Superpositions of coherent states: Squeezing and dissipation

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In this paper we discuss the nonclassical properties of quantum superpositions of coherent states of light. Using general expressions for the Wigner functions of superposition states we analyze the consequences of quantum interference between coherent states. We describe in detail nonclassical properties of a superposition of two coherent states. In particular, we study the oscillatory behavior of the photon number distribution of the even and odd coherent states. We show under which conditions a superposition of two coherent states can exhibit second- and fourth-order squeezing or sub-Poissonian photon statistics. We examine the sensitivity of nonclassical effects such as oscillations in the photon number distribution or second-order squeezing to dissipation. We demonstrate that quantities such as the photon number distribution and interferences in phase space are highly sensitive to even a quite small dissipative coupling, because they depend on all moments of the field observables, and higher moments decay more rapidly than lower moments. Quantities such as quadrature squeezing, on the other hand, are more robust against dissipation because they involve only lower moments. Finally, we find a remarkable effect whereby fourth-order squeezing is generated by damping.

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I. INTRODUCTION

The linear superposition principle is one of the most fundamental features of quantum mechanics [1]. In particular, the interference of quantum amplitudes results from this principle. It has been realized recently that the interference between states of light composing a quantum superposition state gives rise to various nonclassical effects [2–9]. In particular, it has been shown that squeezing (i.e., a reduction of quadrature fluctuations below the level associated with the vacuum [10]), higher-order squeezing [11], as well as sub-Poissonian photon statistics [12] and oscillations of the photon number distribution [9], emerge from a superposition of coherent states.

There have been several proposals recently for the generation of optical superposition states in various nonlinear processes [13–16] and in quantum nondemolition or back-action-evading measurements [17]. It has been pointed out by Yurke and Stoler [13] that, in the presence of low dissipation, a nonlinear system may convert a coherent state (CS), i.e., the state which is associated with the "most" classical state of light one can imagine in the framework of quantum theory [18], into a quantum superposition of macroscopically distinguishable states. As an example of such a nonlinear system, one can imagine a Kerr-like medium (for instance, an optical fibre) modeled as an anharmonic oscillator [13,19–23] with the Hamiltonian ($\hbar=1$)

$$\hat{H}=\omega(\hat{a}^\dagger\hat{a}+\tfrac{1}{2})+\chi(\hat{a}^\dagger\hat{a})^2 , \tag{1.1}$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of a photon of a single-mode electromagnetic field ($[\hat{a},\hat{a}^\dagger]=1$) and the coupling constant χ is related to the dispersive part of the third-order nonlinearity of a Kerr medium. If the light field is initially ($t=0$) in a coherent state $|\alpha\rangle$.

$$|\alpha\rangle=\hat{D}(\alpha)|0\rangle=\exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle ,$$
$$\hat{D}(\alpha)=\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a}) , \tag{1.2}$$

where $|0\rangle$ is the vacuum state of a harmonic oscillator and $\hat{D}(\alpha)$ is a displacement operator, then at time t it evolves into the state

$$|\Psi(t)\rangle=\exp(-|\alpha|^2/2)\times\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp(-i\omega nt-in^2\chi t)|n\rangle . \tag{1.3}$$

This state can be identified with a particular realization of a generalized coherent state introduced by Titulaer and Glauber [24] and later studied by Stoler [25] and Bialynicka-Birula [26]. One can find that, at the time $t_R=\pi/2\chi$, the state given by Eq. (1.3) in an appropriate rotating frame takes the following form (apart from an overall phase factor):

$$|\alpha\rangle_{\rm YS}=\frac{1}{\sqrt{2}}(|\alpha\rangle+e^{i\pi/2}|-\alpha\rangle) , \tag{1.4}$$

i.e., the initial coherent state has evolved into a quantum superposition of two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ which are 180° out of phase with respect to each other. In what follows we will refer to the state (1.4) as to the Yurke-Stoler (YS) coherent state [13].

Recently Phoenix and Knight [27] have shown that a single-mode electromagnetic field interacting with a single two-level atom described in the framework of the Jaynes-Cummings model (JCM) [28] evolves into an almost pure state at one-half of the revival time t_R. Gea-Banacloche [16,29] has pointed out that, in the JCM, a single-mode field, which is initially prepared in the coherent state, can nonunitarily evolve into a quantum superposition of two component states (the other example of the generation of quantum superpositions by nonunitary evolution is that of quantum nondemolition measurements [17]). Depending on the initial photon number, one can find that at one-half of the revival time of the atomic inversion the field evolves into an almost pure state [16,27,29], which can be approximately described either as an even coherent state,

$$|\alpha\rangle_e = A_e^{1/2}(|\alpha\rangle + |-\alpha\rangle) ,$$
$$A_e^{-1} = 2[1+\exp(-2|\alpha|^2)] , \quad (1.5)$$

or as an odd coherent state,

$$|\alpha\rangle_o = A_o^{1/2}(|\alpha\rangle - |-\alpha\rangle) ,$$
$$A_o^{-1} = 2[1-\exp(-2|\alpha|^2)] . \quad (1.6)$$

The Yurke-Stoler state and the even and odd coherent states belong to a wider class of quantum superposition states given by the relation

$$|\Psi\rangle = A^{1/2}\left[\sum_{j=1}^{N} e^{i\varphi_j}|\alpha_j\rangle\right] ,$$
$$A^{-1} = \sum_{j,k=1}^{N} e^{i(\varphi_j-\varphi_k)}\langle\alpha_k|\alpha_j\rangle . \quad (1.7)$$

The main purpose of our paper is to clarify the origin of squeezing and other nonclassical effects in a superposition of coherent states of light given by Eq. (1.7) and to study the influence of damping on these effects. For the most part, we are concerned with small values of α appropriate to the Jaynes-Cummings model rather than to nonlinear optical interactions.

Recently Schleich *et al.* [9] have made an extensive study of the nonclassical state produced from a superposition of two coherent states; the state they employ is a special case of Eq. (1.7),

$$|\Psi\rangle = N\frac{1}{\sqrt{2}}(|\alpha e^{i\phi/2}\rangle + |\alpha e^{-i\phi/2}\rangle) , \quad (1.8)$$

with equal angles φ_j. As we will show, the interference between the component states is a sensitive function of the relative phase difference and can generate a number of novel effects. In addition, we show that interferences between states of different amplitudes are very sensitive to dissipation and we show that, for our superpositions considered here, dissipation very rapidly destroys effects such as oscillations in the photon number distribution dependent on these interferences. We show that, in contrast, quadrature squeezing is relatively robust against dissipation. We explain this differential sensitivity in terms of the various decay rates of moments of observables on which the relevant physical quantities depend. Such effects have not, to our knowledge, been dealt with previously.

This paper is organized as follows: In Sec. II, we briefly summarize some basic definitions which we use in the course of the paper. In Sec. III, we develop a general formalism to describe superpositions of coherent states. In Sec. IV, we discuss the properties of statistical mixtures of two coherent states. In Sec. V, we will analyze the influence of damping on nonclassical properties of superposition of coherent states.

II. NOTATIONS AND BASIC DEFINITIONS

A. The density operator, pure states and statistical mixtures

If $|\Psi_i\rangle$ is a state vector of a quantum-mechanical system (for instance, a harmonic oscillator) and $\hat{M}$ is an arbitrary operator, then the expectation value of $\hat{M}$ is given by

$$\langle\hat{M}\rangle = \langle\Psi_i|\hat{M}|\Psi_i\rangle \quad (2.1)$$

or

$$\langle\hat{M}\rangle = \mathrm{Tr}[\hat{M}|\Psi_i\rangle\langle\Psi_i|] . \quad (2.2)$$

If we know that the quantum-mechanical system is in the state $|\Psi_i\rangle$ with a probability p_i, where

$$\sum_i p_i = 1, \quad p_i \geq 0 , \quad (2.3)$$

we should thus average (2.1) over that probability distribution:

$$\langle\hat{M}\rangle = \sum_i p_i \mathrm{Tr}[\hat{M}|\Psi_i\rangle\langle\Psi_i|] . \quad (2.4)$$

We can define the density operator $\hat{\rho}$ as

$$\hat{\rho} = \sum_i p_i|\Psi_i\rangle\langle\Psi_i| , \quad (2.5)$$

so that

$$\langle\hat{M}\rangle = \mathrm{Tr}[\hat{\rho}\hat{M}] . \quad (2.6)$$

The density operator $\hat{\rho}$ is Hermitian

$$\hat{\rho}^\dagger = \hat{\rho} \quad (2.7)$$

and its trace is equal to unity:

$$\mathrm{Tr}\hat{\rho} = 1 . \quad (2.8)$$

The diagonal matrix elements of $\hat{\rho}$ are real and positive in any representation.

If we know the state of the system precisely, i.e., we know that the system is in the state $|\Psi\rangle$ (i.e., $p_i = \delta_{ij}$), then we have

$$\hat{\rho}^2 = \hat{\rho} \quad (2.9)$$

and

$$\mathrm{Tr}\hat{\rho}^2 = 1 . \quad (2.10)$$

When the last condition is satisfied, we say that the quantum-mechanical system is in a pure state.

For a mixed state (statistical mixture), we do not know the state of the system precisely, and the state of the system is characterized by the $\hat{\rho}$ operator (2.5), for which

$$\mathrm{Tr}\hat{\rho}^2 < 1 . \quad (2.11)$$

To measure the degree of purity of a quantum-mechanical state, it is often convenient to use the concept of the entropy S [27], defined as

$$S=\mathrm{Tr}[\hat{\rho}\ln\hat{\rho}] .$$

For a pure state the entropy is equal to zero, while for a statistical mixture the entropy is strictly positive. The superposition of coherent states (1.7) is an example of a pure state with the density matrix

$$\hat{\rho}=A\left[\sum_{i,j=1}^{N}\exp[i(\varphi_i-\varphi_j)]|\alpha_i\rangle\langle\alpha_j|\right]$$
$$=A\left[\sum_{i=1}^{N}|\alpha_i\rangle\langle\alpha_i|+\sum_{\substack{i=1\\ i\neq j}}^{N}\exp[i(\varphi_i-\varphi_j)]|\alpha_i\rangle\langle\alpha_j|\right] , \tag{2.12}$$

while the density operator

$$\hat{\rho}=\sum_{i=1}^{N}p_i|\alpha_i\rangle\langle\alpha_i|,\quad \sum_{i=1}^{N}p_i=1 \tag{2.13}$$

describes a statistical mixture of coherent states $|\alpha_i\rangle$.

B. Characteristic functions and quasiprobability distributions

The state of the quantum-mechanical system is characterized by the set of expectation values of the system operators. In particular, those of a harmonic oscillator are described by the mean values (moments) of the boson operators $\hat{a}$ and $\hat{a}^\dagger$. Generally, the moments of the bosonic operators are given in the normally ordered form $\langle(\hat{a}^\dagger)^m\hat{a}^n\rangle$, antinormally ordered form $\langle\hat{a}^n(\hat{a}^\dagger)^m\rangle$, and symmetrical form $\langle\{(\hat{a}^\dagger)^m\hat{a}^n\}\rangle$, [30], and can be evaluated with the help of normal, antinormal, and symmetric characteristic functions [31]:

$$\langle(\hat{a}^\dagger)^m\hat{a}^n\rangle=\frac{\partial^{(m+n)}}{\partial\xi^m\partial(-\xi^*)^n}C^{(n)}(\xi)\bigg|_{\xi=0} , \tag{2.14a}$$

$$\langle\hat{a}^n(\hat{a}^\dagger)^m\rangle=\frac{\partial^{(m+n)}}{\partial\xi^m\partial(-\xi^*)n}C^{(a)}(\xi)\bigg|_{\xi=0} , \tag{2.14b}$$

$$\langle\{(\hat{a}^\dagger)^m\hat{a}^n\}\rangle=\frac{\partial^{(m+n)}}{\partial\xi^m\partial(-\xi^*)^n}C^{(w)}(\xi)\bigg|_{\xi=0} , \tag{2.14c}$$

where

$$C^{(n)}(\xi)=\mathrm{Tr}[\hat{\rho}\exp(\xi\hat{a}^\dagger)\exp(-\xi^*\hat{a})] , \tag{2.15a}$$

$$C^{(a)}(\xi)=\mathrm{Tr}[\hat{\rho}\exp(-\xi^*\hat{a})\exp(\xi\hat{a}^\dagger)] , \tag{2.15b}$$

$$C^{(w)}(\xi)=\mathrm{Tr}[\hat{\rho}\exp(\xi\hat{a}^\dagger-\xi^*\hat{a})]=\mathrm{Tr}[\hat{\rho}\hat{D}(\xi)] . \tag{2.15c}$$

The displacement operator $\hat{D}(\xi)$ is given by Eq. (1.2) and

$$\mathrm{Tr}[\hat{\rho}\hat{D}(\xi)]=\frac{1}{\pi}\int d^2\alpha\langle\alpha|\hat{\rho}\hat{D}(\xi)|\alpha\rangle . \tag{2.16}$$

Using the Baker-Hausdorff formula [32] we can show that the characteristic functions defined by Eqs. (2.15) are related as follows:

$$C^{(w)}(\xi)=\exp(-|\xi|^2/2)C^{(n)}(\xi)$$
$$=\exp(|\xi|^2/2)C^{(a)}(\xi) . \tag{2.17}$$

Instead of the three different characteristic functions mentioned above, we can use the s-parametrized characteristic function introduced by Cahill and Glauber [31],

$$C(\xi,s)=\mathrm{Tr}[\hat{\rho}\exp(\xi\hat{a}^\dagger-\xi^*\hat{a}+s|\xi|^2/2)] , \tag{2.18}$$

from which we find

$$C(\xi,s=1)=C^{(n)}(\xi) ,$$
$$C(\xi,s=-1)=C^{(a)} , \tag{2.19}$$
$$C(\xi,s=0)=C^{(w)}(\xi) .$$

It can be shown that the $\hat{\rho}$ matrix may be determined uniquely from the characteristic functions $C(\xi,s)$, which means that the knowledge of the density operator is equivalent to the knowledge of the complete set of moments of the system operators and vice versa [33].

The quasiprobability distributions may be defined as Fourier transforms of the characteristic functions

$$W(\beta,s)=\frac{1}{\pi^2}\int C(\xi,s)\exp(\beta\xi^*-\beta^*\xi)d^2\xi . \tag{2.20}$$

The quasiprobability distribution $W(\beta,s)$ with $s=1$ is the Glauber-Sudarshan P function, $W(\beta,0)$ is the Wigner function [32,34], and $W(\beta,-1)=\langle\alpha|\hat{\rho}|\alpha\rangle/\pi$ is the Q function. The mean values of the s-ordered products $\langle[(\hat{a}^\dagger)^m\hat{a}^n]_s\rangle$ can be obtained by proper integration with weight $W(\beta,s)$ in the complex β plane. For instance, the symmetric mean value

$$\langle\{(\hat{a}^\dagger)^m\hat{a}^n\}\rangle\equiv\langle[(\hat{a}^\dagger)^m\hat{a}^n]_{s=0}\rangle$$

can be obtained by using the Wigner function $W(\beta)\equiv W(\beta,0)$:

$$\langle\{(\hat{a}^\dagger)^m\hat{a}^n\}\rangle=\int d^2\beta\beta^{*m}\beta^nW(\beta) . \tag{2.21}$$

The quasiprobability distribution functions are uniquely defined by the $\hat{\rho}$ operator, and in this sense they characterize the state of the quantum-mechanical system. That is, they implicitly contain information about all moments of the system operators.

From the Wigner function (2.20) we can also evaluate the photon number distribution P_n defined as

$$P_n=\langle n|\hat{\rho}|n\rangle . \tag{2.22}$$

To do so, we note that the $\hat{\rho}$ matrix can be expressed as an inverse Fourier transform of the characteristic function [31]:

$$\hat{\rho}=\frac{1}{\pi}\int\mathrm{Tr}[\hat{\rho}\hat{D}(\xi)]\hat{D}^{-1}(\xi)d^2\xi$$
$$=\frac{1}{\pi}\int d^2\xi\, C^{(w)}(\xi)\hat{D}^{-1}(\xi) , \tag{2.23}$$

and therefore implicitly involves all moments of the relevant field observables; as we shall see later, this has important consequences for the survival of interferences and nonclassical properties when dissipation is present. On the other hand, the characteristic function can be expressed as an inverse Fourier transform of the Wigner function:

$$C^{(w)}(\xi)=\int d^2\alpha \exp(\alpha^*\xi-\alpha\xi^*)W(\beta) \ . \tag{2.24}$$

Inserting Eq. (2.24) into the relation (2.23), we obtain for P_n the following expression:

$$P_n=\pi\int d^2\beta \, W(\beta)W_n(\beta) \ , \tag{2.25}$$

where $W_n(\beta)$ is the Wigner function of the number state $|n\rangle$ [31]:

$$W_n(\beta)=\frac{2(-1)^n}{\pi}\exp(-2|\beta|^2)\mathcal{L}_n(4|\beta|^2) \tag{2.26}$$

and $\mathcal{L}_n(x)$ is the Laguerre polynomial of order n [35].

C. Sub-Poissonian photon statistics

One of the best known nonclassical effects is the generation of sub-Poisonian photon statistics of the light field [12]. A coherent field $|\alpha\rangle$, which can be regarded as a field with the "most" classical behavior, yields Poissonian photon statistics, i.e., the variance $\langle(\Delta\hat{n})^2\rangle$ is equal to the mean photon number $\langle\hat{n}\rangle=|\alpha|^2$. If the variance of $\hat{n}$ is less than $\langle n\rangle$, then a state of light has no classical description via the P function [see definition (2.20) with $s=1$]. This because the P function is not a probability density [12]. Following Mandel [12], we can introduce the Q parameter

$$Q=\frac{\langle(\Delta\hat{n})^2\rangle-\langle\hat{n}\rangle}{\langle\hat{n}\rangle} \ , \tag{2.27}$$

which characterizes the departure from Poissonian photon statistics. The state of light is called Poissonian if $Q=0$. When $Q>0$, the state is called super-Poissonian. If Q has some value between 0 and -1 then the state is called sub-Poissonian. For the number state of light $|n\rangle$, $Q=-1$.

D. Second-order quadrature squeezing

In order to study light squeezing, we introduce two quadrature operators $\hat{a}_1$ and $\hat{a}_2$ corresponding to the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ of the field mode under consideration:

$$\hat{a}_1=\frac{\hat{a}+\hat{a}^\dagger}{2}, \quad \hat{a}_2=\frac{\hat{a}-\hat{a}^\dagger}{2i} \ . \tag{2.28}$$

Operators $\hat{a}^\dagger$ and $\hat{a}$ obey the ordinary bosonic commutation relation $[\hat{a},\hat{a}^\dagger]=1$ (we adopt $\hbar=1$) and, hence,

$$[\hat{a}_1,\hat{a}_2]=2iC, \quad C=\tfrac{1}{4} \ . \tag{2.29}$$

One of the consequences of the commutation relation (2.29) is the uncertainty relation for the variances of the quadrature operators:

$$\langle(\Delta\hat{a}_1)^2\rangle\langle(\Delta\hat{a}_2)^2\rangle\ge C^2=\tfrac{1}{16} \ , \tag{2.30}$$

where the variance of the operator $\hat{a}_i$ is defined as $\langle(\Delta\hat{a}_i)^2\rangle=\langle\hat{a}_i^2\rangle-\langle\hat{a}_i\rangle^2$ and is related to the normally ordered variance $\langle:(\Delta\hat{a}_i)^2:\rangle$ as follows:

$$\langle(\Delta\hat{a}_i)^2\rangle=C+\langle:(\Delta\hat{a}_i)^2:\rangle \ . \tag{2.31}$$

The state for which the equality in Eq. (2.30) holds is called the minimum uncertainty state (MUS). For instance, the vacuum and the coherent states of light are examples of MUS. For these states the variances in both quadratures are equal to $\frac{1}{4}$. The state is called squeezed if the variance of the quadrature operator is less than the vacuum fluctuations (i.e., less than $\frac{1}{4}$). Alternatively, one can say that the state is squeezed if the normally ordered variance is less than zero. It is not necessary for the squeezed state to be a MUS. Nevertheless, the squeezed vacuum state (see definition below) is a MUS.

The variances of the quadrature operators can be written as

$$\langle(\Delta\hat{a}_1)^2\rangle=\tfrac{1}{4}+\tfrac{1}{2}[\langle\hat{a}^\dagger\hat{a}\rangle+\mathrm{Re}\langle\hat{a}^2\rangle-2(\mathrm{Re}\langle\hat{a}\rangle)^2] \ , \tag{2.32a}$$

$$\langle(\Delta\hat{a}_2)^2\rangle=\tfrac{1}{4}+\tfrac{1}{2}[\langle\hat{a}^\dagger\hat{a}\rangle-\mathrm{Re}\langle\hat{a}^2\rangle-2(\mathrm{Im}\langle\hat{a}\rangle)^2] \ , \tag{2.32b}$$

from which it follows that squeezing can appear only if the expectation values $\langle\hat{a}\rangle$ and/or $\langle\hat{a}^2\rangle$ are nonzero (of course, this is not a sufficient condition for observation of squeezing). The nonzero values of $\langle\hat{a}\rangle$ and/or $\langle\hat{a}^2\rangle$ are associated with the off-diagonal terms of the density matrix in the number-state basis. In other words, they appear due to the quantum interference between the number states $|n-1\rangle$ and $|n\rangle$ and $|n-2\rangle$ and $|n\rangle$, respectively (see Refs. [2,8]).

To measure the degree of quadrature squeezing, we can introduce two squeezing parameters $S_i^{(2)}$:

$$\begin{aligned} S_1^{(2)} &= \frac{\langle(\Delta\hat{a}_1)^2\rangle-C}{C} \\ &= \frac{1}{C}\langle:(\Delta\hat{a}_1)^2:\rangle \\ &= 2[\langle\hat{a}^\dagger\hat{a}\rangle+\mathrm{Re}\langle\hat{a}^2\rangle-2(\mathrm{Re}\langle\hat{a}\rangle)^2] \end{aligned} \tag{2.33a}$$

and

$$\begin{aligned} S_2^{(2)} &= \frac{\langle(\Delta\hat{a}_2)^2\rangle-C}{C} \\ &= \frac{1}{C}\langle:(\Delta\hat{a}_2)^2:\rangle \\ &= 2[\langle\hat{a}^\dagger\hat{a}\rangle-\mathrm{Re}\langle\hat{a}^2\rangle-2(\mathrm{Im}\langle\hat{a}\rangle)^2] \ . \end{aligned} \tag{2.33b}$$

The squeezing condition now reads $S_i^{(2)}<0$, and the maximum squeezing corresponds to $S_i^{(2)}=-1$ or, equivalently, $\langle:(\Delta\hat{a}_i)^2:\rangle=-\frac{1}{4}$.

E. Higher-order squeezing

Quadrature squeezing as discussed above is based on properties of the second-order moments of quadrature operators which are related to the field fluctuations. As discussed by Hong and Mandel [11], the higher-order moments of the field can exhibit a nonclassical behavior called higher-order squeezing, that is, when the Nth-order moment of the quadrature operator $\langle(\Delta\hat{a}_i)^N\rangle$ is smaller than its value in a completely coherent state of the field. In what follows we shall consider only the moments of even order because only in those cases will the state necessarily exhibit nonclassical behavior [11].

The Nth moment of $\Delta\hat{a}_1$ can be written as [11]

$$\langle(\Delta\hat{a}_i)^N\rangle=\sum_{l=0}^{N/2-1}\frac{N^{(2l)}}{l!}\left[\frac{C}{2}\right]^l\langle:(\Delta\hat{a}_i)^{N-2l}:\rangle+C^{N/2}(N-1)!! , \qquad (2.34)$$

where for even N we use the notation $N^{(r)}=N(N-1)\cdots(N-r+1)$. Due to the fact that all the normally ordered moments $\langle:(\Delta\hat{a}_i)^m:\rangle$ vanish for a coherent state, the field is squeezed to order N if

$$\langle(\Delta\hat{a}_i)^N\rangle<C^{N/2}(N-1)!! , \qquad (2.35)$$

which means that the squeezing condition can be written as

$$\sum_{l=0}^{N/2-1}\frac{N^{(2l)}}{l!}\left[\frac{C}{2}\right]^l\langle:(\Delta\hat{a}_i)^{N-2l}:\rangle<0 . \qquad (2.36)$$

In order to measure the degree of Nth-order squeezing, we introduce the squeezing parameters $S_i^{(N)}$:

$$S_i^{(N)}=\frac{\langle(\Delta\hat{a}_i)^N\rangle-C^{N/2}(N-1)!!}{C^{N/2}(N-1)!!}=\sum_{l=0}^{N/2-1}\frac{N^{(2l)}(C/2)^l\langle:(\Delta\hat{a}_i)^{N-2l}:\rangle}{l!C^{N/2}(N-1)!!} . \qquad (2.37)$$

The Nth-order-squeezing condition is $S_i^{(N)}<0$. The maximum (100%) Nth-order squeezing corresponds to $S_i^{(N)}=-1$.

III. SUPERPOSITIONS OF COHERENT STATES

Let us consider the superposition $|\Psi\rangle$ of coherent states $|\alpha\rangle$ given by the relation (1.7), i.e.,

$$|\Psi\rangle=A^{1/2}\left[\sum_{j=1}^{N}e^{i\varphi_j}|\alpha_j\rangle\right], \quad \alpha_j=\alpha_j^x+i\alpha_j^y ,$$

where $A^{1/2}$ is the normalization constant. The phases φ_j are arbitrary and, as we will show later, their values determine whether the quantum interference among the coherent states $|\alpha_j\rangle$ is constructive or destructive [36,37], which results in observable effects (e.g., squeezing or sub-Poissonian photon statistics).

The normalization constant $A^{1/2}$ can be written as

$$A=\left[N+2\sum_{\substack{j,k=1\\k>j}}^{N}\exp(-\tfrac{1}{2}|\alpha_j-\alpha_k|^2)\cos[(\varphi_k-\varphi_j)+\alpha_j\otimes\alpha_k]\right]^{-1} , \qquad (3.1)$$

where we have used the notation $\otimes$ for the antisymmetric product of two two-dimensional vectors (α_j^x,α_k^y) and (α_k^x,α_k^y):

$$\alpha_j\otimes\alpha_k\equiv\alpha_j^x\alpha_k^y-\alpha_j^y\alpha_k^x . \qquad (3.2a)$$

The symmetric product of two two-dimensional vectors, which we will use later, is defined as

$$\alpha_j\cdot\alpha_k\equiv\alpha_j^x\alpha_k^x+\alpha_j^y\alpha_k^y . \qquad (3.2b)$$

The density matrix $\hat{\rho}$ corresponding to the pure state (1.7) is

$$\hat{\rho}=A\left[\sum_{j=1}^{N}|\alpha_j\rangle\langle\alpha_j|+\sum_{\substack{k,j=1\\k\neq j}}^{N}e^{i(\varphi_j-\varphi_k)}|\alpha_j\rangle\langle\alpha_k|\right] . \qquad (3.3)$$

The characteristic function $C^{(w)}(\xi)$, given by Eq. (2.15c), related to the density matrix (3.3) is

$$C^{(w)}(\xi)=A\left[\sum_{j=1}^{N}C_j^{(w)}(\xi;\alpha_j)+\sum_{\substack{j,k=1\\j\neq k}}^{N}C_{jk}^{(w)}(\xi;\alpha_j,\alpha_k)\right] , \qquad (3.4)$$

where $C_j^{(w)}(\xi;\alpha)$ is the characteristic function for the coherent state $|\alpha_j\rangle$:

$$C_j^{(w)}(\xi;\alpha_j)=\langle\alpha_j|\hat{D}(\xi)|\alpha_j\rangle=\exp(-\tfrac{1}{2}|\xi|^2+2i\alpha_j\otimes\xi) \qquad (3.5)$$

and

$$C_{jk}^{(w)}(\xi;\alpha_j,\alpha_k)=\langle\alpha_k|\hat{D}(\xi)|\alpha_j\rangle e^{i(\varphi_j-\varphi_k)}=\exp[i(\varphi_j-\varphi_k)]\times\exp[-\tfrac{1}{2}|\xi-(\alpha_k-\alpha_j)|^2+i\alpha_j\otimes\alpha_k+i\alpha_k\otimes\xi+i\alpha_j\otimes\xi] . \qquad (3.6)$$

The corresponding Wigner function $W(\beta;\alpha_1,\ldots,\alpha_N)$ [see Eq. (2.20)] can be written in the form

$$W(\beta;\alpha_1,\ldots,\alpha_N)=A\left[\sum_{j=1}^{N}W_j(\beta;\alpha_j)+2\sum_{\substack{j,k=1\\j>k}}^{N}W_{jk}(\beta;\alpha_j,\alpha_k)\right] , \qquad (3.7)$$

where $W_j(\beta;\alpha)$ is the Wigner function corresponding to the coherent state $|\alpha_j\rangle$:

$$W_j(\beta;\alpha_j)=\frac{2}{\pi}\exp(-2|\beta-\alpha_j|^2) , \qquad (3.8)$$

and $W_{jk}(\beta;\alpha_j,\alpha_k)$ is the quasiprobability distribution emerging from the quantum interference between the coherent states $\exp(i\varphi_j)|\alpha_j\rangle$ and $\exp(i\varphi_k)|\alpha_k\rangle$:

$$W_{jk}(\beta;\alpha_j,\alpha_k)=\frac{2}{\pi}\exp[-\tfrac{1}{2}|\alpha_j-\alpha_k|^2 -2(\alpha_j-\beta)\cdot(\alpha_k-\beta)] \times\cos[\varphi_k-\varphi_j-\alpha_k\otimes\alpha_j +2(\alpha_k-\beta)\otimes(\alpha_j-\beta)] \ . \quad (3.9a)$$

After some algebra we can rewrite the function $W_{jk}(\beta;\alpha_j,\alpha_k)$ in the form

$$W_{jk}(\beta;\alpha_j,\alpha_k)=\frac{2}{\pi}\exp\left[-2\left|\beta-\frac{\alpha_j+\alpha_k}{2}\right|^2\right] \times\cos[\varphi_k-\varphi_j+2\beta\otimes\alpha_k -2\beta\otimes\alpha_j+\alpha_k\otimes\alpha_j] \ . \quad (3.9b)$$

The oscillatory behavior of the interference part given by Eq. (3.9b) of the Wigner function plays a crucial role in the appearance of nonclassical effects (for details see Ref. [6]). For comparison purposes we write down the Wigner function of the statistical mixture described by the $\hat{\rho}$ matrix in Eq. (2.13):

$$W(\beta;\alpha_1,\ldots,\alpha_N)=\sum_{j=1}^{N}p_jW_j(\beta,\alpha_j) \ ,$$

where $W(\beta;\alpha_j)$ is given by Eq. (3.8). This Wigner function does not contain a quantum interference term and therefore does not describe nonclassical effects.

For the photon number distribution (2.25) of the superposition of the coherent states (1.7), we find the following expressions:

$$P_n=A\left[\sum_{j=1}^{N}P_n^{(j)}+2\sum_{\substack{j,k=1\\ j>k}}P_n^{(jk)}\right] \ , \quad (3.10)$$

where

$$P_n^{(j)}=\exp(-|\alpha_j|^2)\frac{|\alpha_j|^2}{n!} \quad (3.11)$$

is the Poissonian distribution corresponding to the coherent state $|\alpha_j\rangle$, while $P_n^{(jk)}$ arises due to the interference between the states $|\alpha_j\rangle$ and $|\alpha_k\rangle$ and takes the form

$$P_n^{(jk)}=\frac{\exp[-(1/2)(|\alpha_k|^2+|\alpha_j|^2)]}{n!}|\alpha_j|^n|\alpha_k|^n \times\cos(\varphi_j-\varphi_k-n\varphi_{jk}) \ , \quad (3.12)$$

with φ_{jk} defined as

$$\tan(\varphi_{jk})=\frac{\alpha_j\otimes\alpha_k}{\alpha_j\cdot\alpha_k} \ . \quad (3.13)$$

The oscillatory behavior of the photon number distribution emerging from the quantum interference term (3.12) has no classical analog (see also Sec. IV) and is absent in the case of the statistical mixture (2.13), for which the photon number distribution reads

$$P_n=\sum_{j=1}^{N}p_jP_n^{(j)} \ . \quad (3.14)$$

As seen from Eq. (3.7), the quantum-interference terms $W_{jk}(\beta,\alpha_j,\alpha_k)$ in the Wigner function of the superposition of N coherent states arise due to the interference between the pairs of coherent states $|\alpha_j\rangle$ and $|\alpha_k\rangle$. Therefore, in what follows we will study nonclassical effects which appear in a superposition of just two coherent states.

IV. SUPERPOSITION OF A PAIR OF COHERENT STATES

In the previous section we have derived general expressions for the Wigner functions of superpositions of N coherent states (1.7). To clarify our analysis, in what follows we study the nonclassical properties associated with a superposition of two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ (for simplicity we suppose α to be real):

$$|\Psi\rangle=\{2[1+\cos\varphi\exp(-2\alpha^2)]\}^{-1/2} \times[|\alpha\rangle+e^{i\varphi}|-\alpha\rangle] \ . \quad (4.1)$$

Obviously, if $\varphi=0$, then Eq. (4.1) describes the even CS (1.5); when $\varphi=\pi$, then (4.1) is equal to the odd CS; and, finally, for $\varphi=\pi/2$ the state (4.1) is equal to the Yurke-Stoler CS (1.4). Here we note that the state (4.1) is an eigenstate of the square of the annihilation operator, i.e.,

$$\hat{a}^2|\Psi\rangle=\alpha^2|\Psi\rangle \ . \quad (4.2)$$

A. Nonclassical properties of even CS

The Wigner function of the even CS (1.5) can be obtained from Eqs. (3.7)–(3.9) and is

$$W(\beta)=\frac{1}{\pi[1+\exp(-2\alpha^2)]} \times\{\exp[-2(x-\alpha)^2-2y^2] +\exp[-2(x+\alpha)^2-2y^2] +2\exp[-2x^2-2y^2]\cos(4y\alpha)\} \ , \quad (4.3)$$

where $x=\mathrm{Re}\beta$, $y=\mathrm{Im}\beta$. The last term on the rhs of Eq. (4.3) arises from quantum interference between the states $|\alpha\rangle$ and $|-\alpha\rangle$ and is responsible for the nonclassical behavior of the even CS (for details see also Ref. [6]). This is clearly seen when we compare the Wigner function (4.3) of the even CS and the Wigner function of the statistical mixture of the states $|\alpha\rangle$ and $|-\alpha\rangle$ described by the density matrix

$$\hat{\rho}=\tfrac{1}{2}|\alpha\rangle\langle\alpha|+\tfrac{1}{2}|-\alpha\rangle\langle-\alpha| \ , \quad (4.4)$$

for which we find

$$W(\beta)=\frac{1}{\pi}\{\exp[-2(x-\alpha)^2-2y^2] +\exp[-2(x+\alpha)^2-2y^2]\} \ . \quad (4.5)$$

Both Wigner functions (4.3) and (4.5) are plotted in Fig. 1. From this figure it follows that the quantum interference between states $|\alpha\rangle$ and $|-\alpha\rangle$ leads to an additional peak of the Wigner function (4.3). Moreover, this Wigner function can take negative values, while the Wigner function of the statistical mixture (4.4) is strictly positive.

Using Eq. (2.25), we find that the photon number distribution of the statistical mixture (4.4) is Poissonian, i.e., $P_n = \exp(-\alpha^2)\alpha^{2n}/n!$. On the other hand, the quantum interference between the states $|\alpha\rangle$ and $|-\alpha\rangle$ generates an oscillatory behavior of the photon number distribution characteristic of the even CS (see Fig. 2)

$$P_n = \frac{2\exp(-|\alpha|^2)}{1+\exp(-2|\alpha|^2)}\frac{|\alpha|^{2n}}{n!} \quad \text{if } n=2m \ , \tag{4.6a}$$

$$P_n = 0 \quad \text{if } n=2m+1 \ . \tag{4.6b}$$

The oscillations in P_n are very similar to those exhibited by the squeezed vacuum discussed by Schleich and coworkers [36].

As it has been shown recently [4,6–9], the oscillatory behavior of the interference part of the Wigner function (4.3) not only results in oscillations of the photon number distribution, but it also gives rise to quadrature squeezing. We can easily find the variances of the quadrature operators $\hat{a}_i$ in the even CS.

$$\langle(\Delta\hat{a}_1)^2\rangle = \frac{1}{4} + \frac{\alpha^2}{1+\exp(-2\alpha^2)} \tag{4.7a}$$

and

$$\langle(\Delta\hat{a}_2)^2\rangle = \frac{1}{4} - \frac{\alpha^2\exp(-2\alpha^2)}{1+\exp(-2\alpha^2)} \ , \tag{4.7b}$$

from which we find reduction of fluctuations in the $\hat{a}_2$ quadrature [i.e., the y direction in phase space—see Fig. 1(a) and Ref. [6]). Simultaneously the fluctuations in $\hat{a}_1$ are enhanced. The squeezing parameters $S_i^{(2)}$ for the even CS are

$$S_1^{(2)} = \frac{4\alpha^2}{1+\exp(-2\alpha^2)} \tag{4.8a}$$

and

$$S_2^{(2)} = -\frac{4\alpha^2\exp(-2\alpha^2)}{1+\exp(-2\alpha^2)} \ . \tag{4.8b}$$

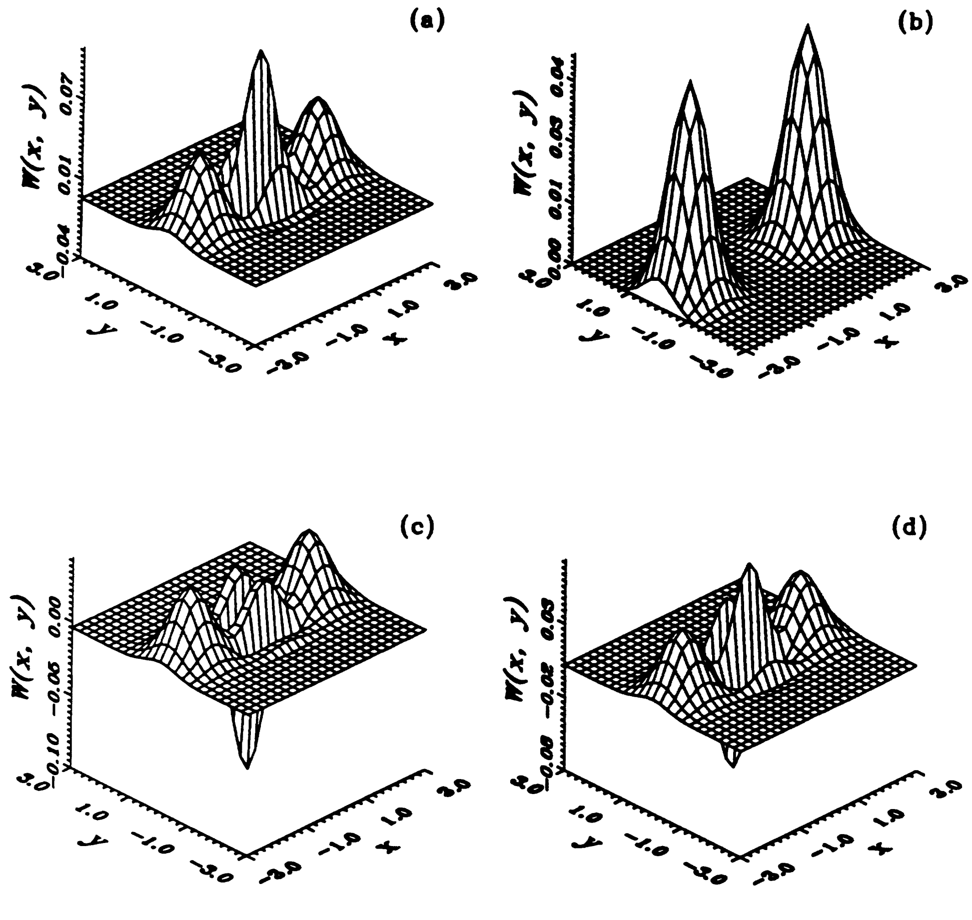

FIG. 1. (a) Wigner functions of the even CS, (b) the statistical mixture given by Eq. (4.4), (c) the odd CS, and (d) the Yurke-Stoler CS with $\alpha=2$. The role of the quantum interference between the coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ is obvious.

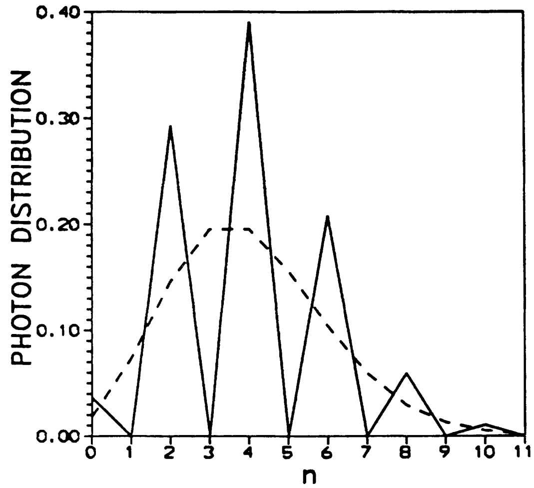

FIG. 2. Photon number distribution of the even CS with $\alpha=2$ (solid line). The dashed line corresponds to the Poissonian distribution of the statistical mixture (4.4). From this figure it is seen that the oscillatory behavior of the interference part of the Wigner function of the even CS is closely related to oscillations of the photon number distribution.

We see that $\hat{a}_2$ is squeezed for any intensity $\bar{n}$ of the even CS, which is related to α^2 through

$$\bar{n}=\alpha^2\frac{1-\exp(-2\alpha^2)}{1+\exp(-2\alpha^2)} .$$

From Fig. 3 we find that the maximum squeezing in the case of the even CS appears for small values of $\bar{n}$. For the statistical mixture (4.4) we obtain the following values for the variances of the quadrature operators $\hat{a}_i$:

$$\langle(\Delta\hat{a}_1)^2\rangle=\tfrac{1}{4}+\alpha^2, \quad \langle(\Delta\hat{a}_2)^2\rangle=\tfrac{1}{4} , \tag{4.9}$$

which means that, for the statistical mixture (4.4), the fluctuations in the $\hat{a}_1$ quadrature are larger than the vacuum (or coherent-state) fluctuations, while the fluctuations in $\hat{a}_2$ quadrature are equal to their vacuum value. It is interesting to note that the fourth-order moments of the operators $\Delta\hat{a}_1$ and $\Delta\hat{a}_2$ in the statistical mixture (4.4) are, respectively.

$$\langle(\Delta\hat{a}_1)^4\rangle=\tfrac{3}{16}+\tfrac{3}{2}\alpha^2+\alpha^4, \quad \langle(\Delta\hat{a}_2)^4\rangle=\tfrac{3}{16} . \tag{4.10}$$

From the above we can conclude that the fourth-order moment of $\Delta\hat{a}_1$ in the case of the statistical mixture is increased compared to its coherent-state value. Generally, the higher-order variances of the quadrature operator $\hat{a}_2$ in the statistical mixture (4.4) are equal to their coherent-state values, while the higher-order variances of the operator $\hat{a}_1$ in the statistical mixture (4.4) are larger.

We now evaluate the fourth-order variances for the even CS which are given by the following relations:

$$\langle(\Delta\hat{a}_1)^4\rangle=\frac{3}{16}+\frac{3\alpha^2}{2[1+\exp(-2\alpha^2)]}+\frac{\alpha^4}{1+\exp(-2\alpha^2)} \tag{4.11a}$$

and

$$\langle(\Delta\hat{a}_2)^4\rangle=\frac{3}{16}-\frac{3\alpha^2\exp(-2\alpha^2)}{2[1+\exp(-2\alpha^2)]}+\frac{\alpha^4\exp(-2\alpha^2}{1+\exp(-2\alpha^2)} . \tag{4.11b}$$

The corresponding squeezing parameters $S_i^{(4)}$ may be written as

$$S_1^{(4)}=\frac{16\alpha^2}{3[1+\exp(-2\alpha^2)]}(\alpha^2+\tfrac{3}{2}) \tag{4.12a}$$

and

$$S_2^{(4)}=\frac{16\alpha^2\exp(-2\alpha^2)}{3[1+\exp(-2\alpha^2)]}(\alpha^3-\tfrac{3}{2}) . \tag{4.12b}$$

We can conclude that the even CS is not only second-order squeezed, but it also exhibits fourth-order squeezing, for weak fields with $\alpha^2<\frac{3}{2}$. Moreover, the degree of fourth-order squeezing is even larger than the degree of second-order squeezing (see Fig. 3). We should underline here that, in the case of the even CS, there is a close relation between the presence of squeezing and the shape of the Wigner function. As seen from Fig. 1(a), the Wigner function itself is "squeezed" in phase space in the y-direction corresponding to the reduction of fluctuations in the $\hat{a}_2$ quadrature (for details see Ref. [6]).

Finally, we note that the even CS has super-Poissonian photon statistics for any value of $\bar{n}$, that is, the Mandel Q parameter [see Eq. (2.27)]

$$Q=\frac{4\alpha^2\exp(-2\alpha^2)}{1-\exp(-4\alpha^2)}>0 \tag{4.13}$$

is positive for any value of α^2.

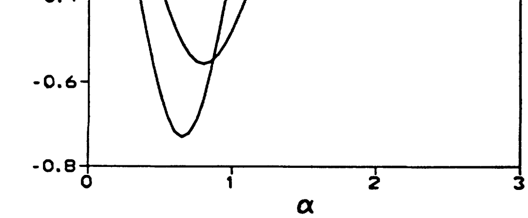

FIG. 3. Squeezing parameters $S_2^{(2)}$ and $S_2^{(4)}$ for the even CS vs the parameter α. We see that the highest degree of squeezing can be observed for small intensities of the light field.

B. The odd coherent state

The odd CS differs from the even CS "just" by a phase factor $e^{i\varphi}$ in expression (4.1). For the even CS we have $\varphi=0$, while for odd CS, $\varphi=\pi$. This subtle difference leads to completely different nonclassical properties of these states.

The Wigner function of the odd CS is equal to the Wigner function of the even CS apart from the fact that overall normalization constant is different and the interference term is multiplied by -1:

$$W(\beta)=\frac{1}{\pi[1-\exp(-2\alpha^2)]}\{\exp[-2(x-\alpha)^2-2y^2]+\exp[-2(x+\alpha)^2-2y^2]-2\exp(-2x^2-2y^2)\times\cos(4y\alpha)\} . \tag{4.14}$$

This Wigner function is plotted in Fig. 1(c), which reveals the striking difference between the Wigner functions of the even CS and odd CS. This difference is also reflected in the photon number distribution of odd CS, for which only an odd number of photons have a nonzero probability of being observed. As we have shown above, in the case of the even CS, the probability of finding an odd number of photons is equal to zero [this is, of course, the reason for calling the states (1.5) and (1.6) even CS and odd CS, respectively]. The photon number distribution of the odd CS can be written as

$$P_n=\frac{2\alpha^{2n}\exp(-|\alpha|^2)}{[1-\exp(-2|\alpha|^2)]n!} \quad \text{if } n=2m+1 , \tag{4.15a}$$

$$P_n=0 \quad \text{if } n=2m . \tag{4.15b}$$

The Mandel Q parameter (2.27) of the odd CS is

$$Q=-\frac{4\alpha^2\exp(-2\alpha^2)}{1-\exp(-4\alpha^2)}<0 , \tag{4.16}$$

from which it follows that the odd CS has sub-Poissonian photon statistics. It is interesting to note that the odd CS has the maximum degree of sub-Poissonian statistics at low intensities $\bar{n}$ of the field, that is, for small values of the parameter α^2:

$$\bar{n}=\alpha^2\frac{1+\exp(-2\alpha^2)}{1-\exp(-2\alpha^2)} .$$

In particular, in the limit $\alpha^2\to 0$ (i.e., $\bar{n}\to 1$), we find that $Q\to -1$ (see Fig. 4).

As we have shown earlier, the even CS has super-Poissonian photon statistics, but simultaneously it exhibits second-order squeezing. The odd CS has sub-Poissonian photon statistics, but it does not exhibit second-order squeezing. The (second-order) squeezing parameters $S_i^{(2)}$ for the odd CS are

$$S_1^{(2)}=\frac{4\alpha^2}{1-\exp(-2\alpha^2)}>0 \tag{4.17a}$$

and

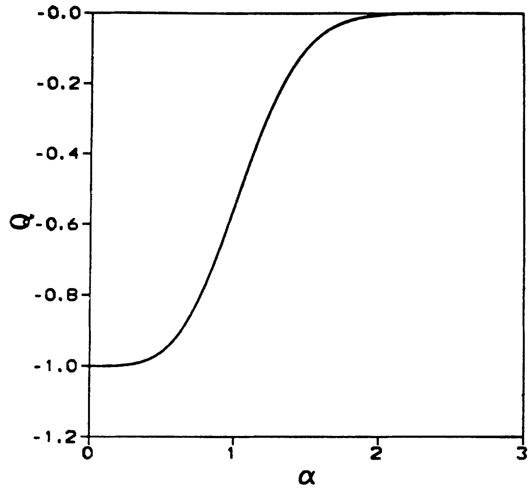

FIG. 4. Mandel Q parameter of the odd CS vs the parameter α. The highest degree of sub-Poissonian photon statistics ($Q\to-1$) is obtained for $\alpha\to 0$, that is for $\bar{n}\to 1$.

$$S_2^{(2)}=\frac{4\alpha^2\exp(-2\alpha^2)}{1-\exp(-2\alpha^2)}>0. \tag{4.17b}$$

In spite of the fact that there is no second-order squeezing in the case of the odd CS, fourth-order squeezing can occur, as we can see from the squeezing parameters $S_i^{(4)}$:

$$S_1^{(4)}=\frac{16\alpha^2}{3[1-\exp(-2\alpha^2)]}(\alpha^2+\tfrac{3}{2}) \tag{4.18a}$$

and

$$S_2^{(4)}=-\frac{16\alpha^2\exp(-2\alpha^2)}{3[1-\exp(-2\alpha^2)]}(\alpha^2-\tfrac{3}{2}) . \tag{4.18b}$$

The fourth-order squeezing in the $\hat{a}_2$ quadrature appears for $\alpha^2>\frac{3}{2}$ when $S_2^{(4)}<0$. Nevertheless, the degree of fourth-order squeezing in the odd CS is significantly smaller than that for the even CS.

C. The Yurke-Stoler coherent state

No way to generate the even and odd CS from a coherent state by a unitary transformation is presently known (although the nonunitary field reduced matrix in the Jaynes-Cummings model evolves to such a state [27,29]). On the other hand, the Yurke-Stoler CS given by Eq. (1.4) appears as a result of a nonlinear interaction of coherent light with a Kerr-like medium [13] and is equal to a unitarily transformed coherent state with the unitary operator given as the exponential of a polynomial in the number operator $\hat{n}$. Therefore, the photon number distribution of the Yurke-Stoler (CS) must remain Poissonian (i.e., $Q=0$).

The Wigner function for the Yurke-Stoler CS is [see Fig. 1(d)]

$$W(\beta)=\frac{1}{\pi}\{\exp[-2(x-\alpha)^2-2y^2]+\exp[-2(x+\alpha)^2-2y^2]-2\exp(-2x^2-2y^2)\sin(4y\alpha)\} . \quad (4.19)$$

In spite of the fact that the photon number distribution of the Yurke-Stoler CS is not altered by the quantum interference between the component states $|\alpha\rangle$ and $|-\alpha\rangle$ [described by the third term in the rhs of Eq. (4.19)], this interference results in the appearance of the second- and fourth-order squeezing of quadrature operators. After some algebra we find for the squeezing parameters $S_i^{(2,4)}$ the following expressions:

$$S_1^{(2)}=4\alpha^2 , \quad (4.20a)$$

$$S_2^{(2)}=-4\alpha^2\exp(-4\alpha^2) , \quad (4.20b)$$

and

$$S_1^{(4)}=\tfrac{16}{3}\alpha^2(\alpha^2+\tfrac{3}{2}) , \quad (4.21a)$$

$$S_2^{(4)}=\tfrac{16}{3}\alpha^2\exp(-4\alpha^2)\{\alpha^2[4-3\exp(-4\alpha^2)]-\tfrac{3}{2}\} . \quad (4.21b)$$

The maximum degree of the second- and fourth-order squeezing in the Yurke-Stoler CS is smaller than in the case of the even CS. This can be seen by comparing Figs. 3 and 5. We can conclude that the superposition of two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ exhibits different nonclassical effects which depend on the particular choice of the phase φ. In other words, the phase φ dictates the character of the quantum interference between $|\alpha\rangle$ and $|-\alpha\rangle$. This dependence on the relative phase in the quantum superposition should be distinguished from the role of the relative phase ϕ of the coherent states with complex amplitudes, studied by Schleich *et al.* [9], which also changes the quantum interference.

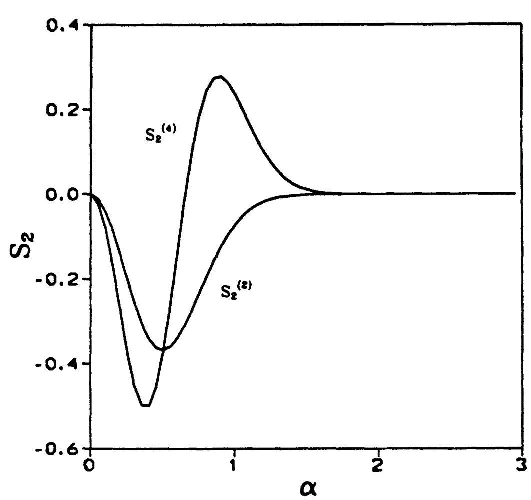

FIG. 5. Squeezing parameters $S_2^{(2)}$ and $S_2^{(4)}$ for the Yurke-Stoler CS vs the parameter α. We see that the highest degree of squeezing can be observed for small intensities of the light field. The highest degree of squeezing in the case of the Yurke-Stoler CS is smaller compared with the highest degree of squeezing of the even CS (see Fig. 3).

D. One-dimensional continuous superpositions of coherent states

We have shown earlier that the even CS (1.5) exhibits a large amount of second-order (quadrature) squeezing. The degree of squeezing can be increased if we add to the superposition (1.5) another pair of coherent states $|\beta\rangle$ and $|-\beta\rangle$ (β and α are supposed to be real)

$$|\Psi\rangle=A^{1/2}[p_\alpha(|\alpha\rangle+|-\alpha\rangle)+p_\beta(|\beta\rangle+|-\beta\rangle)] ,$$

where $p_{\alpha,\beta}$ are some numerical parameters and A is the corresponding normalization constant. In fact, it can be shown [4,6] that a one-dimensional continuous superposition of the type

$$|\xi\rangle=C_F\int_{-\infty}^{\infty}F(\alpha,\xi)d\alpha|\alpha\rangle , \quad (4.22a)$$

where

$$C_F^{-2}=\int\int_{-\infty}^{\infty}(\alpha,\xi)F(\alpha',\xi)\times\exp[-(\alpha-\alpha')^2/2]d\alpha d\alpha' , \quad (4.22b)$$

with properly chosen weight functions $F(\alpha,\xi)$, can exhibit a large degree of squeezing. If $F(\alpha,\xi)$ is taken to be the Gaussian function [6]

$$F(\alpha,\xi)=\exp\left[-\frac{(1-\xi)}{2\xi}\alpha^2\right] , \quad (4.23)$$

then the state (4.22a) is equal to the squeezed vacuum state, that is

$$|\xi\rangle=C_F\int_{-\infty}^{\infty}F(\alpha,\xi)\hat{D}(\alpha)|0\rangle d\alpha=\hat{S}(\xi)|0\rangle , \quad (4.24)$$

where $\hat{S}(\xi)$ is the squeeze operator

$$\hat{S}(\xi)=\exp\left[\frac{r}{2}(\hat{a}^\dagger)^2-\frac{r}{2}\hat{a}^2\right], \quad \xi=\tanh r , \quad (4.25)$$

and $\hat{S}(\xi)|0\rangle$ is the squeezed vacuum state.

On the other hand, the odd CS exhibits a large degree of sub-Poissonian statistics ($Q<0$). One can construct a superposition of coherent states $||\alpha|e^{-i\varphi}\rangle$ with equal amplitudes $|\alpha|$ and suitably chosen distribution of their phases φ in such a way that the degree of sub-Poissonian statistics can be increased (for details see Ref. [4]). It has been shown by Gardiner [38] that, in the continuous limit of coherent states on the circle, one can find the following relation:

$$|n\rangle=A_n(|\alpha|)\int_{-\pi}^{\pi}d\varphi\, e^{-in\varphi}||\alpha|e^{i\varphi}\rangle , \quad (4.26a)$$

where the normalization constant $A_n(|\alpha|)$ is

$$A_n(|\alpha|)=\sqrt{n!}|\alpha|^{-4}e^{|\alpha|^2/2} . \quad (4.26b)$$

In other words, a continuous superposition of coherent states on the circle can represent the number state $|n\rangle$, that is, the state with highest degree of sub-Poissonian statistics $Q=-1$.

V. INFLUENCE OF DAMPING ON QUANTUM INTERFERENCE

In this section we will study the decay of a field mode initially prepared in a superposition of coherent states. To be specific, we shall assume that the density operator $\hat{\rho}$ for the field mode obeys a zero-temperature master equation in the Born-Markov approximation. This equation in the interaction picture can be written as (see, for instance, Ref. [39])

$$\frac{\partial\hat{\rho}}{\partial t}=\frac{\gamma}{2}(2\hat{a}\hat{\rho}\hat{a}^{\dagger}-\hat{a}^{\dagger}\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^{\dagger}\hat{a})\ , \tag{5.1}$$

where γ is the decay constant. Such a model was previously studied by Walls and Milburn [39] and Savage and Walls [40]. These authors have shown that the off-diagonal terms in the field density operator, expressed in a coherent-state basis, are weighted with a time-dependent factor which rapidly suppresses these coherences. The effect of the decay on observable quantities was studied by Phoenix [41] who has shown that mean values of observable quantities, arising from off-diagonal coherence, do not decay on a faster time scale than other terms arising from the diagonal terms. As it will be seen later, this is the reason why squeezing in the superposition of coherent states decays on the same time scale as the mean photon number. On the other hand, we will show that the photon number distribution P_n, which is related to the diagonal matrix elements in the number-state basis, is very sensitive to the presence of damping. The same is also true for that part of the Wigner function which corresponds to the quantum interference between coherent states. One of our tasks is therefore to explain why some quantities seem to be so sensitive to dissipation whereas others do not show this extraordinary sensitivity.

Following Barnett and Knight [42] we can write the solution of the master equation (5.1) in the form

$$\hat{\rho}(t)=\exp(\hat{L}t)\exp\left[\frac{\hat{J}}{\gamma}(1-e^{-\gamma t})\right]\hat{\rho}(0)\ , \tag{5.2}$$

where the two operators $\hat{L}$ and $\hat{J}$ are defined by their action on the density operator, i.e.,

$$\hat{J}\hat{\rho}=\gamma\hat{a}\hat{\rho}\hat{a}^{\dagger} \tag{5.3a}$$

and

$$\hat{L}\hat{\rho}=-\frac{\gamma}{2}(\hat{a}^{\dagger}\hat{a}\hat{\rho}+\hat{\rho}\hat{a}^{\dagger}\hat{a})\ . \tag{5.3b}$$

Using the formal solution (5.2) for the $\hat{\rho}$ matrix, one can find the time-dependent expression for the density matrix of the superposition of two coherent states

$$|\Psi\rangle=A^{1/2}[|\alpha_i\rangle+|\alpha_j\rangle] \tag{5.4}$$

at $t>0$ (for details see Refs. [39,41]):

$$\hat{\rho}(t)=A\sum_{i,j=1}^{2}\langle\alpha_i|\alpha_j\rangle^{1-\mu}|\mu^{1/2}\alpha_j\rangle\langle\mu^{1/2}\alpha_i|\ , \tag{5.5}$$

where $\mu=\exp(-\gamma t)$.

The last expression reflects the fact that the off-diagonal terms of the density matrix are rapidly dephased at a rate governed by the separation of the coherent states. The interference terms in the Wigner function are proportional to the off-diagonal terms of the $\hat{\rho}$ matrix which means that the Wigner function of the pure quantum superposition state decays very rapidly towards the Wigner function of the statistical mixture. This is very noticeable in Fig. 6 in which the Wigner function of the initial even coherent state is plotted for various values of μ.

A similar sensitivity of the quantum interference terms

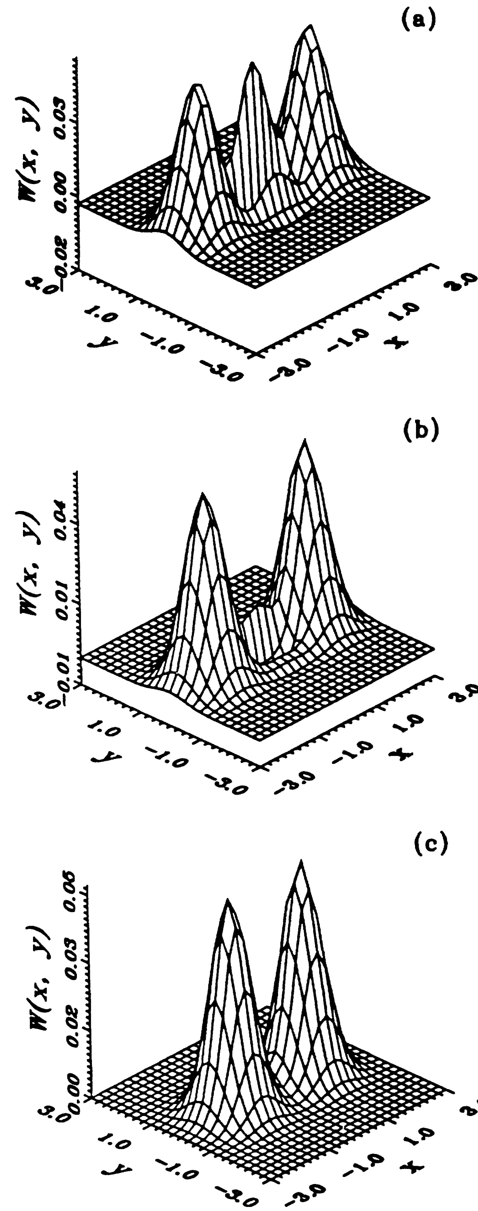

FIG. 6. The Wigner function of the initial even CS influenced by damping. The transition from the Wigner function of the even CS [Fig. 1(a)] towards the Wigner function of the mixture of two coherent states (c) can be observed. The value of α is 2 and (a) $\gamma t=0.1$, (b) $\gamma t=0.3$, and (c) $\gamma t=1.0$.

to decay can also be observed in the case of the photon number distribution. As we noted earlier, the photon number distribution of the even coherent state exhibits oscillations (see Fig. 2), while the statistical mixture of states $|\alpha\rangle$ and $|-\alpha\rangle$ has a Poissonian distribution. The oscillations of the photon number distribution have their origin in the quantum interference described by the off-diagonal terms (in the coherent-state basis) of the density matrix [see Eq. (2.25)]. From this it follows that the oscillations of P_n disappear very rapidly and that only the terms corresponding to the statistical mixture give the dominant contribution to P_n (see Fig. 7).

From the above we can conclude that both the Wigner function and the photon number distribution are very sensitive to the rapid destruction of the off-diagonal coherences. On the other hand, one can find (see, for instance, Ref. [41]) that normally ordered expectation values $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle$ are given by

$$\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle = A \sum_{i,j=1}^{2} \langle \alpha_i | \alpha_j \rangle (\mu^{1/2}\alpha_j)^n (\mu^{1/2}\alpha_i^*)^m , \tag{5.6}$$

which means that the terms arising from the off-diagonal coherences in the superposition state (5.4) decay on the same time scale as those arising from the diagonal elements [41]. We now understand that the intensity of the

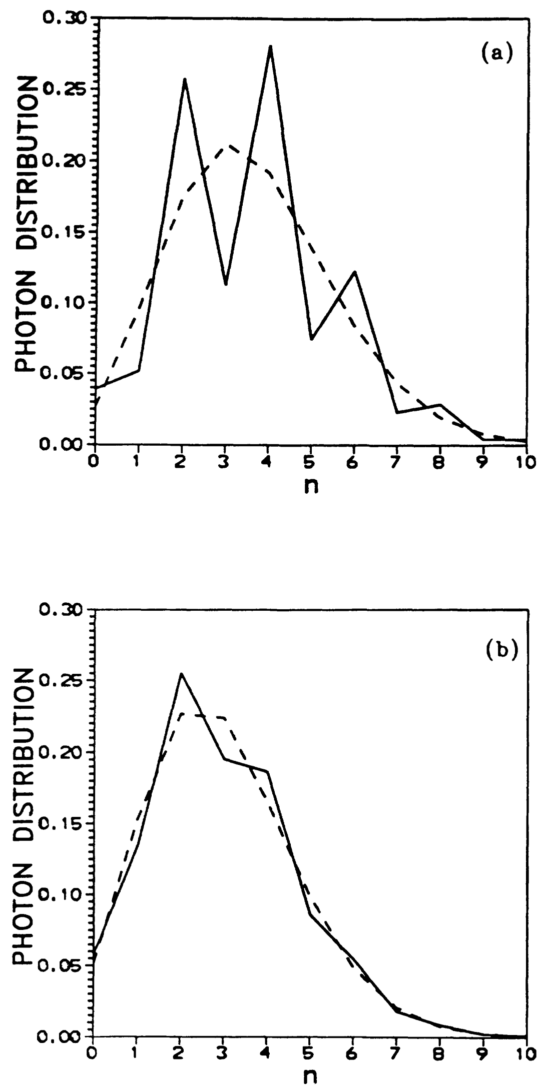

FIG. 7. The photon number distribution of the initial even CS (see Fig. 2) influenced by the damping. The initial value is $\alpha=2$ and (a) $\gamma t=0.1$, (b) $\gamma t=0.3$. It is clearly seen that, with increasing γt, the photon number distribution of the even CS is transformed into the photon number distribution of the statistical mixture. Dashed lines describe photon number distributions of the states obtained from the statistical mixture (4.4) under the influence of the decay mechanism.

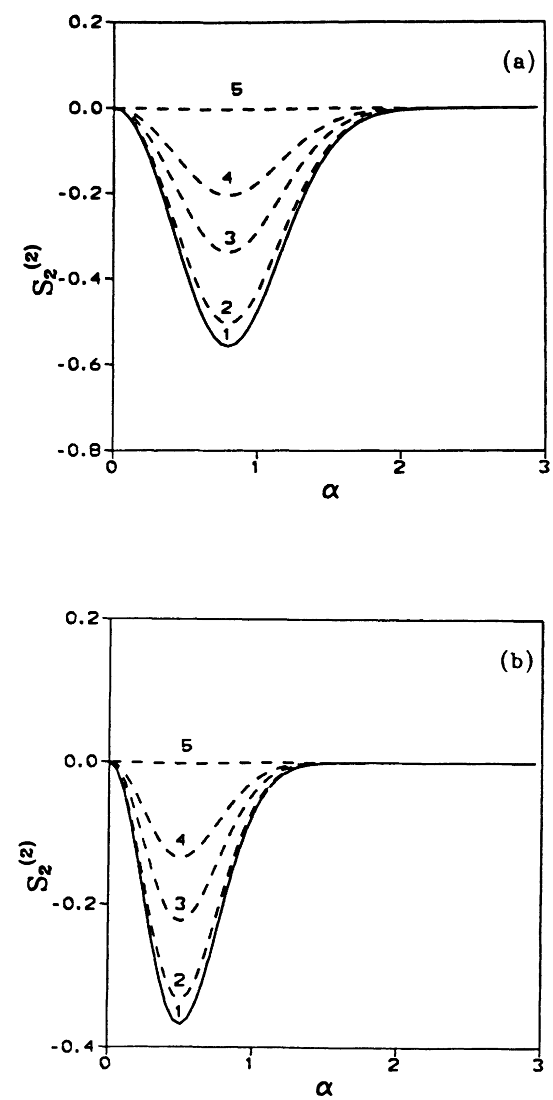

FIG. 8. Squeezing parameters $S_2^{(2)}$ for (a) the even CS and (b) the Yurke-Stoler CS vs the parameter α for various values of γt: $\gamma t=0.0$ (line 1), $\gamma t=0.1$ (line 2), $\gamma t=0.3$ (line 3), $\gamma t=1.0$ (line 4), and $\gamma t=5.0$ (line 5). We see that a considerable amount of squeezing can be observed even for a value for a value $\gamma t=1.0$ (line 4) when the interference term in the Wigner function has totally disappeared.

field mode as well as the mean values $\mathrm{Re}(\langle \hat{a}^2 \rangle)$ and $(\mathrm{Re}\langle \hat{a} \rangle)^2$ decay with the same rate, which means that, for the variances of the quadrature operators, we obtain

$$\langle [\Delta \hat{a}_i(t)]^2 = \tfrac{1}{4} + \mu \langle :[\Delta \hat{a}_i(0)]^2 : \rangle \ . \tag{5.7}$$

For the squeezing parameter $S_i^{(2)}$ we find

$$S_i^{(2)}(t) = \mu S_i^{(2)}(0) \ , \tag{5.8}$$

From which it follows that second-order squeezing is deteriorated by the presence of damping at the same rate as the mean photon number [for which we can find that $\bar{n}(t) = \mu \bar{n}(0)$]. In Fig. 8(a) we plot the squeezing parameter $S_i^{(2)}$ of the even coherent state for various values of γt. We can see that squeezing is much more robust with respect to damping than the oscillations of the photon number distribution or the interference term in the Wigner function. For instance, for $\gamma t = 0.3$ one can observe a considerable degree of quadrature squeezing, while the Wigner function of the even CS for this value of γt is almost identical to the Wigner function of the statistical mixture (see Fig. 6). The same effect can be observed also in the case of the Yurke-Stoler CS [see Fig. 8(b)].

In order to understand more clearly why the second-order squeezing decays at a different rate than the Wigner function, we rewrite the Wigner function in terms of normally ordered moments of the creation and annihilation operators $\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle = \mathrm{Tr}[\hat{\rho}(\hat{a}^\dagger)^m \hat{a}^n]$. To do so, we rewrite the displacement operator $\hat{D}(\xi)$ in normally ordered form:

$$\hat{D}(\xi) = \exp(-\tfrac{1}{2}|\xi|^2) \sum_{n,m=0}^{\infty} \frac{(\xi \hat{a}^\dagger)^m}{\sqrt{m!}} \frac{(-\xi^* \hat{a})^n}{\sqrt{n!}} \ , \tag{5.9}$$

from which we obtain the following expression for the characteristic function $C^{(w)}$:

$$C^{(w)}(\xi) = \exp(-\tfrac{1}{2}|\xi|^2) \sum_{n,m=0}^{\infty} \frac{\xi^m (-\xi^*)^n}{\sqrt{m!n!}} \langle (\hat{a}^\dagger)^m \hat{a}^n \rangle \ . \tag{5.10}$$

The Wigner function (2.20) can now be rewritten as

$$W(\beta) = \sum_{n,m=0}^{\infty} \frac{\langle (\hat{a}^\dagger)^m \hat{a}^n \rangle}{\sqrt{m!n!}} W_{mn}(\beta) \ , \tag{5.11}$$

where the function $W_{mn}(\beta)$ is defined as

$$W_{mn}(\beta) = \frac{1}{\pi^2} \int d^2\xi \exp(-\tfrac{1}{2}|\xi|^2 + \beta \xi^* - \beta^* \xi) \xi^m (-\xi^*)^n \ , \tag{5.12a}$$

and can be evaluated by partial differentiation over the parameters v and z from the generating function

$$W_{mn}(\beta) = \frac{2}{\pi} \left[\frac{\partial}{\partial v} + i \frac{\partial}{\partial z} \right]^m \left[-\frac{\partial}{\partial v} + i \frac{\partial}{\partial z} \right]^n \exp\left[-2 \left[x - \frac{i}{2} v \right]^2 - 2 \left[y + \frac{i}{2} z \right]^2 \right] \Bigg|_{v,z=0} \tag{5.12b}$$

with $x = \mathrm{Re}\beta$, $y = \mathrm{Im}\beta$. In this way we directly see that the Wigner function depends on all moments of the field observables. But these higher-order moments depend on powers of the damping factor μ (as can be seen by consideration of the appropriate Heisenberg equations) and this is responsible for the sensitivity to dissipation. The Wigner function [and $P(n)$] contains a sum of terms, each one of which decays more rapidly than its predecessor. Low-order expectation values depend only on the appropriate low-order terms in the Wigner function expansion and are thus insensitive to the more rapid decay of the higher-order terms.

If we suppose that initially the light field is in a superposition of two coherent states (5.4), then taking into account the damping process, the Wigner function $W(\beta,t)$ at time t will take the form

$$W(\beta,t) = A \sum_{m,n=0}^{\infty} \left[\sum_{i,j=1}^{2} (\alpha_j|\alpha_i\rangle \frac{\alpha_i^n (\alpha_j^*)^m}{\sqrt{m!n!}} \right] \mu^{(m+n)/2} W_{mn}(\beta, t=0) \ , \tag{5.13}$$

from which it is clearly seen that the Wigner function always decays faster than the second-order squeezing. This is because it contains the damping factors $\mu^{(m+n)/2}$ related to higher-order normally ordered products of creation and annihilation operators. The last expression for the Wigner function $W(\beta,t)$ can be rewritten in the form:

$$W(\beta,t) = A \sum_{i,j=1}^{2} \langle \alpha_j|\alpha_i \rangle \frac{2}{\pi} \exp\left[\mu(\alpha_j - \alpha_i) \cdot \frac{\partial}{\partial \eta} - i\mu(\alpha_j + \alpha_i) \otimes \frac{\partial}{\partial \eta} \right] \exp(-2|\beta|^2 + 2i\eta \otimes \beta + \tfrac{1}{2}|\eta|^2)|_{\eta=0} \ , \tag{5.14}$$

where $\eta = v + iz$ and $\partial/\partial\eta = \partial/\partial v + i\partial/\partial z$. We see that the off-diagonal terms of the Wigner function $(i \neq j)$ decay faster than the diagonal ones. This is due to the fact that off-diagonal terms contain an additional damping factor arising from the term $\exp[\mu(\alpha_j - \alpha_i) \cdot \partial/\partial\eta]$, which is equal to unity for diagonal terms.

Decay of the fourth-order squeezing

Earlier in this section we have shown that second-order squeezing, which arises due to the quantum interference between two coherent states, decays linearly with the variable μ [see Eq. (5.8)]. We illustrate this fact in Fig. 8

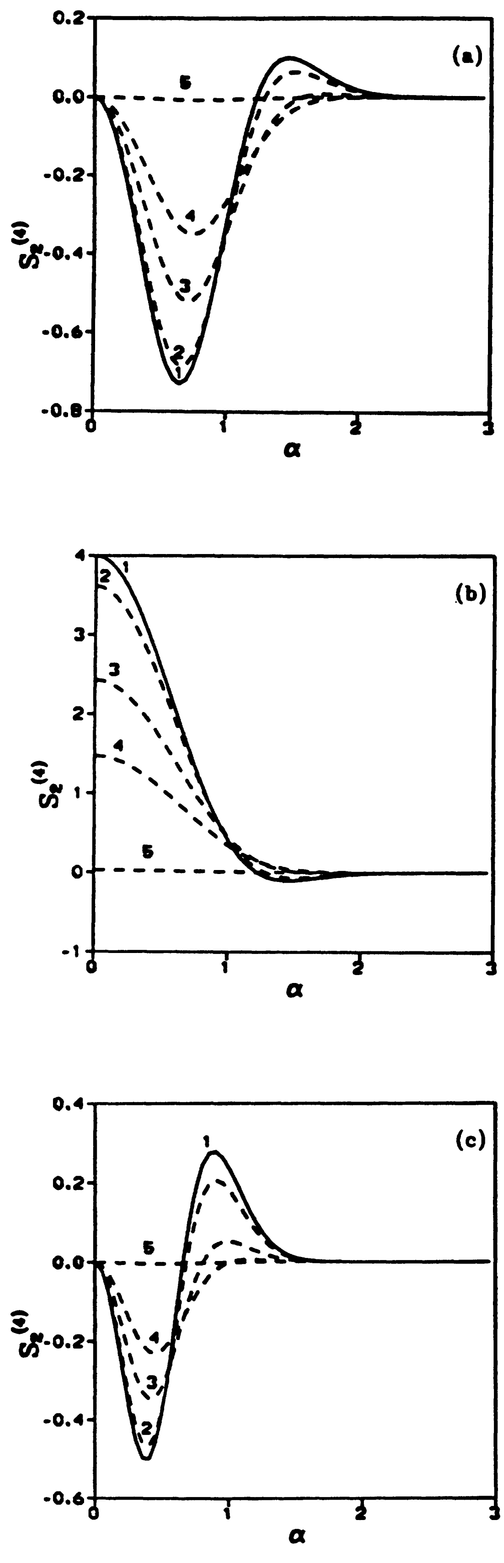

FIG. 9. Squeezing parameters $S_2^{(4)}$ for (a) the even CS, (b) the odd CS, and (c) the Yurke-Stoler CS vs the parameter α for various values of γt: $\gamma t=0.0$ (line 1), $\gamma t=0.1$ (line 2), $\gamma t=0.3$ (line 3), $\gamma t=1.0$ (line 4), and $\gamma t=5.0$ (line 5). We see that, in the case of the even CS and the Yurke-Stoler CS, fourth-order squeezing can be generated via the damping mechanism.

where the parameters $S_2^{(2)}$ are plotted for the even CS and Yurke-Stoler CS vs α for various values of μ. We see how the decay leads to deterioration of second-order squeezing.

The fourth-order squeezing parameters $S_i^{(4)}$ can be written in terms of normally ordered moments of quadrature operators:

$$S_i^{(4)}=\tfrac{16}{3}[\tfrac{3}{2}\langle :(\Delta\hat{a}_i)^2:\rangle+\langle :(\Delta\hat{a}_i)^4:\rangle] . \tag{5.15}$$

As seen from Eq. (5.6), if the light field is initially in the superposition state (5.4), then in the presence of damping we find for the squeezing parameter at time t (corresponding to a particular value of μ) the expression

$$S_i^{(4)}(t)=\tfrac{16}{3}\{\tfrac{3}{2}\mu\langle :[\Delta\hat{a}_i(0)]^2:\rangle+\mu^2\langle :[\Delta\hat{a}_i(0)]^4:\} , \tag{5.16}$$

which means that the two parts of the rhs of Eq. (5.16) decay at different rates, and the fourth-order normally ordered variance $\langle :[\Delta\hat{a}_i(0)]^4:\rangle$ decays faster than the second-order normally ordered variance. This can lead to a remarkable result: the decay mechanism itself can generate fourth-order squeezing. Namely, let us suppose the light field to be initially in the even CS (1.5). The initial degree of fourth-order squeezing in this case is given by Eq. (4.12) from which it is seen that fourth-order squeezing is absent for $\alpha^2>\frac{3}{2}$. If we take into account the decay mechanism, then we find, for $S_2^{(4)}$,

$$S_2^{(4)}(t)=\frac{16\mu\alpha^2\exp(-2\alpha^2)}{3[1+\exp(-2\alpha^2)]}(\mu\alpha^2-\tfrac{3}{2}) , \tag{5.17}$$

from which it follows that, at $t>0$, fourth-order squeezing occurs for $\mu\alpha^2<\frac{3}{2}$, that is, at $t>0$ the fourth-order squeezing can appear for such values of α^2 for which $S_2^{(4)}(t=0)>0$.

Fourth-order squeezing parameters $S_2^{(4)}(t)$ for the odd CS and Yurke-Stoler CS are, respectively,

$$S_2^{(4)}(t)=-\frac{16\mu\alpha^2\exp(-2\alpha^2)}{3[1-\exp(-2\alpha^2)]}(\mu\alpha^2-\tfrac{3}{2}) \tag{5.18}$$

and

$$S_2^{(4)}(t)=\tfrac{16}{3}\mu\alpha^2\exp(-4\alpha^2) \times\{\mu\alpha^2[4-3\exp(-4\alpha^2)]-\tfrac{3}{2}\} . \tag{5.19}$$

All these parameters are plotted in Fig. 9 as functions of α for various values of μ. We can conclude that, in spite of the fact that damping leads to deterioration of second-order squeezing [see Eq. (5.8)], it can produce considerable fourth-order squeezing for particular values of coherent amplitudes of the coherent states composing the quantum superposition [see Figs. 9(a) and 9(c)]. The necessary condition under which one can observe an enhancement of fourth-order squeezing via damping is that, at the initial moment of the evolution, the state under consideration is second-order squeezed. From here it directly follows that, in the case of the odd CS, the fourth-order squeezing cannot be enhanced by the damping mechanism [see Fig. 9(b)]. Finally, we note that the maximum (global) degree of fourth-order squeezing of the quantum superposition of coherent states cannot be enhanced by damping.

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