Pattern form and homoclinic structure in Zakharov equations

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The relations between the homoclinic structure and spatial coherent pattern in Zakharov equations (ZE's) are discussed. Our results present Kolmogorov-Arnold-Moser curves and homoclinic crossing for ZE's, which exhibit the property of a near-integrable system, and Hamiltonian chaos in the ZE's is revealed.

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It is well known that Zakharov equations (ZE's) are the most popular model to describe strong Langmuir turbulence in a plasma, a subject which has been studied extensively. But until now, the study for the complex behavior of the ZE's is mainly limited in the dissipation case because the study of Hamiltonian chaos is difficult, and the discussion of the integrable problem of the ZE's is subtle. In our previous work [1], it is shown that the modulation intabilities belong to a class of periodic solutions of the ZE's. The interest in such periodic solutions is connected with the study of chaos for the ZE's that describe homoclinic structure (HS) in pattern dynamics. This is an active research area recently [2].

HS in the nonlinear Schrödinger equation and the sine-Gordon equation has been studied in recent years [2,3]. The ZE's are more complex than they are. In order to understand the origin of strong Langmuir turbulence, it is very important to study the HS of the ZE's. In this Brief Report, we are investigating the HS related to a typical pattern of the ZE's. Our results present Kolmogorov-Arnold-Moser (KAM) curves and homoclinic crossings that indicate the existence of Hamiltonian chaos; this is a well-known phenomenon in a near-integrable finite-degree-freedom system [4].

We consider the ZE's in the one-dimensional case [5]:

$$i\partial_t E + \partial_x^2 E = nE \; , \qquad \partial_t^2 n - \partial_x^2 n = \partial_x^2 |E|^2 \; , \tag{1}$$

where $E(x,t)$ is a slowly varying envelope of the high-frequency electric field and $n(x,t)$ is a low-frequency quasineutral density perturbation. In the linearized form, Eqs. (1) lead to the following dispersion relation:

$$(\omega^2 - k^2)(\omega^2 - k^4) = 2E_0^2 k^4 \; , \tag{2}$$

where E_0 is an amplitude of the homogeneous Langmuir field, k is a wave number of the disturbance, and the time dependence is proportional to $\exp(-i\omega t)$. If k lies in the range

$$0 < k < \sqrt{2} E_0 \equiv k_c \tag{3}$$

the system will evolve from a spatially homogeneous state to a localized structure, which is known as modulation instability [6].

Equations (1) have at least two kinds of fixed points (FP's)

$$(E,n) = O$$

the (0,0) FP, and

$$(E,n) = Q$$

the $(E_0, 0)$ FP, where E_0 is a real constant. The stability of these FP's can be examined by the linearized Eqs. (1) and solving the eigenvalue equation. Here we consider a simple case that only one wave number k_0 in the system satisfies Eq. (3), say, according to the condition

$$k_0 \equiv 0.95 k_c \; . \tag{4}$$

In this case, around the FP's, E and n are eigenfunctions of a spatial operator ∂_x^2 which has the form

$$\partial_x^3 E = -k_0^2 E \; , \qquad \partial_x^2 n = -k_0^2 n \; . \tag{5}$$

The analyses of the stability of the FP's for Eqs. (1) show that O is a center, and Q is the saddle and center in their own subspace. Combining the above facts and the periodic solutions of Eqs. (1), we believe that the HS exists in the ZE's. In order to study it we do a local analysis for the FP Q in detail. We divide the complex variable E of Eqs. (1) into real and imaginary components

$$E = v + iu \; .$$

Now we rewrite Eqs. (1) as

$$\begin{aligned} \partial_t v &= -\partial_x^2 u + nu \; , \\ \partial_t u &= \partial_x^2 v - nv \; , \\ \partial_t n &= m \; , \\ \partial_t m &= \partial_x^2 n + \partial_x^2 (u^2 + v^2) \; . \end{aligned} \tag{6}$$

Combining Eqs. (5), we have the linearized form in a neighborhood U of Q:

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$$\dot{\mathbf{w}} = Df(Q)\mathbf{w} , \tag{7}$$

where $Df(Q)$ is the Jacobian matrix at Q which has the form

$$Df(Q)=\begin{bmatrix} 0 & k_0^2 & 0 & 0 \\ -k_0^2 & 0 & -E_0 & 0 \\ 0 & 0 & 0 & 1 \\ -2E_0k_0^2 & 0 & -k_0^2 & 0 \end{bmatrix}$$

and

$$\mathbf{w}=\begin{bmatrix} v \\ u \\ n \\ m \end{bmatrix} ,$$

the eigenvalue equation $\|Df(Q)-\lambda I\|=0$ gives eigenvalues

$$\begin{aligned} \lambda_1 &= \{\tfrac{1}{2}k_0^2[(1-k_0^2)^2+8E_0^2]^{1/2}-(k_0^2+1)\}^{1/2}\equiv\lambda_0 , \\ \lambda_2 &= -\lambda_0 , \\ \lambda_3 &= i\{\tfrac{1}{2}k_0^2[(1-k_0^2)^2+8E_0^2]^{1/2}+(k_0^2+1)\}^{1/2}\equiv i\omega_0 , \\ \lambda_4 &= -i\omega_0 , \end{aligned} \tag{8}$$

where λ_0 and ω_0 are real if Eq. (4) is satisfied. In our case, the solution of Eq. (7) can be written as follows:

$$\mathbf{f}=C_1e^{\lambda_0 t}\mathbf{f}_1+C_2e^{-\lambda_0 t}\mathbf{f}_2+C_3e^{i\omega_0 t}\mathbf{f}_3+C_4e^{-i\omega_0 t}\mathbf{f}_4 ,$$

where C_i, $i=1,2,3,4$ are arbitrary constants, and $\mathbf{f}_0$ are eigenvectors which have the form

$$\mathbf{f}_i=\begin{bmatrix} k_0^2 \\ \lambda_i \\ \dfrac{-(k_0^4+\lambda_i^2)}{E_0} \\ \dfrac{-\lambda_i(k_0^4+\lambda_i^2)}{E_0} \end{bmatrix} \quad \text{for } i-1,2,3,4 . \tag{9}$$

Obviously the FP Q is the saddle and the center in the (f_1,f_2) and (f_3,f_3) respective subspace.

A very important thing is that the periodic solutions for the ZE's, which are known as the Fermi-Pasta-Ulam (FPU) recurrence phenomena [7], are the phase shifts [1]. Then the HS exists only in phase space $(\rho,\partial_t\rho,n,\partial_t n)$, where $\rho=(\mathrm{Re}E)^2+(\mathrm{Im}E)^2$, rather than $(\mathrm{Re}E,\mathrm{Im}E,n,\partial_t n)$. Next, we construct a phase space which is beneficial to investigate HS. Substituting E by $\sqrt{\rho}e^{i\phi}$ we have another linear form at FP Q for the ZE's

$$\dot{y}=Ty , \tag{10}$$

where

$$y=\begin{bmatrix} \rho \\ \mu \\ n \\ m \end{bmatrix}, \quad \mu\equiv\partial_t\rho, \quad m=\partial_t n ,$$

and

$$T=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_0^4 & 0 & -2E_0^2k_0^2 & 0 \\ 0 & 0 & 0 & 1 \\ -k_0^2 & 0 & -k_0^2 & 0 \end{bmatrix} .$$

The local structure of Eq. (10) is qualitatively similar to Eqs. (6). We divide the subspace spanned by the eigenvectors into two classes, the saddle subspace

$$E_S=\mathrm{span}(y_1,y_2)$$

and the center subspace,

$$E_C=\mathrm{span}(y_2,y_4) ,$$

where y_1,y_2 from Eq. (10) are the eigenvectors whose eigenvalues have negative and positive real parts and y_3,y_4 from Eq. (10) are those whose eigenvalues have zero real parts. We will put global information of the system into those subspaces in the following manner:

$$y_j - s_{j1}\rho+s_{j2}\partial_t\rho+s_{j3}n+s_{j4}\partial_t n \tag{11}$$

for $j=1,2,3,4$ where s_{ji} are the elements of matrix S which satisfies

$$STS^{-1}=\begin{bmatrix} T_1' & 0 \\ 0 & T_2' \end{bmatrix} ,$$

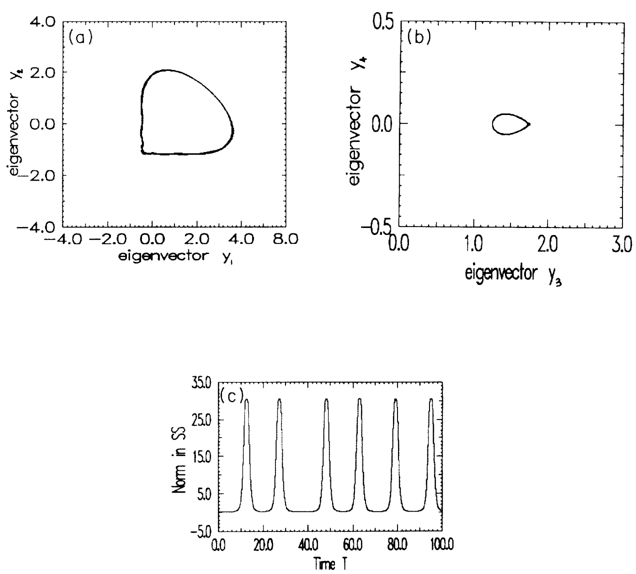

FIG. 1. $C_1=1.0$, $C_2=1.0$. (a) Phase trajectory in the saddle subspace. Notice this orbit forms the KAM curves. (b) Phase trajectory in the center subspace. Motion in this subspace is excited by the motion in the saddle subspace. (c) Time evolution of the norm $(y_1^2+y_2^2)$ in the saddle subspace.

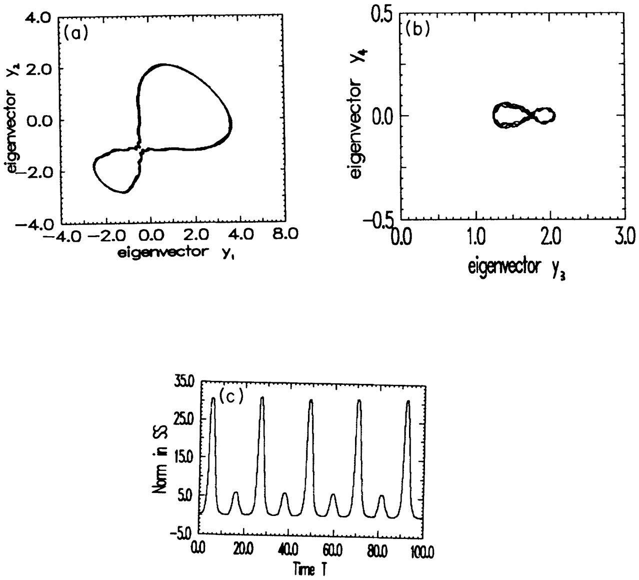

FIG. 2. The same as in Fig. 1, but the adjustable constants are $C_1=1.0$, $C_1=-1.0$.

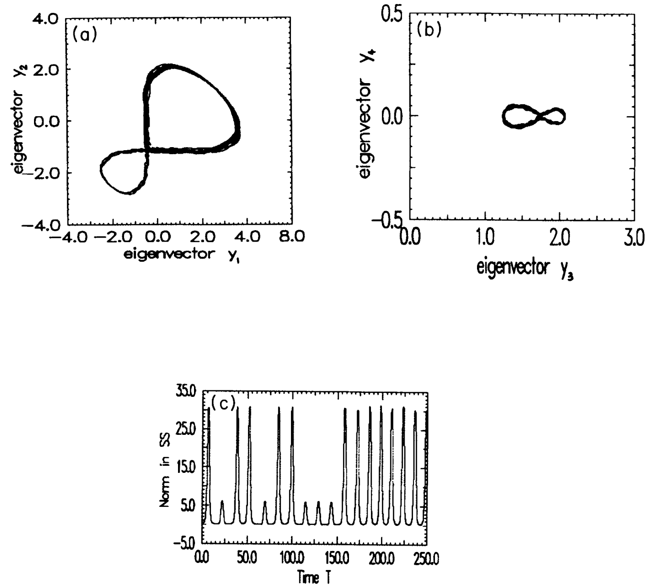

FIG. 3. $C_1=1.0$, $C_2=-0.1$. (a) Phase trajectory in the saddle subspace. Notice irregular homoclinic crossings. (b) Phase trajectory in center subspace. (c) Time evolution of the norm $(y_1^2+y_2^2)$ in the saddle subspace. Notice the irregular period corresponds to homoclinic crossings.

where T_1' and T_2' are the 2×2 matrix, respectively. Numerical solution of Eqs. (1) was obtained with periodic boundary conditions. The spatial length L is chosen in such a way that $k_0 L = 2\pi$, and the number of grids is 64. The initial values are given as

$$\psi=\psi_0+(C_1\mathbf{f}_1+C_2\mathbf{f}_2)E_0\cos(k_0x)/500.0 ,$$

where

$$\psi_0=\begin{bmatrix}\mathrm{Re}E\\ \mathrm{Im}E\\ n\\ m\end{bmatrix}_{t=0}=\begin{bmatrix}E_0\\ 0\\ 0\\ 0\end{bmatrix}, \quad E_0=2.0 ;$$

$\mathbf{f}_1$ and $\mathbf{f}_2$ are given by Eq. (9) and C_1, C_2 are adjustable constants. The dynamic variables are recorded according to Eq. (11) at $x=0$. Our initial conditions ensure that phase trajectories will locally lie in saddle subspace (SS) E_S. The long-time evolution of Eqs. (1) is obtained with C_1 and C_2 given in the following manner.

Case (i). $C_1=1.0$, $C_2=1.0$, the long-time evolution of the system is displayed in SS E_S and center subspace (CS) E_C, which are shown in Figs. 1(a) and 1(b), respectively.

Case (ii). $C_1=1.0$, $C_2=-1.0$, the same long-time evolution is shown in Fig. 2. Summing up Figs. 1 and 2, we see that there are two kinds of recurrence motions, one-loop and double-loop orbits in SS E_S. This property is the same as particle motion in a double-well potential with different level sets. Clearly, this behavior exhibits the HS in the ZE's. Around the FP Q, the SS E_S and CS E_C are invariant subspaces. Beyond local analysis, it is not true. Figures 1(b) and 2(b) show that there is a low level excitation in CS E_S because the motion in SS E_S and in CS E_C influence each other. This coupling typically leads to the creation of "chaotic" orbits for the homoclinic orbits [8]. This is why the ZE's are the near-integrable system. The phase trajectories in Figs. 1(a) and 2(a) are not broken up by the addition of the coupling. These orbits form the KAM curves in the SS E_S the same as a finite dimensional system [4].

Now we discuss the case that the orbit is closing to the homoclinic orbit.

Case (iii). $C_1=1.0$, $C_2=-0.1$, the long-time evolution is displayed in Fig. 3. Comparing Fig. 3(c) to 2(c) and 1(c), it is shown that the time period of orbit in case (iii) is longer than that in cases (i) and (ii). It is easy to understand because the orbits in case (iii) are closing homoclinic orbits whose periods are infinite in relation to cases (i) and (ii). Under the influence of motion in CS E_C Fig. 3(a) shows that the motion in case (iii) is complex in SS E_S. The evolution of the system can now fall "on both sides" of the saddle point—one route corresponding to a rotation and the other to a libration. The system must again and again follow one route or the other. It leads to undetermined consequences. This feature which is known as homoclinic crossings is presented in Fig. 3(c). That indicates the existence of a Hamiltonian chaos for the ZE's.

The FPU recurrence is a very interesting phenomenon that exists in an integrable system, such as nonlinear Schrödinger equation, the Kerteweg–de Vries equation, etc. [9]. Recently, Akhmedive *et al.* studied the FPU problem in a near-integrable system [10]. They conclude that a near-integrable system can lead to pseudorecurrence rather than exact recurrence. This view is not quite right because there exist KAM curves. In our problem cases (i) and (ii) lead to an exact recurrence and case (iii) leads to pseudorecurrence. A more detailed investigation will be presented later.

In conclusion, although there are no inertial manifolds to reduce the dimension in the Hamiltonian system, coherent spatial structure can lead to the degeneracy of degrees of freedom of the system. Some properties of finite dimensional systems, such as KAM curves and homoclinic crossings, are discovered in the ZE's as they relate to typical pattern formation. FPU recurrence in a near-integrable system is also discussed. That Hamiltonian chaos exists in the ZE's is very interesting for a better understanding of Langmuir turbulence.

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