, we have carried out the calculations of the nonclassical depths for various specific quantum states, such as the photon-number state, the binomial state, the two-photon thermal state, the squeezed-vacuum state, etc. We note that the nonclassical depths of the photon-number state and the squeezed-vacuum state have been determined another way in Ref. [7].

## ACKNOWLEDGMENT

This work was supported by the U.S. Navy, Office of Naval Research, under Grant No. N00014-89-J-1050.

## APPENDIX A

We consider an arbitrary density matrix expressed in terms of Fock's states as

$$\hat{\rho}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho(n,m)|n\rangle\langle m|\ . \quad (A1)$$

We can use this density matrix to calculate the moments as

$$\mu_{k,l}=\mathrm{Tr}[\hat{\rho}(\hat{a}^\dagger)^l(\hat{a})^k]$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho(n,m)\langle m|(\hat{a}^\dagger)^l(\hat{a})^k|n\rangle$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho(n,m)\left[\frac{n!m!}{(n-k)!(m-l)!}\right]^{1/2}\delta_{n-k,m-1}\ . \quad (A2)$$

Using this expression in Eq. (2.20) we obtain the $P$ function as

$$P(z,\bar{z})=\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}\rho(n,m)\sqrt{n!}\begin{cases}\displaystyle\sum_{l=0}^{m}\frac{\delta^{(n-m+l,l)}(z,\bar{z})}{(m-l)!l!(n-m+l)!} & \text{if } n\geq m\\ \displaystyle\sum_{k=0}^{n}\frac{\delta^{(k,m-n+k)}(z,\bar{z})}{(n-k)!k!(m-n+k)!} & \text{if } n<m\ .\end{cases} \tag{A3}$$

On the other hand, Sudarshan, in his original paper on coherent-state representation [4], gives a formula for the $P$ function as

$$P^{s}(r,\theta)=\pi\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\rho(n,m)\sqrt{n!m!}}{(n+m)!(2\pi r)}\exp[r^2+i(m-n)\theta]\left[\left(-\frac{\partial}{\partial r}\right)^{n+m}\delta(r)\right], \tag{A4}$$

where a factor $\pi$ is added in order to be in conformity with our notation here. Equations (A3) and (A4) are quite different in appearance, and it is the purpose of this Appendix to show that they are equivalent in the sense of *new generalized functions.* This will be done by way of the $R$ function.

We first calculate the $R$ function according to Eq. (A3) as

$$R(r,\theta,\tau)=\frac{1}{\tau}e^{-r^2/\tau}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho(n,m)\exp[i(m-n)\theta]A_{n,m} \tag{A5}$$

with

$$A_{n,m}=\begin{cases}(-1)^m\left[\dfrac{m!}{n!}\right]^{1/2}\dfrac{(1-\tau)^m}{\tau^n}r^{n-m}L_m^{n-m}(r^2/\tau(1-\tau)) & \text{if } n\geq m\\ (-1)^n\left[\dfrac{n!}{m!}\right]^{1/2}\dfrac{(1-\tau)^n}{\tau^m}r^{m-n}L_n^{m-n}(r^2/\tau(1-\tau)) & \text{if } n<m\ ,\end{cases} \tag{A6}$$

where $L_n^{\alpha}(x)$ is the associated Laguerre polynomial.

We now calculate the $R$ function according to Eq. (A4). We have

$$R^{s}(r,\theta,\tau)=\frac{1}{2\pi\tau}e^{-r^2/\tau}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\rho(n,m)\frac{\sqrt{n!m!}}{(n+m)!}\int_0^{2\pi}d\theta'\exp[i(m-n)\theta']I_{m+n}\ , \tag{A7}$$

where

$$I_k\equiv\int_0^{\infty}dr'\exp\left[-\frac{1-\tau}{\tau}(r')^2+\frac{2rr'}{\tau}\cos(\theta-\theta')\right]\left[\left(-\frac{\partial}{\partial r'}\right)^{k}\delta(r')\right]$$

$$=\begin{cases}\displaystyle\sum_{l=0}^{k/2}C(k,2l)\left[-2\frac{1-\tau}{\tau}\right]^{k/2-l}\left[\frac{2r}{\tau}\cos(\theta-\theta')\right]^{2l} & \text{for even } k\\ \displaystyle\sum_{l=0}^{(k-1)/2}C(k,2l+1)\left[-2\frac{1-\tau}{\tau}\right]^{(k-1)/2-l}\left[\frac{2r}{\tau}\cos(\theta-\theta')\right]^{2l+1} & \text{for odd } k\ .\end{cases} \tag{A8}$$

With the understanding that $C(k,j)$ vanishes unless $k$ and $j$ are either both even numbers or both odd numbers, these coefficients satisfy the following partial-difference equation:

$$C(k,j)=C(k-1,j-1)+(j+1)C(k-1,j+1) \tag{A9}$$

under the boundary condition

$$C(k,k)=1\ . \tag{A10}$$

The solution to Eq. (A9) can be expressed as

$$C(k,j)=\frac{k!}{j!(k-j)!!}\ . \tag{A11}$$

Substituting Eqs. (A8) and (A11) into Eq. (A7), we are left with an integral of the form

$$\int_0^{2\pi} d\theta' \exp[i(m-n)\theta'][2\cos(\theta-\theta')]^{2l} = \begin{cases} \dfrac{2\pi(2l!\exp[i(m-n)\theta}{[l+(n-m)/2]![l-(n-m)/2]!} & \text{if } l \geq |n-m|/2 \\ 0 \quad \text{if } l < |n-m|/2 \end{cases} \tag{A12}$$

if $n+m$ is even, or of the form

$$\int_0^{2\pi} d\theta' \exp[i(m-n)\theta'][2\cos(\theta-\theta')]^{2l+1} = \begin{cases} \dfrac{2\pi(2l+1)!\exp[i(m-n)\theta]}{[l+(n-m+1)/2]![l-(n-m-1)/2]!} & \text{if } l \geq (|n-m|-1)/2 \\ 0 \quad \text{if } l < (|n-m|-1)/2 \end{cases} \tag{A13}$$

if $n+m$ is odd. Combining Eqs. (A7), (A8), (A11), (A12), and (A13) and considering four different possibilities, depending on whether $n \geq m$ or $n < m$ and whether $n+m$ is even or odd, we obtain the same expression for Eq. (A7) as given in Eq. (A5) together with Eq. (A6). Therefore we conclude that

$$R^s(r,\theta,\tau) = R(r,\theta,\tau) \ . \tag{A14}$$

Since the last equation holds for any positive value of $\tau$, we should have

$$\lim_{\tau\to 0} R^s(r,\theta,\tau) = \lim_{\tau\to 0} R(r,\theta,\tau) \ ; \tag{A15}$$

and we also have

$$\lim_{\tau\to 0} \frac{1}{\tau} \exp[-(z-w)(\bar{z}-\bar{w})/\tau] = \delta(z-w,\bar{z}-\bar{w}) \ , \tag{A16}$$

which is a reproducing kernel. Then we see that Eq. (A15) implies

$$\langle \delta(z-w,\bar{z}-\bar{w}), P^s(w,\bar{w}) \rangle = \langle \delta(z-w,\bar{z}-\bar{w}), P(w,\bar{w}) \rangle \ . \tag{A17}$$

These are convolution products of two distributions. Such products are impossible in the sense of the original definition by Schwartz [18], but they are justified in the sense of new generalized function [19]. Since the $z$ in Eq. (A17) can be any complex number, we conclude that $P^s(z,\bar{z})$ and $P(z,\bar{z})$ are equivalent.

## APPENDIX B

Cahill and Glauber [11] have derived a very useful integral identity in the complex domain

$$\frac{1}{\pi} \int \exp(\alpha\bar{z}+\beta z-\gamma|z|^2) d^2z = \frac{1}{\gamma} \exp\left[\frac{\alpha\beta}{\gamma}\right] \ . \tag{B1}$$

We are particularly interested in a special case of the identity when $\beta = -\bar{\alpha}$, which gives the Fourier transform of a Gaussian function $f(z,\bar{z}) = \exp(-\gamma|z|)$ as

$$\tilde{f}(\alpha,\bar{\alpha}) = \frac{1}{\pi} \int \exp(\alpha\bar{z}-\bar{\alpha}z-\gamma|z|^2) d^2z = \frac{1}{\gamma} \exp(-\alpha\bar{\alpha}/\gamma) \ . \tag{B2}$$

Using the above relation, we can easily obtain the Fourier transform of the function

$$f_k(z,\bar{z}) = |z|^{2k} \exp(-\gamma|z|^2) \tag{B3}$$

as

$$\begin{aligned} \tilde{f}_k(\alpha,\bar{\alpha}) &= \frac{1}{\gamma} \left[\frac{\partial}{\partial\alpha}\right]^k \left[-\frac{\partial}{\partial\bar{\alpha}}\right]^k \exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right] \\ &= k! \left[\frac{1}{\gamma}\right]^{k+1} \exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right] L_k\left[\frac{\alpha\bar{\alpha}}{\gamma}\right] \ . \end{aligned} \tag{B4}$$

Using Eq. (B4), we can further derive the Fourier transform of the function

$$g_n(z,\bar{z}) = \exp(-\gamma|z|^2) L_n(|z|^2) \tag{B5}$$

as

$$\begin{aligned} \tilde{g}_n(\alpha,\bar{\alpha}) &= \exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right] \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \left[\frac{1}{\gamma}\right]^{k+1} \sum_{j=0}^{k} \frac{1}{j!} \begin{bmatrix} k \\ j \end{bmatrix} \left[-\frac{\alpha\bar{\alpha}}{\gamma}\right]^j \\ &= \frac{1}{\gamma} \exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right] \sum_{j=0}^{n} \frac{1}{j!} \begin{bmatrix} n \\ j \end{bmatrix} \left[-\frac{\alpha\bar{\alpha}}{\gamma}\right]^j \sum_{k=j}^{n} \begin{bmatrix} n-j \\ k-j \end{bmatrix} \left[-\frac{1}{\gamma}\right]^k \\ &= \frac{1}{\gamma} \exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right] \sum_{j=0}^{n} \frac{1}{j!} \begin{bmatrix} n \\ j \end{bmatrix} \left[\frac{\alpha\bar{\alpha}}{\gamma^2}\right]^j \left[1-\frac{1}{\gamma}\right]^{n-j} \\ &= \frac{1}{\gamma} \left[1-\frac{1}{\gamma}\right]^n \exp\left[-\frac{\alpha\bar{\alpha}}{\gamma}\right] L_n\left[\frac{\alpha\bar{\alpha}}{\gamma(1-\gamma)}\right] \ . \end{aligned} \tag{B6}$$

## APPENDIX C

The purpose of this Appendix is to derive a general expression for the quantum state $\hat{b}^k|0\rangle$ in terms of photon-number states, where $\hat{b}$ is the operator defined by Eq. (3.33). To begin, let us work out the first few cases as follows:

$$\hat{b}|0\rangle = e^{i\theta}\sinh r|1\rangle ,$$
$$\hat{b}^2|0\rangle = (e^{i\theta}\sinh r)(\cosh r)|0\rangle + (e^{i\theta}\sinh r)^2\sqrt{2!}|2\rangle ,$$
$$\hat{b}^3|0\rangle = 3(e^{i\theta}\sinh r)^2(\cosh r)|1\rangle + (e^{i\theta}\sinh r)^3\sqrt{3!}|3\rangle ,$$
$$\hat{b}^4|0\rangle = 3(e^{i\theta}\sinh r)^2(\cosh r)^2|0\rangle + 6(e^{i\theta}\sinh r)^3(\cosh r)\sqrt{2!}|2\rangle + (e^{i\theta}\sinh r)^4\sqrt{4!}|4\rangle ,$$
$$\vdots \tag{C1}$$

These few cases reveal a pattern to suggest the following general form:

$$\hat{b}^{2j}|0\rangle = \sum_{n=0}^{j} C(2j,2n)(e^{i\theta}\sinh r)^{j+n} \times(\cosh r)^{j-n}\sqrt{(2n)!}|2n\rangle \tag{C2}$$

and

$$\hat{b}^{2j+1}|0\rangle = \sum_{n=0}^{j} C(2j+1,2n+1)(e^{i\theta}\sinh r)^{j+n+1} \times(\cosh r)^{j-n}\sqrt{(2n+1)!}|2n+1\rangle . \tag{C3}$$

On the other hand, the expression for $\hat{b}^{2j+1}|0\rangle$ can also be obtained by applying $\hat{b}$ to Eq. (C2) as

$$\hat{b}(\hat{b}^{2j}|0\rangle) = \sum_{n=0}^{j} [C(2j,2n)+(2n+2)C(2j,2n+2)] \times(e^{i\theta}\sinh r)^{j+n+1} \times(\cosh r)^{j-n}\sqrt{(2n+1)!}|2n+1\rangle , \tag{C4}$$

where we have imposed the condition

$$C(k,m)=0 \quad \text{for } k<m . \tag{C5}$$

Comparison of Eq. (C3) with Eq. (C4) yields the recurrence relation

$$C(2j+1,2n+1) = C(2j,2n)+(2n+2)C(2j,2n+2) . \tag{C6}$$

Similarly, we can also apply $\hat{b}$ to $\hat{b}^{2j-1}|0\rangle$ given by Eq. (C3) and compare the result with Eq. (C2) to obtain the relation

$$C(2j,2n) = C(2j-1,2n-1)+(2n+1)C(2j-1,2n+1) . \tag{C7}$$

With the understanding that $C(k,m)$ vanishes unless $k$ and $m$ are either both even numbers or both odd numbers, Eqs. (C6) and (C7) can be summarized by the partial-difference equation

$$C(k,m) = C(k-1,m-1)+(m+1)C(k-1,m+1) , \tag{C8}$$

which is identical to Eq. (A9) with the solution given in Eq. (A11).


[1] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, 1950).
[2] E. P. Wigner, Phys. Rev. **40**, 749 (1932).
[3] R. J. Glauber, Phys. Rev. Lett. **10**, 84 (1963).
[4] E. C. G. Sudarshan, Phys. Rev. Lett. **10**, 277 (1963).
[5] J. E. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).
[6] F. J. Narcowich, J. Math. Phys **28**, 2873 (1987).
[7] C. T. Lee, Phys. Rev. A **44**, 2775 (1991).
[8] R. D. Morten and A. M. Krall, SIAM J. Math. Anal. **9**, 604 (1978).
[9] R. Loudon, Rep. Prog. Phys. **43**, 58 (1980).
[10] For a short introduction to the theory of generalied functions of one and several complex variables, see, for example, I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. I, Appendix B.
[11] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1857 (1969).
[12] M. Hillery, Phys. Lett. **111A**, 409 (1985).
[13] See, for example, F. Oberhettinger, *Fourier Transforms of Distributions and Their Inverses* (Academic, New York, 1973).
[14] C. T. Lee, Phys. Rev. A **31**, 1213 (1985).
[15] K. E. Cahill and R. J. Glauber, Phys. Rev. **177**, 1882 (1969).
[16] D. Stoler, Phys. Rev. D **1**, 3217 (1970).
[17] N. Bogoliubov, J. Phys. (Moscow) **11**, 292 (1947).
[18] L. Schwartz, C. R. Acad. Sci. **239**, 847 (1954).
[19] J. F. Colombeau, *New Generalized Functions and Multiplication of Distributions,* North-Holland Mathematical Studies, No. 84 (North-Holland, New York, 1984).