Cumulative beam breakup in linear accelerators with periodic beam current

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An analytic formalism of cumulative beam breakup in linear accelerators is developed. This formalism is applied to both low-velocity ion accelerators and high-energy electron accelerators. It includes arbitrary velocity, acceleration, focusing, initial conditions, beam-cavity resonances, finite bunch length, and arbitrary charge distribution within the bunches, and variable cavity geometry and spacing along the accelerator. For both direct-current beams and beams comprised of δ-function bunches, both the steady-state and transient displacements of the beam are calculated, and scaling laws are determined for the transient beam breakup. The steady-state transverse displacement of particles between bunches is also calculated since, if allowed to impinge on the accelerating structures, these particles could cause activation over long periods of continuous-wave operation. The formalism is then applied to high-current ion accelerators by studying the effects of finite bunch length and arbitrary charge distribution within the bunches. The role of focusing in controlling cumulative beam breakup is quantified in each of these cases. Additionally, the effects of random initial conditions and a distribution of deflecting-mode frequencies in the cavities are also quantified.

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I. INTRODUCTION

The cumulative-beam-breakup (BBU) instability in linear accelerators results when the beam is offset transversely and couples to deflecting dipole modes in the accelerating structures [1, 2]. These modes deflect the trailing portion of the beam, which subsequently couples more strongly to the modes in the next cavity, and the process evolves likewise along the rest of the accelerator. Consequently, the transverse displacement grows, with accompanying degradation of beam quality and possible beam loss to the cavity walls.

Cumulative BBU has been studied principally in conjunction with the design of electron accelerators [3–9], yet it is also of interest in connection with linear accelerators (linacs) for high-current ion beams [10, 11]. For example, in superconducting linacs for high-current ion beams, the constituent cavities will be short and electromagnetically decoupled, and cumulative BBU is therefore expected to be the dominant transverse instability [12, 13]. Moreover, superconducting linacs inherently run in the continuous-wave (cw) mode, and the steady-state properties of their beams are fundamentally important. This is opposite to the case of ultrarelativistic electron linacs, such as those being considered for use in linear colliders, in which relatively few bunches comprise the beam so that only the transient dynamics are relevant [9]. Pulsed linacs for acceleration of relativistic electrons have received considerable attention recently with regard to BBU [9, 14], but cw linacs for the acceleration of high-current ion beams have received very little attention.

This paper concerns cumulative BBU in a linac with smoothly varying parameters which drives a beam of arbitrary velocity and acceleration. Our initial objective was to determine how to control BBU under conditions associated with the long-term cw operation of high-current ion linacs. To do so required the development of an alternative formalism to include finite bunch length and arbitrary charge distribution within the bunches. However, this formalism could be applied to a much broader range of BBU problems, and this single formalism could be used to unify previous work on the cumulative BBU of direct-current beams with previous work on the cumulative BBU of beams comprised of δ-function bunches. Moreover, it could be used to identify and solve various aspects of the problem which were previously left unstudied. Therefore we shall present the formalism in detail, and we shall apply it to problems of interest in linear accelerators spanning the entire spectrum of applications, ranging from low-velocity cw ion linacs to ultrarelativistic pulsed electron linacs. Consequently, we shall be concerned with both the steady-state displacement and the transient displacement of the beam. The steady-state displacement predominates after times long compared to the decay time of the transverse wake fields in the cavities. Otherwise, the transient displacement predominates. In cw linacs, the transient amplitude of the displacement can be controlled by slowly increasing the current during turn-on [15, 16], and the steady-state displacement is therefore of most interest in connection with cw linacs.

In low-velocity accelerators the bunches have finite length, and they can occupy a significant fraction of the radio-frequency (rf) period of the deflecting mode. In turn, their coupling to the wake fields will differ from that of the nearly δ-function bunches which constitute the beams of high-velocity accelerators, and the bunches will deform as they travel down the linac. Therefore the dynamics of cumulative BBU in low-velocity ion accelerators will generally differ from that in high-velocity electron accelerators. We shall calculate the effects of finite bunch length and arbitrary charge distribution within each bunch on the steady-state BBU of a coasting beam. In using a coasting beam to model the problem, we ignore the effects of the synchrotron oscillations within each bunch which occur in the presence of acceleration. In addition, we calculate the effects of random initial conditions and a distribution of deflecting-mode frequencies in the cavities.

The deflecting wake fields between bunches are potentially important, for they will influence any unbunched particles traveling between bunches. These particles may be viewed as constituting a diffuse "longitudinal halo" which responds to the deflecting wake fields but contributes negligibly to them. The deflecting fields may drive the halo into the accelerating structures and thereby activate them. This is a particularly important concern in connection with the design of high-current ion linacs which are envisioned to run cw over long operational lifetimes. Examples include linacs for irradiation of fusion materials [17], for tritium production [18], and for transmutation of nuclear wastes [19]. We shall therefore calculate the steady-state displacement of the longitudinal halo and determine how to control it.

II. EQUATION OF TRANSVERSE MOTION

A. Integro-differential BBU equation

Cumulative beam breakup is analyzed within the framework of a model which incorporates a number of approximations. The cavities which comprise the linac are considered to have negligible length, to be electromagnetically decoupled from one another, and to be the only sources of deflecting fields apart from the focusing elements. These approximations imply that the deflecting fields are localized, i.e., they propagate neither backward nor forward along the accelerator. The approximations may oversimplify the study of beam breakup in certain low-velocity ion accelerators in which each bunch travels slower than the fields and interacts in the future with the deflecting fields it generated in the past. This scenario characterizes regenerative beam breakup [2] and is beyond the scope of our analysis. The approximations may also oversimplify the study of beam breakup in linacs comprised of cavities which are electromagnetically coupled. BBU in the presence of coupling between cavities has been investigated previously [20] and is also beyond the scope of our analysis. The approximations are accurate, however, in a number of practical linacs including, for example, superconducting ion linacs which characteristically are comprised of short, electromagnetically decoupled cavities [11]. In addition, a "continuum approximation" is invoked in which the discrete kicks in transverse momentum imparted by the cavities are considered to be smoothed along the linac.

The equation of transverse motion consolidates the equations governing the excitation of the deflecting fields by the beam and the force imparted on the beam by the fields. With the approximations enumerated above, and allowing for arbitrary velocity and acceleration, the equation of transverse motion is [8, 14]

$$\left[\frac{1}{\beta\gamma}\frac{\partial}{\partial\sigma}\left(\beta\gamma\frac{\partial}{\partial\sigma}\right)+\kappa^2\right]x(\sigma,\zeta) = \varepsilon\int_0^{\zeta} d\zeta'\, w(\zeta-\zeta')\, F(\zeta')\, x(\sigma,\zeta')\,. \quad (2.1)$$

Here, $\sigma = s/\mathcal{L}$ is a dimensionless spatial variable defined in terms of position along the linac s and total length of the linac $\mathcal{L}$; $\beta(\sigma)$ is the particle velocity in units of the speed of light c; $\gamma(\sigma)$ is the particle energy in units of the particle rest-mass energy mc^2; $\zeta = \omega(t - \int ds/\beta c)$ is the time, made dimensionless by use of an angular frequency ω, measured after the arrival of the head of the beam at s; $\kappa(\sigma)$ is the net transverse focusing wave number multiplied by $\mathcal{L}$; $x(\sigma,\zeta)$ is the transverse displacement of the beam centroid from the axis; $\varepsilon(\sigma)$ is a dimensionless quantity which represents the strength of the BBU interaction; $F(\zeta) = I(\zeta)/\bar{I}$ is the form factor for the current defined in terms of the beam current $I(\zeta)$ and average beam current $\bar{I}$; and $w(\zeta)$ is the wake function. Because of causality and the assumed localization of the deflecting fields, $w(\zeta < 0) = 0$. Implicit in the continuum approximation are the assumptions that the distance between neighboring cavities L is small compared to the betatron wavelength associated with focusing, i.e., $\kappa L/\mathcal{L} \ll 1$, and that BBU growth occurs over a length scale which is long compared to L.

A special case arises if a single deflecting mode is present whose frequency is the same in every cavity. This constitutes the worst case for cumulative BBU because a spread in deflecting-mode frequencies suppresses BBU (as will be shown in Sec. VI B), but it is nevertheless of practical interest for the study of BBU control. In this special case, we can choose ω to represent the angular frequency of the deflecting mode. Then the strength of the BBU interaction, represented by $\varepsilon(\sigma)$, is given by

$$\varepsilon(\sigma) = \left(\frac{\bar{I}\mathcal{Z}e}{2\beta\gamma mc}\right)\left(\frac{\Gamma_\perp}{\omega}\right)\left(\frac{\mathcal{L}^2}{L}\right)\,, \quad (2.2)$$

a product of quantities determined by beam properties, cavity properties, and the linac configuration, respectively. $\mathcal{Z}e$ is the net charge of the accelerated particle. The geometry of the cavities determines $\Gamma_\perp(\sigma)$, which is given by

$$\Gamma_\perp(\sigma) = \frac{2}{\epsilon_0\omega}\,\frac{\left|\int_0^l e^{-i\omega z/\beta c}\,\dfrac{\partial E_z(0,0,z)}{\partial x}\,dz\right|^2}{\int_V \mathbf{E}^2(\mathbf{x})\,d\mathbf{x}}\,. \quad (2.3)$$

$\mathbf{E}$ and E_z refer to the rf electric field of the deflecting mode. The integrals are over the inside of the accelerating cavity, and ϵ_0 is the permittivity of free space. The dimensionless "wake function" $w_1(\zeta)$ for the single deflecting mode is given by

$$w_1(\zeta) = u(\zeta)e^{-\zeta/2Q}\sin\zeta\ , \quad (2.4)$$

where $u(\zeta)$ is the unit step function, and Q is the quality factor of the deflecting mode under consideration.

B. Fourier-analyzed BBU equation

We now consider cumulative BBU for the case of a periodic current. The current is composed of an infinite series of bunches with identical but arbitrary length and charge distribution. The bunches are separated in time by period τ, which is a multiple of the rf period of the accelerating mode. For simplicity, we ignore longitudinal synchrotron oscillations within the bunches. A strict treatment of an accelerated beam would have to include these oscillations, but they are not present in a coasting beam, and they become less important in slowly accelerated beams and in relativistic beams. The equation of transverse motion then becomes

$$\left[\frac{1}{\beta\gamma}\frac{\partial}{\partial\sigma}\left(\beta\gamma\frac{\partial}{\partial\sigma}\right)+\kappa^2\right]x(\sigma,\zeta) = \varepsilon\int_{-\infty}^{\zeta} d\zeta'\, w(\zeta-\zeta')\,F(\zeta')\,x(\sigma,\zeta')\ , \quad (2.5)$$

with

$$F(\zeta) = \sum_{k=-\infty}^{+\infty} F_k e^{ik(2\pi/\omega\tau)\zeta}\ . \quad (2.6)$$

Since the current is assumed to be periodic and turned on at $\zeta = -\infty$, the lower limit of the integral in Eq. (2.5) is $-\infty$ and not 0 as it was in Eq. (2.1). This does not restrict Eq. (2.5) to an analysis of the steady-state BBU since no assumption has been made about the initial conditions at $\sigma = 0$, the entrance of the accelerator. For example, a beam which is turned on at $\zeta = 0$ can be treated by taking $x(0,\zeta<0) = 0$ and $\partial x(0,\zeta<0)/\partial\sigma = 0$ because the beam will not generate a deflecting field as long as the current travels on axis.

As shown in Appendix A, upon introducing the Fourier transforms of $x(\sigma,\zeta)$ and $w(\zeta)$,

$$\tilde{x}(\sigma,Z) = \int_{-\infty}^{+\infty} e^{-iZ\zeta}x(\sigma,\zeta)\,d\zeta\ ,$$
$$\tilde{w}(Z) = \int_{-\infty}^{+\infty} e^{-iZ\zeta}w(\zeta)\,d\zeta\ , \quad (2.7)$$

Eq. (2.5) becomes

$$\left[\frac{1}{\beta\gamma}\frac{\partial}{\partial\sigma}\left(\beta\gamma\frac{\partial}{\partial\sigma}\right)+\kappa^2\right]\tilde{x}(\sigma,Z) = \varepsilon\tilde{w}(Z)\sum_{k=-\infty}^{+\infty} F_k\,\tilde{x}\left(\sigma, Z-k\frac{2\pi}{\omega\tau}\right)\ . \quad (2.8)$$

For the special case of the single deflecting mode,

$$\tilde{w}(Z) = \tilde{w}_1(Z) = \left(1 - Z^2 + \frac{1}{4Q^2} + \frac{iZ}{Q}\right)^{-1}\ . \quad (2.9)$$

Upon introducing a "normalized displacement,"

$$\xi(\sigma,\zeta) \equiv \sqrt{\beta\gamma}\,x(\sigma,\zeta)\ , \quad (2.10)$$

Eq. (2.8) takes the form

$$\left(\frac{\partial^2}{\partial\sigma^2}+K^2\right)\tilde{\xi}(\sigma,Z) = \varepsilon\tilde{w}(Z)\sum_{k=-\infty}^{+\infty} F_k\,\tilde{\xi}\left(\sigma, Z-k\frac{2\pi}{\omega\tau}\right)\ , \quad (2.11)$$

where

$$K^2(\sigma) \equiv \kappa^2(\sigma) + \frac{1}{4(\beta\gamma)^4}\left[(\gamma^2+2)\left(\frac{d\gamma}{d\sigma}\right)^2 - 2\gamma(\beta\gamma)^2\frac{d^2\gamma}{d\sigma^2}\right]\ . \quad (2.12)$$

Equation (2.11) is a difference-differential equation for the Fourier transform of the displacement of periodic bunches of arbitrary length and current distribution. It replaces, and is equivalent to, the integro-differential equation Eq. (2.1), which is more commonly used. We shall use Eq. (2.11) as the starting point for our investigations of cumulative BBU.

III. DIRECT-CURRENT BEAM

A. General solution

For the case of a dc beam, the Fourier components of the form factor of the beam current are $F_0 = 1$ and $F_{k\neq 0} = 0$, with which Eq. (2.11) becomes

$$\left(\frac{\partial^2}{\partial\sigma^2}+K^2\right)\tilde{\xi}(\sigma,Z) = \varepsilon\tilde{w}(Z)\tilde{\xi}(\sigma,Z)\ . \quad (3.1)$$

Equation (3.1) can be solved in closed form using the WKBJ approximation. This solution, expressed in terms of the functionals

$$\left.\begin{matrix}\mathcal{C}[f(\sigma,Z)]\\ \mathcal{S}[f(\sigma,Z)]\end{matrix}\right\} \equiv \sqrt{\frac{f(0,Z)}{f(\sigma,Z)}} \times \left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}\int_0^{\sigma} f(\sigma',Z)\,d\sigma'\ , \quad (3.2)$$

and the quantity

$$\lambda^2(\sigma,Z) \equiv K^2(\sigma) - \varepsilon(\sigma)\tilde{w}(Z)\ , \quad (3.3)$$

and written in (σ,ζ) space, is

$$\xi(\sigma,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dZ\, e^{iZ\zeta}\left(A(Z)\mathcal{C}[\lambda(\sigma,Z)] + B(Z)\frac{\mathcal{S}[\lambda(\sigma,Z)]}{\lambda(0,Z)}\right)\ . \quad (3.4)$$

$A(Z)$ and $B(Z)$ are determined from the initial conditions at $\sigma = 0$:

$$A(Z) = \int_{-\infty}^{+\infty} e^{-iZ\zeta}\xi(0,\zeta)\,d\zeta\ ,$$
$$B(Z) = \int_{-\infty}^{+\infty} e^{-iZ\zeta}\frac{\partial\xi(0,\zeta)}{\partial\sigma}\,d\zeta\ . \tag{3.5}$$

Letting $\xi' = \partial\xi/\partial\sigma$, the general solution for the transverse displacement of a dc beam can also be written in the form

$$\xi(\sigma,\zeta) = \int_{-\infty}^{+\infty} c(\sigma,\mu)\xi(0,\zeta-\mu)\,d\mu + \int_{-\infty}^{+\infty} s(\sigma,\mu)\xi'(0,\zeta-\mu)\,d\mu\ , \tag{3.6}$$

with

$$c(\sigma,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\zeta Z}\mathcal{C}[\lambda(\sigma,Z)]\,dZ\ ,$$
$$s(\sigma,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\zeta Z}\frac{\mathcal{S}[\lambda(\sigma,Z)]}{\lambda(0,Z)}\,dZ\ . \tag{3.7}$$

The functions $c(\sigma,\zeta)$ and $s(\sigma,\zeta)$ are the inverse Fourier transforms of $\mathcal{C}[\lambda(\sigma,Z)]$ and $\frac{\mathcal{S}[\lambda(\sigma,Z)]}{\lambda(0,Z)}$, respectively, which satisfy $c(\sigma,\zeta<0) = s(\sigma,\zeta<0) = 0$ because of causality.

B. Steady-state solution

We consider the steady-state displacement induced by a "misaligned beam" for which $\xi(0,\zeta) = \xi_0$ and $\xi'(0,\zeta) = \xi_0'$. According to Eq. (3.5), these initial conditions yield

$$A(Z) = 2\pi\xi_0\delta(Z)\ ,$$
$$B(Z) = 2\pi\xi_0'\delta(Z)\ . \tag{3.8}$$

Upon performing the integration over Z in Eq.(3.4), the steady-state displacement is found to be

$$\xi(\sigma,\zeta) = \xi_0\mathcal{C}[\lambda(\sigma,0)] + \xi_0'\frac{\mathcal{S}[\lambda(\sigma,0)]}{\lambda(0,0)}\ . \tag{3.9}$$

Here and throughout the paper, we shall be concerned about the stability of the beam. If a misaligned beam experiences a net transverse force at position σ which is directed toward the axis, the beam is said to be locally "stable." Otherwise, it is said to be locally "unstable." According to Eq. (3.9), the beam is locally stable at σ with respect to steady-state BBU if $\lambda(\sigma,0)$ is real. The beam is locally unstable at σ if $\lambda(\sigma,0)$ is imaginary.

In the special case of a coasting beam, $K(\sigma) = \kappa$ and $\lambda(\sigma,Z) = \lambda(Z)$ are independent of σ, and the transverse displacement along the linac is

$$x(\sigma,\zeta) = x_0\cos[\lambda(0)\sigma] + x_0'\frac{\sin[\lambda(0)\sigma]}{\lambda(0)}\ , \tag{3.10}$$

where, from Eq. (3.3),

$$\lambda^2(0) = \kappa^2 - \varepsilon\tilde{w}(0)\ . \tag{3.11}$$

If we assume a single deflecting mode, then from Eq. (2.9),

$$\lambda_1^2(0) = \kappa^2 - \varepsilon\frac{4Q^2}{4Q^2+1}\ . \tag{3.12}$$

Since generally $Q^2 \gg 1$, it follows that the coasting dc beam is stable with respect to steady-state BBU if the focusing strength κ^2 is larger than the BBU coupling constant ε. Otherwise, it is unstable. Stability is a global property in the case of a coasting beam because λ has the same value at all points along the linac.

In the case of a sinusoidal time dependence of the offset at the accelerator entrance, $\xi(0,\zeta) = \xi_0\cos\left(\frac{\omega'}{\omega}\zeta\right)$ and $\xi'(0,\zeta) = 0$, the transverse displacement is given by

$$\xi(\sigma,\zeta) = \xi_0\mathrm{Re}\{e^{i(\omega'/\omega)\zeta}\mathcal{C}[\lambda(\sigma,\omega'/\omega)]\}\ . \tag{3.13}$$

The local stability of the beam is now determined by $\lambda(\sigma,\omega'/\omega)$ rather than by $\lambda(\sigma,0)$, which characterizes the misaligned beam. This is the signature of the different boundary conditions.

In the case of a beam whose offset at $\sigma = 0$ includes many frequencies, stability is not as easily defined in terms of the sign of the transverse force. However, as indicated by Eqs. (3.4) and (3.6), the various Fourier components of the initial offset evolve independently of each other along the linac, and the beam will be stable if each Fourier component is stable. Thus, because of the linearity of Eqs. (3.4) and (3.6), the stability of the beam will be determined by $\lambda(\sigma,Z)$ for all the values of Z where $A(Z)$ and $B(Z)$ defined in Eq. (3.5) are different from 0.

C. Transient solution

A beam turned on at $\zeta = 0$ can be modeled by assuming $\xi(\zeta<0) = \xi'(\zeta<0) = 0$, and Eq. (3.6) becomes

$$\xi(\sigma,\zeta) = \frac{1}{2\pi}\int_0^\zeta d\mu\int_{-\infty}^{+\infty} dZ\,e^{iZ\mu} \times\left(\xi(0,\zeta-\mu)\mathcal{C}[\lambda(\sigma,Z)] + \xi'(0,\zeta-\mu)\frac{\mathcal{S}[\lambda(\sigma,Z)]}{\lambda(0,Z)}\right)\ . \tag{3.14}$$

It is convenient to define a different integration variable θ which aids in comparing our results with those of previous authors:

$$\theta \equiv i\left(\frac{1}{2Q} + iZ\right)\ , \tag{3.15}$$

for which

$$\lambda^2(\sigma,\theta) = K^2(\sigma) - \varepsilon(\sigma)\tilde{w}(\theta)\ . \tag{3.16}$$

Q is chosen so that all the poles of $\tilde{w}(\theta)$ lie on or below the line $\mathrm{Im}(\theta) = 0$. As shown in Appendix B, the displacement $\xi(\sigma,\zeta)$ given in Eq. (3.14) can now be expressed in the form

$$\xi(\sigma,\zeta) = \frac{1}{2\pi}\int_0^{\zeta} d\mu\, e^{-(\mu/2Q)} \int_{-\infty}^{+\infty} d\theta\, e^{-i\mu\theta}\left(\xi(0,\zeta-\mu)\mathcal{C}[\lambda(\sigma,\theta)] + \xi'(0,\zeta-\mu)\frac{\mathcal{S}[\lambda(\sigma,\theta)]}{\lambda(0,\theta)}\right) . \tag{3.17}$$

We shall consider two classes of solutions for the transient displacement corresponding to two distinctly different initial conditions. The first class of solutions (I) corresponds to an impulse excitation of the wake field, and the second class of solutions (II) corresponds to the misaligned beam. "Impulse excitation" refers to a δ-function impulse in which only the head of the beam is displaced at the entrance to the linac, i.e., $\xi(0,\zeta) = \xi_0\delta(\zeta)$ and $\xi_0'(0,\zeta) = \xi_0'\delta(\zeta)$. The misaligned beam has initial conditions $\xi(0,\zeta) = \xi_0 u(\zeta)$ and $\xi'(0,\zeta) = \xi_0' u(\zeta)$. From Eq. (3.17), the steady-state displacements for the two classes of solutions are

$$\xi(\sigma,\infty) = \begin{cases} 0, \ \text{class I} & (3.18)\\ \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\theta \frac{1}{i\theta+\frac{1}{2Q}}\left(\xi_0\mathcal{C}[\lambda(\sigma,\theta)] + \xi_0'\frac{\mathcal{S}[\lambda(\sigma,\theta)]}{\lambda(0,\theta)}\right), \ \text{class II.} & (3.19)\end{cases}$$

In the latter case, the integrand has a simple pole at $\theta = i/2Q$, so contour integration around the upper half plane yields directly the steady-state solution Eq. (3.9). The transient displacements for the two classes of solutions are

$$\xi(\sigma,\zeta) = \frac{e^{-\zeta/2Q}}{2\pi}\int_{-\infty}^{+\infty} d\theta\, e^{-i\zeta\theta}\left(\xi_0\mathcal{C}[\lambda(\sigma,\theta)] + \xi_0'\frac{\mathcal{S}[\lambda(\sigma,\theta)]}{\lambda(0,\theta)}\right) , \ \text{class I} \tag{3.20}$$

$$\xi(\sigma,\zeta) - \xi(\sigma,\infty) = -\frac{e^{-\zeta/2Q}}{2\pi}\int_{-\infty}^{+\infty} d\theta \frac{e^{-i\zeta\theta}}{i\theta+\dfrac{1}{2Q}}\left(\xi_0\mathcal{C}[\lambda(\sigma,\theta)] + \xi_0'\frac{\mathcal{S}[\lambda(\sigma,\theta)]}{\lambda(0,\theta)}\right) , \ \text{class II.} \tag{3.21}$$

To provide examples of the applications of Eqs. (3.20) and (3.21), we shall calculate the transient growth of a coasting beam in the presence of a single deflecting mode. In this case, $\tilde{w}_1(\theta)$ is given by

$$\tilde{w}_1(\theta) = \frac{1}{1-\theta^2} , \tag{3.22}$$

and $\lambda_1(\theta)$ is given by

$$\lambda_1^2(\theta) = \kappa^2 - \epsilon\tilde{w}_1(\theta) . \tag{3.23}$$

Our goal is to determine the dependence of $x(\sigma,\zeta)$ on the various physical parameters once BBU has significantly amplified the beam displacement. For this purpose, asymptotic solutions suffice, and these solutions can be generated using the method of steepest descent [1]. When BBU is pronounced, Eqs. (3.20) and (3.21) both take the form

$$x(\sigma,\zeta) - x(\sigma,\infty) \simeq 2\,\mathrm{Re}\left(\int_{-\infty}^{+\infty} d\theta\, g(\theta)e^{f(\sigma,\theta)}\right) , \tag{3.24}$$

where

$$f(\sigma,\theta) \equiv -i[\zeta\theta + \lambda_1(\theta)\sigma], \tag{3.25}$$

and

$$g(\theta) \equiv \begin{cases} \dfrac{e^{-\zeta/2Q}}{4\pi}\left(x_0 + \dfrac{ix_0'}{\lambda_1(\theta)}\right) , \ \text{class I} & (3.26)\\ -\dfrac{e^{-\zeta/2Q}}{4\pi\left(i\theta+\frac{1}{2Q}\right)}\left(x_0 + \dfrac{ix_0'}{\lambda_1(\theta)}\right) , \ \text{class II.} & (3.27)\end{cases}$$

The integral of Eq. (3.24) results in a function exhibiting rapid oscillations in ζ with a slowly varying amplitude. The amplitude is of principal interest to the accelerator designer. Therefore, in what follows, we shall express our results for the approach to steady state in terms of the amplitude. The amplitude will be referred to as the "beam envelope," and the approach to steady state will be designated as $|x(\sigma,\zeta) - x(\sigma,\infty)|$. Because the exponential factor $f(\sigma,\theta)$ is the same for both classes of solutions, the transient BBU growth rates of the beam envelopes will likewise be identical. However, the amplitude function $g(\theta)$ differs for the two classes, and the magnitudes of the beam envelopes will likewise differ.

As shown in Appendix C, the saddle points θ_s are determined approximately from the following quartic equation:

$$(\psi-1)^4 + s_1(\psi-1)^3 - s_2^2\psi = 0 , \tag{3.28}$$

where

$$\psi \equiv \theta_s^2, \quad s_1 \equiv \frac{\varepsilon}{\kappa^2}, \quad s_2 \equiv \frac{\varepsilon\sigma}{\kappa\zeta}. \tag{3.29}$$

The evaluation of $f(\sigma,\theta)$ at the saddle points results in exponential growth $e^{\Gamma\zeta}$, where the transient BBU growth rate Γ is

$$\Gamma = \mathrm{Im}\left[\sqrt{\psi}\left(1 + \frac{s_2^2/s_1}{(\psi-1)^2}\right)\right] , \tag{3.30}$$

in which ψ is the root of the quartic equation generating the largest growth rate.

This expression for the transient growth rate of a dc beam is identical to that found by Lau [8], yet it was calculated using an entirely different approach. Lau calculated the cumulative BBU growth rate of a direct-current, coasting beam in the presence of a single deflecting mode by evaluating the Green's function of the integro-differential equation of motion. By contrast, our calculation of the growth rate is based on solving the Fourier-transformed equation of transverse motion. Lau's formulation is restricted to the study of the class of solutions I corresponding to an impulse excitation of the wake field in which only the head of the pulse is displaced at the entrance to the linac. Our formulation allows the incorporation of arbitrary initial conditions. It is restricted neither to special initial conditions, nor to a coasting beam, nor to a special choice of wake function.

Following Lau, we categorize the transient dc BBU into four domains, each corresponding to different relative strengths of the dimensionless parameters s_1 and s_2. The first two domains, A and B, are characterized by sufficiently small values of s_2, and the saddle points ψ are given by the following approximate form of Eq. (3.28):

$$\psi \simeq 1+\Delta, \quad |\Delta| \ll 1, \quad \Delta^4+s_1\Delta^3-s_2^2 \simeq 0 \ \text{ for small } s_2. \tag{3.31}$$

The last two domains, C and D, are characterized by sufficiently large values of s_2, and the saddle points ψ are given by the following approximate form of Eq. (3.28):

$$|\psi| \gg 1, \quad \psi^4 + s_1\psi^3 - s_2^2\psi \simeq 0 \ \text{ for large } s_2. \tag{3.32}$$

The saddle points, growth rates, and beam envelopes are calculated using the asymptotic methods of Appendix C. For each domain and for classes I and II of solutions, they are as follows:

Domain A: $s_2 \ll s_1^{1/2}$ and $s_2 \ll s_1^2$ (weak focusing, weak BBU coupling, long pulse length). In terms of the dimensionless parameter

$$E_A \equiv \left(\frac{s_2^2}{s_1}\right)^{\frac{1}{3}} \zeta \ \propto \ \sigma^{2/3}\zeta^{1/3}\,, \tag{3.33}$$

the dominant saddle point and growth rate are, respectively,

$$\psi \simeq 1 + \frac{E_A}{\zeta} e^{2\pi i/3}\,, \quad \Gamma \simeq \frac{3\sqrt{3}}{4}\frac{E_A}{\zeta}\,, \tag{3.34}$$

and the beam envelopes (for classes I and II, respectively) are given by

$$|x(\sigma,\zeta)| \simeq \left[\left(x_0 + \frac{\sqrt{3}}{2}\frac{\sigma}{E_A}x_0'\right)^2 + \left(\frac{\sigma}{2E_A}x_0'\right)^2\right]^{1/2} \times \frac{\sqrt{E_A}}{\zeta\sqrt{6\pi}} \exp\left(\frac{3\sqrt{3}}{4}E_A - \frac{\zeta}{2Q}\right)\,, \tag{3.35}$$

$$|x(\sigma,\zeta) - x(\sigma,\infty)| \simeq \left[\left(x_0 + \frac{\sqrt{3}}{2}\frac{\sigma}{E_A}x_0'\right)^2 + \left(\frac{\sigma}{2E_A}x_0'\right)^2\right]^{1/2} \times \frac{\sqrt{E_A}}{\zeta\sqrt{6\pi}}\sqrt{\frac{4Q^2}{4Q^2+1}} \exp\left(\frac{3\sqrt{3}}{4}E_A - \frac{\zeta}{2Q}\right)\,. \tag{3.36}$$

Panofsky and Bander [1] were the first to study this domain and derive the growth rate presented here. They treated extremely relativistic dc beams with the purpose of explaining the pulse shortening observed in connection with the operation of the Stanford Linear Accelerator Center (SLAC) two-mile electron accelerator. Later, in what appears to be the only publication explicitly concerning BBU in nonrelativistic beams, Neil and Cooper [10] extended the analysis of Panofsky and Bander and recovered their results in the limit of a coasting dc beam. Neil and Cooper also considered an unfocused, accelerated dc beam within the framework of the WKBJ approximation, and they applied the corresponding results to estimate BBU in a low-velocity electron injector. Their growth rates are also easily recovered with our formalism, but they correspond to only a subset of the possible solutions implied by Eqs. (3.20) and (3.21).

Domain B: $s_1^2 \ll s_2 \ll 1$ (strong focusing, moderate BBU coupling, moderate pulse length). In terms of the dimensionless parameter

$$E_B \equiv \sqrt{s_2}\zeta \ \propto \ \sigma^{1/2}\zeta^{1/2}\,, \tag{3.37}$$

the dominant saddle point and growth rate are, respectively,

$$\psi \simeq 1 + i\frac{E_B}{\zeta}\left(1 + i\frac{s_1\zeta}{4E_B}\right)\,, \quad \Gamma \simeq \frac{E_B}{\zeta}\,, \tag{3.38}$$

and the beam envelopes are given by (for classes I and II, respectively)

$$|x(\sigma,\zeta)| \simeq \left[x_0^2 + \left(\frac{x_0'}{\kappa}\right)^2\right]^{1/2} \frac{\sqrt{E_B}}{2\zeta\sqrt{2\pi}} \exp\left(E_B - \frac{\zeta}{2Q}\right)\,, \tag{3.39}$$

$$|x(\sigma,\zeta) - x(\sigma,\infty)| \simeq \left[x_0^2 + \left(\frac{x_0'}{\kappa}\right)^2\right]^{1/2} \frac{\sqrt{E_B}}{2\zeta\sqrt{2\pi}}\sqrt{\frac{4Q^2}{4Q^2+1}} \times \exp\left(E_B - \frac{\zeta}{2Q}\right)\,. \tag{3.40}$$

Neil, Hall, and Cooper [3] were the first to study this domain and account for the effect of strong focusing on the exponential growth of dc beams in the beam transport system of induction linacs.

Domain C: $s_2 \gg 1$ and $s_2 \gg s_1^{3/2}$ (strong focusing, strong BBU coupling, short pulse length). In terms of the dimensionless parameter

$$E_C \equiv s_2^{1/3}\zeta \ \propto \ \sigma^{1/3}\zeta^{2/3}\,, \tag{3.41}$$

the dominant saddle point and growth rate are, respectively,

$$\psi \simeq \left(\frac{E_C}{\zeta}\right)^2 e^{2\pi i/3} - \frac{1}{3}s_1\,, \quad \Gamma \simeq \frac{3\sqrt{3}}{4}\frac{E_C}{\zeta}\,, \tag{3.42}$$

and the beam envelopes are given by (for classes I and II, respectively)

$$|x(\sigma,\zeta)| \simeq \left[x_0^2 + \left(\frac{x_0'}{\kappa}\right)^2\right]^{1/2} \frac{\sqrt{E_C}}{\zeta\sqrt{6\pi}} \exp\left(\frac{3\sqrt{3}}{4}E_C - \frac{\zeta}{2Q}\right)\,, \tag{3.43}$$

$$|x(\sigma,\zeta)-x(\sigma,\infty)| \simeq \left[x_0^2+\left(\frac{x_0'}{\kappa}\right)^2\right]^{1/2}\frac{1}{\sqrt{6\pi E_C}} \times \exp\left(\frac{3\sqrt{3}}{4}E_C-\frac{\zeta}{2Q}\right) . \quad (3.44)$$

This domain is of interest for the study of the head-tail instability, i.e., single-bunch BBU, in a linear collider. Chao, Richter, and Yao [14] were the first to consider this domain, and they did so to quantify BBU-induced emittance growth of a single bunch traveling along a next-generation, single-pass linear collider. Neri and Gluckstern [21] later reworked the problem by decomposing the single bunch into a series of closely spaced macroparticles and calculating the displacement of each macroparticle.

Domain D: $s_1^{3/2} \gg s_2 \gg s_1^{1/2}$ *(weak focusing, strong BBU coupling, short pulse length).* In terms of the dimensionless parameter

$$E_D \equiv \left(\frac{s_2^2}{s_1}\right)^{1/4}\zeta \propto \sigma^{1/2}\zeta^{1/2} , \quad (3.45)$$

the dominant saddle point and growth rate are, respectively,

$$\psi \simeq -\left(\frac{E_D}{\zeta}\right)^2 , \quad \Gamma \simeq 2\frac{E_D}{\zeta} , \quad (3.46)$$

and the beam envelopes are given by (for classes I and II, respectively)

$$|x(\sigma,\zeta)| \simeq \left(x_0+\frac{\sigma}{E_D}x_0'\right)\frac{\sqrt{E_D}}{2\zeta\sqrt{\pi}}\exp\left(2E_D-\frac{\zeta}{2Q}\right) , \quad (3.47)$$

$$|x(\sigma,\zeta)-x(\sigma,\infty)| \simeq \left(x_0+\frac{\sigma}{E_D}x_0'\right)\frac{1}{2\sqrt{\pi E_D}} \times \exp\left(2E_D-\frac{\zeta}{2Q}\right) . \quad (3.48)$$

Lau [8] apparently was the first to establish the existence of this domain.

IV. δ-FUNCTION BEAM

This section concerns the evolution of BBU in a beam consisting of δ-function bunches separated by period τ. Beam bunching introduces a new degree of freedom represented by the parameter $\omega\tau$ which quantifies the interaction of the beam with the wake field oscillating with a representative angular frequency ω.

A. General solution

For the case of a beam composed of δ-function bunches, the Fourier components of the form factor are $F_k = 1$ for all k, and Eq. (2.11) becomes

$$\left(\frac{\partial^2}{\partial\sigma^2}+K^2\right)\tilde{\xi}(\sigma,Z) = \varepsilon\tilde{w}(Z)\sum_{k=-\infty}^{+\infty}\tilde{\xi}\left(\sigma,Z-k\frac{2\pi}{\omega\tau}\right) . \quad (4.1)$$

As shown in Appendix D, Eq. (4.1) can be solved in closed form using the WKBJ approximation. This solution, expressed in terms of the functionals $\mathcal{C}[f(\sigma,Z)]$ and $\mathcal{S}[f(\sigma,Z)]$ given in Eq. (3.2), and the auxiliary quantities

$$\tilde{W}(Z) \equiv \sum_{k=-\infty}^{+\infty}\tilde{w}\left(Z-k\frac{2\pi}{\omega\tau}\right) = \omega\tau\sum_{k=0}^{+\infty}w(k\omega\tau)e^{-ik\omega\tau Z} , \quad (4.2)$$

$$\Lambda^2(\sigma,Z) \equiv K^2(\sigma)-\varepsilon(\sigma)\tilde{W}(Z) , \quad (4.3)$$

and written in (σ,ζ) space, is

$$\xi(\sigma,\zeta) = \xi(0,\zeta)\mathcal{C}[K(\sigma)]+\xi'(0,\zeta)\frac{\mathcal{S}[K(\sigma)]}{K(0)} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}dZ\,e^{iZ\zeta}\frac{\tilde{w}(Z)}{\tilde{W}(Z)}\left[A(Z)\{\mathcal{C}[\Lambda(\sigma,Z)]-\mathcal{C}[K(\sigma)]\}+B(Z)\left(\frac{\mathcal{S}[\Lambda(\sigma,Z)]}{\Lambda(0,Z)}-\frac{\mathcal{S}[K(\sigma)]}{K(0)}\right)\right] . \quad (4.4)$$

$A(Z)$ and $B(Z)$ are determined from the initial conditions at $\sigma = 0$:

$$A(Z) = \omega\tau\sum_{k=-\infty}^{+\infty}\xi(0,k\omega\tau)e^{-ik\omega\tau Z} ,$$
$$B(Z) = \omega\tau\sum_{k=-\infty}^{+\infty}\xi'(0,k\omega\tau)e^{-ik\omega\tau Z} . \quad (4.5)$$

The beam displacement is defined for all values of ζ, which includes, but is not limited to, $\zeta = m\omega\tau$ representing bunch m. In particular, Eq. (4.4) can be used to calculate the transverse displacement of unbunched particles located outside the bunches. These unbunched particles comprise the "longitudinal halo." They do not contribute to the beam breakup, but they experience the deflecting fields generated by the bunches.

It is convenient for some calculations to decompose the integral over Z into a sum of integrals over intervals of length $2\pi/\omega\tau$ centered on the bunches. Upon changing the integration variable in the manner indicated in Appendix D, the result is

$$\xi(\sigma,\zeta) = \xi(0,\zeta)\mathcal{C}[K(\sigma)] + \xi'(0,\zeta)\frac{\mathcal{S}[K(\sigma)]}{K(0)}$$
$$+\frac{1}{2\pi}\int_{-\pi/\omega\tau}^{\pi/\omega\tau} dZ\, e^{i\zeta Z}\chi(Z,\zeta)\left[A(Z)\{\mathcal{C}[\Lambda(\sigma,Z)] - \mathcal{C}[K(\sigma)]\} + B(Z)\left(\frac{\mathcal{S}[\Lambda(\sigma,Z)]}{\Lambda(0,Z)} - \frac{\mathcal{S}[K(\sigma)]}{K(0)}\right)\right], \quad (4.6)$$

where

$$\chi(Z,\zeta) = \frac{1}{\tilde{W}(Z)}\sum_{k=-\infty}^{+\infty} e^{ik(2\pi/\omega\tau)\zeta}\,\tilde{w}\left(Z + k\frac{2\pi}{\omega\tau}\right). \quad (4.7)$$

This function is periodic in Z with period $2\pi/\omega\tau$ and in ζ with period $\omega\tau$, and it satisfies $\chi(Z, m\omega\tau) = 1$. The remainder of this section is directed toward applications of Eq. (4.6).

B. Steady-state solution and longitudinal halo

Steady-state BBU is of greater concern than transient BBU in connection with the design of linacs for long-term cw operation. We therefore consider, as an example, the steady-state displacement induced by a misaligned beam for which $\xi(0,\zeta) = \xi_0$ and $\xi'(0,\zeta) = \xi_0'$. According to Eq. (4.5), these initial conditions yield

$$A(Z) = 2\pi\xi_0 \sum_{k=-\infty}^{+\infty} \delta\left(Z - k\frac{2\pi}{\omega\tau}\right),$$
$$B(Z) = 2\pi\xi_0' \sum_{k=-\infty}^{+\infty} \delta\left(Z - k\frac{2\pi}{\omega\tau}\right). \quad (4.8)$$

Upon integrating over Z in Eq. (4.6), the steady-state transverse displacement is found to be

$$\xi(\sigma,\zeta) = \xi_0\mathcal{C}[K(\sigma)] + \xi_0'\frac{\mathcal{S}[K(\sigma)]}{K(0)} + \chi(0,\zeta)\left[\xi_0\{\mathcal{C}[\Lambda(\sigma,0)] - \mathcal{C}[K(\sigma)]\} + \xi_0'\left(\frac{\mathcal{S}[\Lambda(\sigma,0)]}{\Lambda(0,0)} - \frac{\mathcal{S}[K(\sigma)]}{K(0)}\right)\right]. \quad (4.9)$$

It is periodic in ζ with period $\omega\tau$. The "amplitude function" $\chi(0,\zeta)$ represents the relative displacement of particles in and between the bunches. To find the displacement of the bunches themselves, we set $\zeta = 0$. The result is

$$\xi(\sigma,\zeta) = \xi_0\mathcal{C}[\Lambda(\sigma,0)] + \xi_0'\frac{\mathcal{S}[\Lambda(\sigma,0)]}{\Lambda(0,0)}. \quad (4.10)$$

In the special case of a coasting beam, $K(\sigma) = \kappa$ and $\Lambda(\sigma,Z) = \Lambda(Z)$ are independent of σ, and the steady-state displacement along the linac is

$$x(\sigma,\zeta) = x_0\cos\kappa\sigma + x_0'\frac{\sin\kappa\sigma}{\kappa} + \chi(0,\zeta)\left[x_0\{\cos[\Lambda(0)\sigma] - \cos\kappa\sigma\} + x_0'\left(\frac{\sin[\Lambda(0)\sigma]}{\Lambda(0)} - \frac{\sin\kappa\sigma}{\kappa}\right)\right]. \quad (4.11)$$

The steady-state displacement of the bunches themselves is given by

$$x(\sigma,0) = x_0\cos[\Lambda(0)\sigma] + x_0'\frac{\sin[\Lambda(0)\sigma]}{\Lambda(0)}. \quad (4.12)$$

Thus the steady-state BBU of a coasting beam comprised of δ-function bunches is globally stable if $\Lambda(0)$ is real.

In the presence of a single deflecting mode with the wake function $w_1(\zeta)$, $\tilde{W}(Z)$ is given by

$$\tilde{W}_1(Z) = \frac{\omega\tau}{2}\,\frac{\sin\omega\tau}{\cosh\left[\omega\tau\left(\frac{1}{2Q} + iZ\right)\right] - \cos\omega\tau}. \quad (4.13)$$

The functions $\chi(0,\zeta)$ and $\Lambda(0)$ found from $w_1(\zeta)$ can be expressed in terms of functions p and q which contain the resonances between the frequencies $1/\tau$ of the beam bunches and $\omega/2\pi$ of the deflecting mode:

$$p(\omega\tau,Q) = \sum_{k=1}^{+\infty} e^{-k(\omega\tau/2Q)}\sin k\omega\tau = \frac{1}{4}\,\frac{\sin\omega\tau}{\sinh^2\frac{\omega\tau}{4Q} + \sin^2\frac{\omega\tau}{2}}, \quad (4.14)$$

$$q(\omega\tau,Q) = \sum_{k=1}^{+\infty} e^{-k(\omega\tau/2Q)}\cos k\omega\tau = \frac{1}{4}\,\frac{\cos\omega\tau - e^{-\omega\tau/2Q}}{\sinh^2\frac{\omega\tau}{4Q} + \sin^2\frac{\omega\tau}{2}}. \quad (4.15)$$

The results are, for $0 \le \zeta/\omega\tau \le 1$,

$$\chi_1(0,\zeta) = e^{-\zeta/2Q}\left(\frac{1+q}{p}\sin\zeta + \cos\zeta\right) \quad (4.16)$$

and

$$\Lambda_1(0) = \sqrt{\kappa^2 - \varepsilon\,\omega\tau p}. \quad (4.17)$$

The stability of steady-state BBU with a single deflecting mode is largely determined by the resonance function p. This function is nearly independent of Q (being approximated by $\frac{1}{2}\cot\frac{\omega\tau}{2}$) except in the vicinity of the zero crossings at $\omega\tau = 2n\pi$. It reaches its peak values at $\omega\tau = 2n\pi\left(1 \pm \frac{1}{2Q}\right)$ where it reaches the extrema $\pm\frac{Q}{2n\pi}$, respectively. At the zero crossings where $\omega\tau = (2n+1)\pi$, it is zero. These limiting values of p are due to the fact that the contributions to the excitation of the wake fields by the bunches have the same signs at the even-integral zero crossings and alternating signs at the odd-integral zero crossings. Thus the cumulative wake field is strongest at the even-integral zero crossings.

Equations (4.9) and (4.11) give the steady-state transverse displacement in the general case and for the coasting beam, respectively, for all values of the dimensionless time ζ, while Eqs. (4.10) and (4.12) give the displacements of the bunches themselves. The relative displacement of the particles located between the bunches and of the bunches themselves is determined by $\chi(0,\zeta)$ defined in Eq. (4.7). Unbunched particles located outside the bunches may be displaced substantially more than the bunches themselves. This is implied in Fig. 1, which provides example plots of the amplitude function $\chi_1(0,\zeta)$ with $Q = 1000$. As specific examples of this occurrence, we list three special cases derived from Eq. (4.11) where we assume a coasting beam with $x_0' = 0$ for simplicity and $\omega\tau/2Q \ll 1$.

Case 1: $\omega\tau = 2n\pi\left(1 + \frac{1}{2Q}\right)$ *(resonance)*:

$$\frac{x(\sigma,\zeta)}{x_0} = \cos\kappa\sigma + (\sin\zeta + \cos\zeta) \times \left[\cosh\left(\sigma\sqrt{\varepsilon Q - \kappa^2}\right) - \cos\kappa\sigma\right] . \quad (4.18)$$

Case 2: $\omega\tau = 2n\pi$ *(even-integral zero crossing of wake function)*:

$$\frac{x(\sigma,\zeta)}{x_0} = \cos\kappa\sigma + \frac{\varepsilon Q}{\kappa}\sigma\sin\kappa\sigma\sin\zeta . \quad (4.19)$$

Case 3: $\omega\tau = (2n+1)\pi$ *(odd-integral zero crossing of wake function)*:

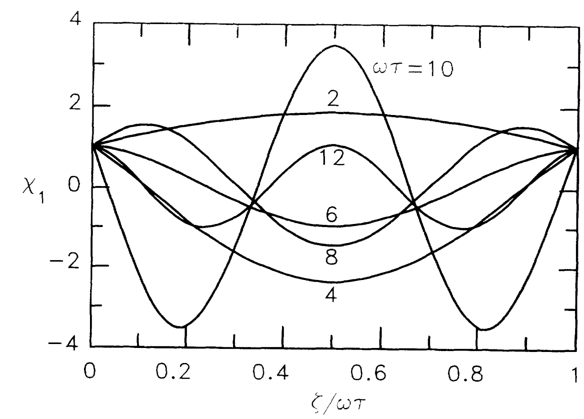

FIG. 1. Plot of amplitude function χ_1 vs $\zeta/\omega\tau$ for δ-function bunches with $Q = 1000$ and for various values of $\omega\tau$. The bunches are located at $\zeta/\omega\tau = 0, 1$.

$$\frac{x(\sigma,\zeta)}{x_0} = \cos\kappa\sigma + \frac{\varepsilon}{\kappa}\frac{(2n+1)\pi}{4}\sigma\sin\kappa\sigma\sin\zeta . \quad (4.20)$$

These cases illustrate clearly the role of focusing in controlling the transverse displacement of the longitudinal halo. For example, case 2 requires very strong focusing, $\kappa \gg \varepsilon Q$, to confine the halo. Considerations of halo control are potentially important in connection with the suppression of activation of the cavity walls due to particle impingement during the long-term cw operation of a high-current linac. It can be noted that in cases 2 and 3 the bunches ($\zeta = 0$) are not deflected by the transverse fields of the deflecting mode, but the halo is. Moreover, with Q being large, the halo is much more strongly displaced in the vicinity of the even-integral zero crossings of the wake function than in the vicinity of the odd-integral zero crossings.

C. Transverse displacement of bunches

We now calculate the displacement $\xi_M(\sigma) \equiv \xi(\sigma, M\omega\tau)$ of bunch M in response to the wake field generated by the bunches which precede bunch M. The longitudinal halo is of no practical interest in connection with transient BBU, and we therefore ignore it in this context. Upon setting $\zeta = M\omega\tau$ in Eq. (4.6) and performing a series of straightforward reductions, we have

$$\xi_M(\sigma) = \frac{\omega\tau}{2\pi}\sum_{m=-\infty}^{+\infty}\int_{-\pi/\omega\tau}^{\pi/\omega\tau} dZ\, e^{im\omega\tau Z}\left(\xi_{M-m}(0)\mathcal{C}[\Lambda(\sigma,Z)] + \xi_{M-m}'(0)\frac{\mathcal{S}[\Lambda(\sigma,Z)]}{\Lambda(0,Z)}\right) . \quad (4.21)$$

It may seem, at first, that the displacement of bunch M, $\xi_M(\sigma)$, depends on the initial displacements of future bunches $\xi_{M'>M}(0)$. This is not so. The integrals with $m < 0$ vanish because the Z dependence of the integrands comes from $\tilde{W}(Z)$, a function containing only negative powers of $e^{i\omega\tau Z}$ as seen in Eq. (4.2). The lower bound of the summation can therefore be set to $m = 0$ rather than $m = -\infty$. Since the current is assumed to be periodic and turned on at $\zeta = -\infty$, the upper limit of the sum in Eq. (4.21) is $+\infty$ and not M. This does not restrict Eq. (4.21) to an analysis of the steady-state BBU since no assumption has been made about the initial conditions at $\sigma = 0$. For example, a beam which is turned on at $\zeta = 0$ can be treated by taking $\xi_k(0) = \xi_k'(0) = 0$ for $k < 0$ because the beam will not generate a deflecting field as long as the current travels on axis.

It is convenient to introduce a different integration variable θ which aids in comparing our results with those of previous authors:

$$\theta \equiv i\omega\tau\left(\frac{1}{2Q}+iZ\right) , \qquad (4.22)$$

for which

$$\Lambda^2(\sigma,\theta) = K^2(\sigma) - \varepsilon(\sigma)\tilde{W}(\theta) . \qquad (4.23)$$

Q is chosen so that all the poles of $\tilde{W}(\theta)$ lie on or below the line $\mathrm{Im}(\theta) = 0$. The beam is assumed to be turned on at $\zeta = 0$. As shown in Appendix E, the displacement $\xi_M(\sigma)$ can now be expressed in the following form:

$$\xi_M(\sigma) = \frac{1}{2\pi}\sum_{m=0}^{M} e^{-m(\omega\tau/2Q)}\int_{-\pi}^{\pi} d\theta\, e^{-im\theta}\left(\xi_{M-m}(0)\mathcal{C}[\Lambda(\sigma,\theta)] + \xi'_{M-m}(0)\frac{\mathcal{S}[\Lambda(\sigma,\theta)]}{\Lambda(0,\theta)}\right) . \qquad (4.24)$$

This result is analogous to the dc BBU result given in Eq. (3.17). It is also analogous to, and has the same form as, the results of previous authors who have investigated BBU in relativistic beams comprised of δ-function bunches for which $\beta = 1$ [4, 6, 7]. The key distinction is that Eq. (4.24) has been derived as a special case of the Fourier-analyzed BBU equation, Eq. (2.11), which obviously is not restricted at the outset to δ-function bunches. Moreover, smooth variation of the parameters, including arbitrary velocity $\beta(\sigma)$, is included within the framework of the continuum and WKBJ approximations.

In parallel with the treatment of dc BBU, we shall consider two classes of solutions for the transient displacement. The first (I) corresponds to an impulse excitation of the wake field, and the associated boundary conditions are $\xi_M(0) = \xi_0\delta_{0,M}$ and $\xi'_M(0) = \xi'_0\delta_{0,M}$. The second (II) corresponds to a misaligned beam, and the associated boundary conditions are $\xi_M(0) = \xi_0$ and $\xi'_M(0) = \xi'_0$ for $M \geq 0$, and $\xi_M(0) = \xi'_M(0) = 0$ for $M < 0$. From Eq. (4.24), the steady-state displacements for the two classes of solutions are found to be

$$\xi_\infty(\sigma) = \begin{cases} 0, \quad \text{class I} & (4.25)\\ \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} d\theta \frac{1}{1-e^{-i(\theta - i\omega\tau/2Q)}} \times\left(\xi_0\mathcal{C}[\Lambda(\sigma,\theta)] + \xi'_0\frac{\mathcal{S}[\Lambda(\sigma,\theta)]}{\Lambda(0,\theta)}\right), \quad \text{class II.} & (4.26)\end{cases}$$

In the latter case, the integral can be evaluated with the contour used in Appendix E modified such that the top of the contour is at $y = i\infty$. The integrand has a simple pole at $\theta = i\frac{\omega\tau}{2Q}$ lying inside this contour, so contour integration yields directly the steady-state solution given by Eq. (4.10). The transient displacements for the two classes of solutions are

$$\xi_M(\sigma) = \frac{e^{-M(\omega\tau/2Q)}}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{-iM\theta}\left(\xi_0\mathcal{C}[\Lambda(\sigma,\theta)] + \xi'_0\frac{\mathcal{S}[\Lambda(\sigma,\theta)]}{\Lambda(0,\theta)}\right) , \quad \text{class I} \qquad (4.27)$$

$$\xi_M(\sigma) - \xi_\infty(\sigma) = -\frac{e^{-(M+1)(\omega\tau/2Q)}}{2\pi}\int_{-\pi}^{\pi} d\theta \frac{e^{-iM\theta}}{e^{i\theta} - e^{-\omega\tau/2Q}}\left(\xi_0\mathcal{C}[\Lambda(\sigma,\theta)] + \xi'_0\frac{\mathcal{S}[\Lambda(\sigma,\theta)]}{\Lambda(0,\theta)}\right), \quad \text{class II.} \qquad (4.28)$$

To provide examples of the application of Eqs. (4.27) and (4.28), we shall calculate the transient growth of a coasting beam in the presence of a single deflecting mode, for which

$$\tilde{W}_1(\theta) = \frac{\omega\tau}{2}\frac{\sin\omega\tau}{\cos\theta - \cos\omega\tau} \qquad (4.29)$$

and

$$\Lambda_1^2(\theta) = \kappa^2 - \varepsilon\tilde{W}_1(\theta) . \qquad (4.30)$$

Our goal is to determine the dependence of $x_M(\sigma)$ on the various physical parameters once BBU has significantly amplified the beam displacement. For this purpose, asymptotic solutions suffice, and these solutions can be generated using the method of steepest descent [1]. When BBU is pronounced, Eqs. (4.27) and (4.28) both take the form

$$x_M(\sigma) - x_\infty(\sigma) \simeq 2\,\mathrm{Re}\left(\int_{-\pi}^{\pi} d\theta\, g(\theta)e^{f(\sigma,\theta)}\right) , \qquad (4.31)$$

where

$$f(\sigma,\theta) \equiv -i[M\theta + \Lambda_1(\theta)\sigma] , \qquad (4.32)$$

and

$$g(\theta) \equiv \frac{e^{-M(\omega\tau/2Q)}}{4\pi}\left(x_0 + \frac{ix'_0}{\Lambda_1(\theta)}\right) , \quad \text{class I} \qquad (4.33)$$

$$g(\theta) \equiv -\frac{e^{-(M+1)(\omega\tau/2Q)}}{4\pi\left(e^{i\theta} - e^{-\omega\tau/2Q}\right)}\left(x_0 + \frac{ix'_0}{\Lambda_1(\theta)}\right) , \quad \text{class II.} \qquad (4.34)$$

The integral of Eq. (4.31) results in a function exhibiting rapid oscillations in M with a slowly varying amplitude. As we did previously for the dc beam, we shall express our results for the approach to steady state in terms of the amplitude, referred to as the "beam envelope" and designated as $|x_M(\sigma) - x_\infty(\sigma)|$. Because the exponential factor $f(\sigma,\theta)$ is the same for both classes of solutions, the transient BBU growth rates of the beam envelopes will likewise be identical. However, the amplitude function $g(\theta)$ differs for the two classes, and the amplitudes of the beam envelopes will likewise differ.

As shown in Appendix F, the saddle points θ_s are determined approximately from the following quartic equation:

$$\delta^4 + s_1\delta^3 - s_2^2\left[1 + (\omega\tau\cot\omega\tau)\delta - \left(\frac{\omega\tau}{2}\right)^2\delta^2\right] \simeq 0\,, \tag{4.35}$$

where

$$\delta(\psi) \equiv \frac{2\left[\cos\omega\tau - \cos(\omega\tau\sqrt{\psi})\right]}{\omega\tau\sin\omega\tau}\,, \tag{4.36}$$

and

$$\psi \equiv \left(\frac{\theta_s}{\omega\tau}\right)^2, \quad s_1 \equiv \frac{\varepsilon}{\kappa^2}, \quad s_2 \equiv \frac{\varepsilon\sigma}{\kappa M\omega\tau}. \tag{4.37}$$

The evaluation of $f(\sigma,\theta)$ at the saddle points results in exponential growth $e^{\Gamma M\omega\tau}$, where the transient BBU growth rate Γ is

$$\Gamma = \mathrm{Im}\left(\sqrt{\psi}\left\{1 + \frac{s_2^2/s_1}{\delta^2\sqrt{\psi}}\left[1 + (\omega\tau\cot\omega\tau)\delta - \left(\frac{\omega\tau}{2}\right)^2\delta^2\right]^{1/2}\right\}\right), \tag{4.38}$$

in which ψ is the root of the quartic equation generating the largest growth rate.

The quartic equation and growth rate calculated here for the beam of δ-function bunches are analogous to their counterparts calculated in Sec. III for the dc beam. Upon replacing θ_s and $M\omega\tau$ by $\omega\tau\theta_s$ and ζ, respectively, in Eqs. (4.35) and (4.38), and then taking the limit $\omega\tau \to 0$, the dc results given by Eqs. (3.28) and (3.30) are recovered. However, the presence of the parameter $\omega\tau$ in Eq. (4.35) enriches the solution space of the beam comprised of δ-function bunches relative to that of the dc beam. There are now six domains rather than four, and these domains are paired. Each pair is associated with one of the terms multiplying s_2 in the saddle-point equation, Eq. (4.35), and each of these terms can dominate the others depending on the values of s_2 and $\omega\tau$. Near the zero crossings of the wake function where $\sin\omega\tau \simeq 0$, the second term will dominate because $|\cot\omega\tau|$ is large. Away from these zero crossings the first term will dominate for sufficiently small values of s_2, and the third term will dominate for sufficiently large values of s_2. The domains comprising each pair are then distinguished from each other by the relative values of s_1 and s_2. The domains with small values of s_2 away from the zero crossings are the counterparts of domains A and B of the dc beam, and their associated BBU dynamics are identical in the dc limit. The domains near the zero crossings are the counterparts of domains C and D of the dc beam, and their associated BBU dynamics are identical in the dc limit (as is expected because $\omega\tau \to 0$ corresponds to a zero of $\sin\omega\tau$). The domains with large values of s_2 away from the zero crossings have no counterparts with the dc beam; these domains are labeled E and F. The saddle points, growth rates, and beam envelopes are calculated using the asymptotic methods of Appendix F. For each domain and for classes I and II of solutions, they are as follows.

Domain A: $s_2 \ll s_1^{1/2}$, $s_2 \ll s_1^2$; $\left(\frac{\omega\tau}{2}\right)^2\left(\frac{s_2^2}{s_1}\right)^{2/3} + \omega\tau|\cot\omega\tau|\left(\frac{s_2^2}{s_1}\right)^{1/3} \ll 1$ *(away from zero crossing, weak focusing, weak BBU coupling, long pulse train).* In terms of the dimensionless parameter

$$E_A \equiv \left(\frac{s_2^2}{s_1}\right)^{1/3} M\omega\tau \propto \sigma^{2/3}(M\omega\tau)^{1/3}\,, \tag{4.39}$$

the dominant saddle point and growth rate are, respectively,

$$\psi \simeq 1 + \delta\,, \qquad \delta \simeq \frac{E_A}{M\omega\tau}e^{2\pi i/3}\,, \quad \Gamma \simeq \frac{3\sqrt{3}}{4}\frac{E_A}{M\omega\tau}\,, \tag{4.40}$$

and the beam envelopes are given by (for classes I and II, respectively)

$$|x_M(\sigma)| \simeq \left[\left(x_0 + \frac{\sqrt{3}}{2}\frac{\sigma}{E_A}x_0'\right)^2 + \left(\frac{\sigma}{2E_A}x_0'\right)^2\right]^{1/2} \times \frac{\sqrt{E_A}}{M\sqrt{6\pi}}\exp\left(\frac{3\sqrt{3}}{4}E_A - M\frac{\omega\tau}{2Q}\right)\,, \tag{4.41}$$

$$|x_M(\sigma) - x_\infty(\sigma)| \simeq \left[\left(x_0 + \frac{\sqrt{3}}{2}\frac{\sigma}{E_A}x_0'\right)^2 + \left(\frac{\sigma}{2E_A}x_0'\right)^2\right]^{1/2} \times \frac{\sqrt{E_A}}{2M\sqrt{6\pi}}\frac{\exp\left(\frac{3\sqrt{3}}{4}E_A - M\frac{\omega\tau}{2Q}\right)}{\sqrt{\sinh^2\frac{\omega\tau}{4Q} + \sin^2\frac{\omega\tau}{2}}}\,. \tag{4.42}$$

These are the results quoted by Gluckstern, Cooper, and Channell [4], who were the first to study this domain in connection with relativistic beams of δ-function bunches. Their work also marked the first attempt to include the rf structure of the beam in a BBU calculation. Gluckstern, Cooper, and Channell derived their results without the aid of the continuum approximation and applied them to calculate BBU in the linac used for Los Alamos' free-electron laser. Decker and Wang [6] revisited the problem using the continuum approximation to calculate BBU for an ultrarelativistic, uniformly accelerated beam.

Though we have chosen to present explicit results only for a coasting beam, arbitrary velocities and acceleration are also included in our formalism by way of the WKBJ approximation.

Domain B: $s_1^2 \ll s_2 \ll 1$; $\left(\frac{\omega\tau}{2}\right)^2 s_2 + \omega\tau|\cot\omega\tau|\sqrt{s_2} \ll 1$ *(away from zero crossing, strong focusing, moderate BBU coupling, moderate pulse train).* In terms of the dimensionless parameter

$$E_B \equiv \sqrt{s_2} M\omega\tau \;\propto\; \sigma^{1/2}(M\omega\tau)^{1/2} , \qquad (4.43)$$

the dominant saddle point and growth rate are, respectively,

$$\psi \simeq 1+\delta , \quad \delta \simeq i\frac{E_B}{M\omega\tau} - \frac{1}{4}s_1 , \qquad \Gamma \simeq \frac{E_B}{M\omega\tau} , \qquad (4.44)$$

and the beam envelopes are given by (for classes I and II, respectively)

$$|x_M(\sigma)| \simeq \left[x_0^2 + \left(\frac{x_0'}{\kappa}\right)^2\right]^{1/2} \frac{\sqrt{E_B}}{2M\sqrt{2\pi}} \times \exp\left(E_B - M\frac{\omega\tau}{2Q}\right) , \qquad (4.45)$$

$$|x_M(\sigma) - x_\infty(\sigma)| \simeq \left[x_0^2 + \left(\frac{x_0'}{\kappa}\right)^2\right]^{1/2} \frac{\sqrt{E_B}}{4M\sqrt{2\pi}} \times \frac{\exp\left(E_B - M\frac{\omega\tau}{2Q}\right)}{\sqrt{\sinh^2\frac{\omega\tau}{4Q} + \sin^2\frac{\omega\tau}{2}}} . \qquad (4.46)$$

Yokoya [5] was the first to study this domain in connection with ultrarelativistic beams comprised of δ-function bunches. His calculations were based on a discrete Laplace transform of the equation of transverse motion, and he included acceleration by way of the WKBJ approximation. Yokoya found that his analytic results, which are analogous to those presented here, agreed well with a numerical solution of the equation of transverse motion. Decker and Wang [6] derived these results for a strongly focused ultrarelativistic beam with uniform acceleration. They revisited the problem in a later paper [7], treating BBU in a linac with a periodic lattice of cavities and focusing elements. Gluckstern, Neri, and Cooper [16] also investigated this domain and found close agreement between the analytic results and simulations.

Domain C: $n_2^2 \equiv s_2^2\omega\tau|\cot\omega\tau|$; $n_2 \gg 1$, $n_2 \gg s_1^{3/2}$; $\csc^2\omega\tau \gg \omega\tau|\cot\omega\tau| n_2^{2/3} \gg |1 - [(\omega\tau/2)n_2^{2/3}]^2|$ *(near zero crossing, strong focusing, strong BBU coupling, short pulse train).* In terms of the dimensionless parameter

$$E_C \equiv s_2^{1/3}\left(\frac{|\tan\omega\tau|}{\omega\tau}\right)^{1/3} M\omega\tau \;\propto\; \sigma^{1/3}(M\omega\tau)^{2/3} , \qquad (4.47)$$

the dominant saddle point and growth rate are, respectively,

$$\psi \simeq \frac{\sin\omega\tau\cos\omega\tau}{\omega\tau}\delta ,$$
$$\delta \simeq \left(\frac{E_C}{M\omega\tau}\right)^2 (\omega\tau\cot\omega\tau)e^{2\pi i/3} - \frac{1}{3}s_1 , \qquad (4.48)$$
$$\Gamma \simeq \frac{\sqrt{3}}{2}\left(|\cos\omega\tau| + \frac{1}{2}\right)\frac{E_C}{M\omega\tau} ,$$

and the beam envelopes are given by (for classes I and II, respectively)

$$|x_M(\sigma)| \simeq \left[x_0^2 + \left(\frac{x_0'}{\kappa}\right)^2\right]^{1/2} \frac{\sqrt{E_C}}{M\sqrt{6\pi}} \exp\left[\frac{\sqrt{3}}{2}\left(|\cos\omega\tau| + \frac{1}{2}\right)E_C - M\frac{\omega\tau}{2Q}\right] , \qquad (4.49)$$

$$|x_M(\sigma) - x_\infty(\sigma)| \simeq \left[x_0^2 + \left(\frac{x_0'}{\kappa}\right)^2\right]^{1/2} \frac{\sqrt{E_C}}{M\sqrt{6\pi}} \frac{\exp\left(\frac{\sqrt{3}}{2}\left(|\cos\omega\tau| + \frac{1}{2}\right)E_C - (M+1)\frac{\omega\tau}{2Q}\right)}{\sqrt{\left(\cos\omega\tau - e^{-\omega\tau/2Q} - \frac{\sqrt{3}}{2}\frac{E_C}{M}|\cos\omega\tau|\right)^2 + \left(\frac{1}{2}\frac{E_C}{M}\cos\omega\tau\right)^2}} . \qquad (4.50)$$

Thompson and Ruth [9] discussed the suppression of BBU in next-generation e^+e^- linear colliders by placing the bunches near zero crossings of the wake function. Using a formal series solution of the equation of transverse motion, they considered cases involving very short bunch trains where BBU is small. Neri and Gluckstern [21] approached the problem of single-bunch BBU by decomposing the bunch into a train of closely spaced macroparticles. The results presented above pass to the results of Neri and Gluckstern in the dc limit. As is discussed in Appendix F, the criteria for validity of the saddle-point calculation are easier to fulfill near the zero crossings where $\omega\tau$ is near an odd-integral multiple of π than near the zero crossings where $\omega\tau$ is near an even-integral multiple of π.

Domain D: $n_2^2 \equiv s_2^2\omega\tau|\cot\omega\tau|$; $s_1^{3/2} \gg n_2 \gg s_1^{1/2}$; $\csc^2\omega\tau \gg \omega\tau|\cot\omega\tau|(n_2/\sqrt{s_1}) \gg |1 -$

$[(\omega\tau/2)(n_2/\sqrt{s_1})]^2|$ *(near zero crossing, weak focusing, strong BBU coupling, short pulse train).* In terms of the dimensionless parameter

$$E_D \equiv \alpha\left(\frac{s_2^2}{s_1}\right)^{1/4}\left(\frac{|\tan\omega\tau|}{\omega\tau}\right)^{1/4} M\omega\tau \,\propto\, \sigma^{1/2}(M\omega\tau)^{1/2}\,, \tag{4.51}$$

where

$$\alpha = \begin{cases} 1 & \text{for } \tan\omega\tau \ge 0 \\ \sqrt{2}/2 & \text{for } \tan\omega\tau < 0, \end{cases} \tag{4.52}$$

the dominant saddle point and growth rate are, respectively,

$$\psi \simeq \frac{\sin\omega\tau\cos\omega\tau}{\omega\tau}\,\delta\,,$$

$$\delta \simeq -\left(\frac{E_D}{M\omega\tau}\right)^2 \omega\tau\cot\omega\tau\,, \tag{4.53}$$

$$\Gamma \simeq (|\cos\omega\tau|+1)\frac{E_D}{M\omega\tau}\,,$$

and the beam envelopes are given by (for classes I and II, respectively)

$$|x_M(\sigma)| \simeq \left(x_0 + \frac{\sigma}{E_D}x_0'\right)\frac{\sqrt{E_D}}{2M\sqrt{\pi}} \times \exp\left((|\cos\omega\tau|+1)E_D - M\frac{\omega\tau}{2Q}\right)\,, \tag{4.54}$$

$$|x_M(\sigma) - x_\infty(\sigma)| \simeq \left(x_0 + \frac{\sigma}{E_D}x_0'\right) \times \frac{\sqrt{E_D}}{2M\sqrt{\pi}}\,\frac{\exp\left((|\cos\omega\tau|+1)E_D - (M+1)\frac{\omega\tau}{2Q}\right)}{\left|\cos\omega\tau - e^{-\omega\tau/2Q} - \frac{E_D}{M}|\cos\omega\tau|\right|}\,. \tag{4.55}$$

Again, as is discussed in Appendix F, the criteria for validity of the saddle-point calculation are easier to fulfill near zero crossings where $\omega\tau$ is near an odd-integral multiple of π than near zero crossings where $\omega\tau$ is near an even-integral multiple of π.

Domain E: $\left(\frac{\omega\tau}{2}\right)^2 s_2|\sin\omega\tau| \gg 1$, $(\omega\tau/2)s_2 \gg s_1$; $[(\omega\tau/2)^2 s_2]^2 \gg 1 + [(\omega\tau)^2/2]s_2|\cot\omega\tau|$ *(away from zero crossing , strong focusing, very strong BBU coupling, short pulse train).* The dominant saddle point is

$$\sqrt{\psi} \simeq \frac{i}{\omega\tau}\ln(-\omega\tau\delta\sin\omega\tau)\,, \quad \delta \simeq i\frac{\omega\tau}{2}s_2 - \frac{1}{2}s_1\,. \tag{4.56}$$

In terms of the dimensionless parameter

$$E_E \equiv M\ln\left[2\left(\frac{\omega\tau}{2}\right)^2 s_2|\sin\omega\tau|\right] \,\propto\, M\,, \tag{4.57}$$

the growth rate is

$$\Gamma \simeq \frac{E_E}{M\omega\tau} + \frac{1}{\omega\tau}\,, \tag{4.58}$$

and the beam envelopes are given by (for classes I and II, respectively)

$$|x_M(\sigma)| \simeq \left[x_0^2 + \left(\frac{x_0'}{\kappa}\right)^2\right]^{1/2}\frac{1}{\sqrt{2\pi M}} \times \exp\left(E_E + M - M\frac{\omega\tau}{2Q}\right)\,, \tag{4.59}$$

$$|x_M(\sigma) - x_\infty(\sigma)| \simeq \left[x_0^2 + \left(\frac{x_0'}{\kappa}\right)^2\right]^{1/2} \times \frac{1}{\sqrt{2\pi M}}\exp\left[\left(1-\frac{1}{2M}\right)E_E + M - (M+1)\frac{\omega\tau}{2Q}\right]\,. \tag{4.60}$$

Domain F: $(\omega\tau/2)^3 s_2^2|\sin\omega\tau| \gg s_1$, $s_1 \gg (\omega\tau/2)s_2$; $[(\omega\tau/2)^3(s_2^2/s_1)]^2 \gg 1 + [(\omega\tau)^3/4](s_2^2/s_1)|\cot\omega\tau|$ *(away from zero crossing, weak focusing, very strong BBU coupling, short pulse train).* The dominant saddle point is

$$\sqrt{\psi} \simeq \frac{i}{\omega\tau}\ln(-\omega\tau\delta\sin\omega\tau)\,, \quad \delta \simeq -\left(\frac{\omega\tau}{2}\right)^2\frac{s_2^2}{s_1}\,. \tag{4.61}$$

In terms of the dimensionless parameter

$$E_F \equiv M\ln\left[2\left(\frac{\omega\tau}{2}\right)^3\frac{s_2^2}{s_1}|\sin\omega\tau|\right] \,\propto\, M\,, \tag{4.62}$$

the growth rate is

$$\Gamma \simeq \frac{E_F}{M\omega\tau} + \frac{2}{\omega\tau}\,, \tag{4.63}$$

and the beam envelopes are given by (for classes I and II, respectively)

$$|x_M(\sigma)| \simeq \left(x_0 + \frac{\sigma}{2M}x_0'\right)\frac{1}{\sqrt{\pi M}} \times \exp\left(E_F + 2M - M\frac{\omega\tau}{2Q}\right)\,, \tag{4.64}$$

$$|x_M(\sigma) - x_\infty(\sigma)| \simeq \left(x_0 + \frac{\sigma}{2M}x_0'\right) \times \frac{1}{\sqrt{\pi M}}\exp\left[\left(1-\frac{1}{2M}\right)E_F + 2M - (M+1)\frac{\omega\tau}{2Q}\right]\,. \tag{4.65}$$

D. Transient BBU simulations

To check the scaling laws for each domain, we compare the analytic results with numerical simulations. The simulations are developed by way of Runge-Kutta integration of the equation of transverse motion of the bunches themselves. According to Eq. (2.1), for a coasting beam comprised of δ-function bunches in the presence of a single deflecting mode, the equation of motion of bunch M is

$$\left(\frac{d^2}{d\sigma^2}+\kappa^2\right)x_M(\sigma)=\varepsilon\omega\tau\sum_{m=0}^{M-1}w_{M-m}x_m(\sigma)\ , \tag{4.66}$$

where

$$w_k \equiv e^{-k(\omega\tau/2Q)}\sin k\omega\tau\ . \tag{4.67}$$

This equation was integrated using the boundary conditions of a misaligned beam, $x_m(0)=x_0$ and $x'_m(0)=0$ for all m, with $Q\rightarrow\infty$. The input parameters used for each domain are listed in Table I. These parameters have no special significance beyond their intended purpose of providing examples with which to check the analytic scaling laws.

Results of the simulations are shown in Fig. 2 for domains A, B, and C, and in Fig. 3 for domains D, E, and F. The simulations are seen to follow the time (M) dependence predicted with the analytic scaling laws calculated for the envelopes. Plots of the spatial (σ) dependence (not shown) for these examples likewise conform to the analytic scaling laws with accuracies comparable to those of Figs. 2 and 3.

A comparison of the results for domains C and D, which are calculated using the same coupling constant ε and frequency ratio $\omega\tau$, reveals that focusing can be a powerful cure of transient BBU when the bunches are placed near the zero crossings of the wake function. In these examples, increasing the focusing strength from $\kappa=0$ to 30 reduces BBU by a factor greater than 10^4 at $M\sim 50$. Domains E and F are characterized by very large BBU coupling and are therefore probably of lim-

TABLE I. Parameters for simulations of transient BBU with δ-function bunches.

Domain	ε	κ	$\omega\tau$
A	0.5	0	10
B	5.0	30	10
C	300	30	$3\pi+\frac{1}{300\pi}$
D	300	0	$3\pi+\frac{1}{300\pi}$
E	3000	60	10
F	20000	0	10

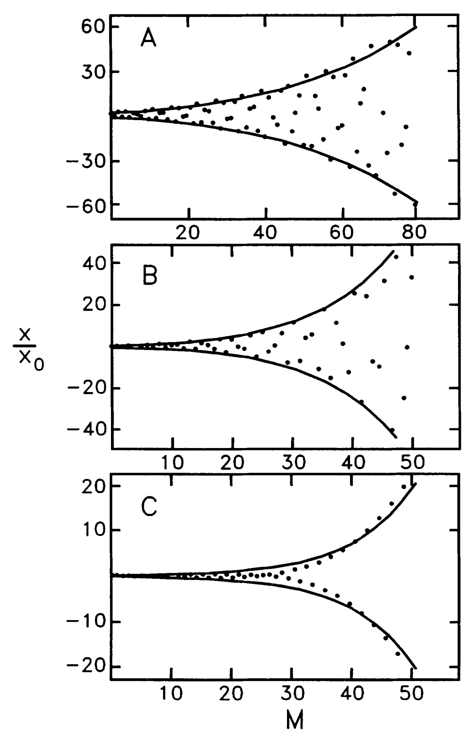

FIG. 2. Transient displacement of δ-function bunches at the exit of the linac vs bunch number M for domains A, B, and C. Transverse displacement, plotted as the ordinate, is normalized with respect to the initial displacement. The dots are the bunch displacements calculated from numerical simulations using the parameters of Table I. The solid curves are the beam envelopes calculated analytically.

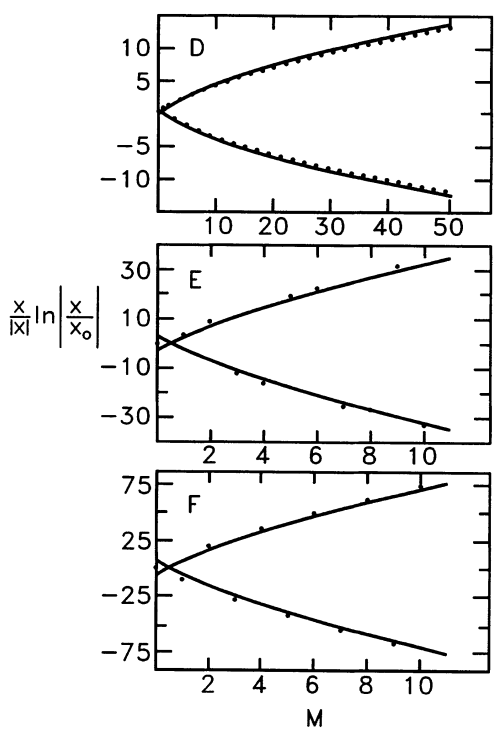

FIG. 3. Transient displacement of δ-function bunches at the exit of the linac vs bunch number M for domains D, E, and F. Transverse displacement, plotted as the ordinate on a logarithmic scale, is normalized with respect to the initial displacement. The dots are the bunch displacements calculated from numerical simulations using the parameters of Table I. The solid curves are the beam envelopes calculated analytically.

ited practical interest. These domains are distinguished by enormous and rapid BBU growth even with very short bunch trains.

With bunch trains which are sufficiently long to experience an exponential decrease in a wake field with finite Q, methods which lower the effective Q of the wake field will also be useful in suppressing BBU. This consideration is most applicable to domains A and B, and also to steady-state BBU. For example, the results of domains A and B and of the steady state have been applied elsewhere to estimate the maximum effective Q of the dominant deflecting mode which can be sustained without significant BBU in two superconducting cw linacs for deuteron beams [22]. The first is a five-cavity section being designed for tests with a deuteron beam of energy 7.5 MeV and current 80 mA, which was modeled assuming no focusing (domain A followed by steady state), and the second is a notional linac for accelerating a 100-mA deuteron beam from 5 to 200 MeV with strong focusing (domain B followed by steady state). For both linacs, it was found that a maximum Q of order 5×10^6 for the deflecting modes in the constituent resonators should sufficiently suppress BBU under worst-case conditions. In actual operation, the external Q of the accelerating mode in these linacs is expected to be of order 10^5 due to the rf coupling required to drive the high-current beams. The rf coupling would probably also generate $Q < 10^6$ for the deflecting modes and thereby control the transverse displacement of the bunches.

V. FINITE BUNCH LENGTH AND ARBITRARY CHARGE DISTRIBUTION

A. Steady-state solution

The behavior of BBU for an arbitrary but periodic beam is described by Eq. (2.11). This equation was solved in closed form for the special cases of a dc beam and of a beam comprised of δ-function bunches. In this section we investigate BBU for the case of a periodic current comprised of an infinite series of bunches of finite length and arbitrary but identical charge distribution.

Equation (2.11) cannot be solved in closed form for arbitrary F_k. We shall therefore make a simplifying assumption. In particular, we shall restrict our investigation to the steady-state behavior of a coasting beam whose misalignment at the entrance of the linac has the same time periodicity as the beam current. With this assumption the problem is completely periodic in time, and the transverse displacement can be expanded in a Fourier series:

$$x(\sigma,\zeta) = \sum_{m=-\infty}^{+\infty} x_m(\sigma) e^{im(2\pi/\omega\tau)\zeta} . \tag{5.1}$$

The Fourier coefficients $x_m(\sigma)$ are related to the Fourier transform $\tilde{x}(\sigma, Z)$ by

$$x_m(\sigma) = \frac{1}{2\pi} \int_{(2\pi m/\omega\tau)-0}^{(2\pi m/\omega\tau)+0} \tilde{x}(\sigma, Z)\, dZ . \tag{5.2}$$

With the above assumptions, integrating Eq. (2.11) from $\frac{2\pi m}{\omega\tau} - 0$ to $\frac{2\pi m}{\omega\tau} + 0$ yields the following system of differential equations:

$$\left(\frac{d^2}{d\sigma^2} + \kappa^2\right) x_m(\sigma) = \varepsilon \tilde{w}_m \sum_{k=-\infty}^{+\infty} F_k x_{m-k}(\sigma) , \tag{5.3}$$

where

$$\tilde{w}_m = \tilde{w}\left(m\frac{2\pi}{\omega\tau}\right) = \int_0^{+\infty} d\zeta\, w(\zeta) e^{-im(2\pi/\omega\tau)\zeta} . \tag{5.4}$$

In the case of a single deflecting mode we have

$$\tilde{w}_{1,m} = \left[1 + \frac{1}{4Q^2} - \left(m\frac{2\pi}{\omega\tau}\right)^2 + i\left(m\frac{2\pi}{\omega\tau}\frac{1}{Q}\right)\right]^{-1} . \tag{5.5}$$

As expected, the system of Eq. (5.3) can be solved exactly in the case of the dc beam where the only nonzero coefficient is $F_0 = 1$, and in the case of the beam of δ-function bunches where $F_k = 1$ for all k.

For a dc beam the system of Eq. (5.3) reduces to an infinite system of uncoupled linear differential equations:

$$\begin{aligned} \frac{d^2}{d\sigma^2} x_m(\sigma) &= (\varepsilon\tilde{w}_m - \kappa^2) x_m(\sigma) \\ &= -\lambda_m^2 x_m(\sigma) , \end{aligned} \tag{5.6}$$

where $\lambda_m^2 = \lambda^2(m\frac{2\pi}{\omega\tau})$ represents the growth of the mth Fourier component of the bunch displacement. If the misalignment offset and angle of incidence at the accelerator entrance are expanded in Fourier series:

$$\begin{aligned} x(0,\zeta) &= \sum_{m=-\infty}^{+\infty} x_{0,m} e^{im(2\pi/\omega\tau)\zeta} , \\ x'(0,\zeta) &= \sum_{m=-\infty}^{+\infty} x'_{0,m} e^{im(2\pi/\omega\tau)\zeta} , \end{aligned} \tag{5.7}$$

then the transverse displacement along the accelerator is given by

$$x(\sigma,\zeta) = \sum_{m=-\infty}^{+\infty} \left(x_{0,m} \cos\lambda_m\sigma + x'_{0,m} \frac{\sin\lambda_m\sigma}{\lambda_m} \right) e^{im(2\pi/\omega\tau)\zeta} . \tag{5.8}$$

This solution is a generalization of the one given in Eq. (3.10) since we have now assumed that the initial offset is periodic and not simply constant as it is in Eq. (3.10).

For a beam composed of δ-function bunches the system of Eq. (5.3) reduces to

$$\left(\frac{d^2}{d\sigma^2} + \kappa^2\right) x_m(\sigma) = \varepsilon \tilde{w}_m \sum_{k=-\infty}^{+\infty} x_k(\sigma) , \tag{5.9}$$

which is an infinite system of coupled linear differential equations; nevertheless, it can still be solved in closed form. If we assume the same Fourier expansion of the initial conditions as before, the solution is

$$x(\sigma,\zeta) = x(0,\zeta)\cos\kappa\sigma + x'(0,\zeta)\frac{\sin\kappa\sigma}{\kappa} + \sum_{m=-\infty}^{+\infty} e^{im(2\pi/\omega\tau)\zeta}\frac{\tilde{w}_m}{\tilde{W}_0}\left[x_0\left(\cos(\Lambda_0\sigma)-\cos\kappa\sigma\right)+x_0'\left(\frac{\sin(\Lambda_0\sigma)}{\Lambda_0}-\frac{\sin\kappa\sigma}{\kappa}\right)\right], \tag{5.10}$$

where

$$\tilde{W}_0 = \sum_{k=-\infty}^{+\infty}\tilde{w}_k\ , \quad \Lambda_0^2 = \kappa^2 - \varepsilon\tilde{W}_0\ , \tag{5.11}$$

and

$$x_0 = \sum_{k=-\infty}^{+\infty} x_{0,k}\ , \quad x_0' = \sum_{k=-\infty}^{+\infty} x_{0,k}' \tag{5.12}$$

are the initial offset and angle of incidence of the bunches themselves. This solution is a generalization of the one given by Eq. (4.11) since we have now allowed for a periodic longitudinal halo between the bunches at the linac entrance. The initial periodic misalignment of the halo is unaffected by the BBU and proceeds along the accelerator under the influence of the focusing fields only. The BBU of the halo depends only on the initial misalignment of the bunches themselves, as would be expected since the bunches alone drive the deflecting wake fields.

B. Recurrence solution and growth factor

For a dc beam or a beam of δ-function bunches the infinite system of Eq. (5.3) could be diagonalized and its eigenvalues and eigenfunctions determined. With arbitrary Fourier coefficients F_k of the beam current, however, this is not the case, and a closed-form solution cannot be found. Nevertheless, a recurrence solution of the system of Eq. (5.3) can be obtained. If we assume

$$x_m(\sigma) = \sum_{j=0}^{\infty} x_{m,j}\frac{\sigma^j}{j!}\ , \tag{5.13}$$

then

$$x_{m,j+2} = \varepsilon\tilde{w}_m\sum_{k=-\infty}^{+\infty}F_k x_{m-k,j} - \kappa^2 x_{m,j}\ . \tag{5.14}$$

If we also assume the beam is misaligned with

$$x(0,\zeta) = x_0\ , \quad x'(0,\zeta) = 0\ , \tag{5.15}$$

then the first terms in the series expansion are

$$x_{0,2} = \varepsilon\tilde{w}_0F_0x_0 - \kappa^2x_0\ , \qquad x_{m,2} = \varepsilon\tilde{w}_mF_mx_0\ , \tag{5.16}$$

and, for small σ, the transverse displacement of the beam is given by

$$x(\sigma,\zeta) \simeq x_0\left[1+\frac{\sigma^2}{2}\left(\varepsilon\sum_{m=-\infty}^{+\infty}\tilde{w}_mF_me^{im(2\pi/\omega\tau)\zeta} - \kappa^2\right)\right] \tag{5.17}$$

The transverse displacement of the center of the bunch ($\zeta = 0$) is given by

$$x(\sigma,0) \simeq x_0\left(1+\frac{\sigma^2}{2}\left(\varepsilon\Omega^2-\kappa^2\right)\right)\ , \tag{5.18}$$

where

$$\Omega^2 \equiv \sum_{m=-\infty}^{+\infty}\tilde{w}_mF_m = \int_0^{+\infty} w(\zeta)F(-\zeta)\,d\zeta\ . \tag{5.19}$$

It is readily apparent that, for the dc beam and the beam of δ-function bunches, Eq. (5.18) gives the first terms in the series expansion of the solutions given by Eqs. (3.10) and (4.12) with $x_0' = 0$.

It is useful at this point to introduce a "growth factor" $G^2(\sigma,\zeta;\kappa)$ to describe the local tendency of the beam to experience BBU. This growth factor is defined as

$$G^2(\sigma,\zeta;\kappa) \equiv \frac{1}{\varepsilon x(\sigma,\zeta)}\frac{d^2}{d\sigma^2}x(\sigma,\zeta)\ . \tag{5.20}$$

A positive growth factor indicates the beam is deflected away from the axis and is therefore locally unstable, while a negative growth factor indicates the beam is deflected toward the axis and is therefore locally stable. Close to the entrance of the accelerator ($\sigma \ll 1$), and for the center of each bunch ($\zeta = 0$), the growth factor is

$$G^2(0,0;\kappa) = \frac{1}{\varepsilon x_0}\frac{d^2}{d\sigma^2}x(0,0) = \Omega^2 - \frac{\kappa^2}{\varepsilon}\ . \tag{5.21}$$

The role of focusing is clear: $\kappa^2 > \varepsilon\Omega^2$ generates a negative growth factor at the accelerator entrance which tends to stabilize the beam with respect to steady-state BBU.

Although the growth factor calculated from Eq. (5.18) is strictly valid only as long as the beam has not deviated significantly from its original misalignment, the value of the growth factor obtained this way has a much wider range of applicability. In particular, for the dc beam and the beam of δ-function bunches, the value of $G^2(0,0;\kappa)$ obtained above is exact anywhere along the linac and for arbitrary BBU coupling and focusing strength. For an arbitrary periodic beam propagating down a linac with parameters which are independent of σ, the growth factor is expected to be almost constant provided the distortion of the beam bunches is small compared to the bunch displacement. This expectation holds because the σ dependence of the growth factor originates from the harmonic content of the beam and vanishes for both the dc beam and the beam of δ-function bunches. Accordingly, the growth factor can be calculated close to the linac entrance where the σ dependence of the transverse displacement is given by Eq. (5.18), and the result of this calculation will be approximately valid over the entire length of the linac. In the numerical simulations described in Sec. V D below, this was found to be true except near the resonances where the deflecting-mode frequency is an even-integer multiple of the bunch frequency. In this circumstance, the bunch distortion could be very large, comparable in magnitude to the bunch displacement, and the growth factor could likewise vary strongly with σ.

As an example, we consider a beam composed of uniform bunches of finite length in the presence of a single deflecting mode. The form factor for the beam current is

$$F(\zeta) = \begin{cases} \dfrac{\omega\tau}{\alpha} \text{ for } |\zeta| < \dfrac{\alpha}{2} \\ 0 \text{ for } \dfrac{\alpha}{2} < |\zeta| < \dfrac{\omega\tau}{2}, \end{cases} \tag{5.22}$$

from which the growth factor of the bunch centroids close to the entrance of the linac is found to be

$$G^2(0,0;\kappa) = -\frac{\kappa^2}{\varepsilon} + \frac{\omega\tau}{\alpha}\frac{4Q^2}{4Q^2+1}\left[\left(2q+\frac{p}{Q}\right)\cos\frac{\alpha}{2}\sinh\frac{\alpha}{4Q} + \left(2p-\frac{q}{Q}\right)\sin\frac{\alpha}{2}\cosh\frac{\alpha}{4Q} + 1 - \left(\cos\frac{\alpha}{2}+\frac{1}{2Q}\sin\frac{\alpha}{2}\right)e^{-\alpha/4Q}\right], \tag{5.23}$$

where p and q are the resonance functions defined in Eqs. (4.14) and (4.15), respectively. Figure 4 shows how $G^2(0,0;\kappa=0)$ varies with the "filling factor" $f \equiv \alpha/\omega\tau$, a quantity which parametrizes the bunch length. For a given value of $\omega\tau$, the growth factor may change sign as the filling factor is increased, which in turn changes the local stability of the bunch centroids with respect to BBU. This circumstance appears again in Fig. 5 where the growth factor is plotted for various values of f in the vicinity of the resonance at $\omega\tau = 4\pi\left(1+\frac{1}{2Q}\right)$. As the filling factor increases, the stability of the bunch centroids qualitatively reverses.

In the limit of finite but short bunches ($\alpha \simeq 0$), we find

$$G^2(0,0;\kappa) \simeq \frac{1}{\varepsilon}\left[\varepsilon\,\omega\tau\left(p+\frac{\alpha}{8}\right) - \kappa^2\right] . \tag{5.24}$$

Thus we find that a slight lengthening of δ-function bunches to make short uniform bunches always exacerbates BBU, independent of the focusing strength κ. As we now show, this is also true both for arbitrary charge distributions as long as they are nearly δ-function distributions, and for arbitrary wake field $w(\zeta)$. The difference in growth factor between two otherwise identical beams, one composed of bunches of finite extent ($\alpha \neq 0$) and one composed of δ-function bunches ($\alpha = 0$) is

$$\Delta G^2 \equiv G^2(\alpha \neq 0) - G^2(\alpha = 0) = \sum_{m=-\infty}^{+\infty} \tilde{w}_m(F_m - 1) . \tag{5.25}$$

If the finite bunches deviate only slightly from δ-function bunches, their associated Fourier components will decrease significantly from unity only for large $|m|$. Thus only large values of $|m|$ will contribute to ΔG^2. As can be seen from Eq. (5.5), for large values of $|m|$, $\tilde{w}_{1,m}$ is nega-

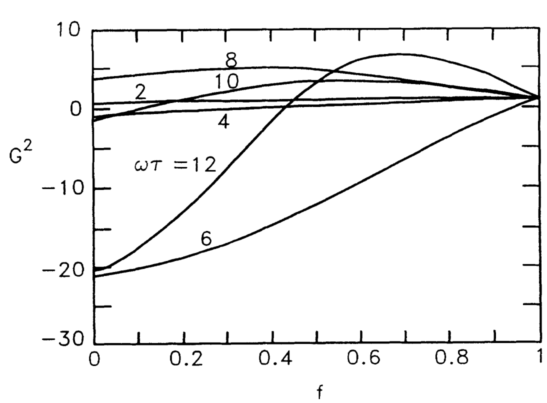

FIG. 4. Growth factor $G^2(0,0;\kappa=0)$ vs filling factor $f = \alpha/\omega\tau$ for uniform bunches of finite length in the presence of a single deflecting mode. Plots are shown for several values of $\omega\tau$ assuming $Q = 1000$. δ-function bunches have $f = 0$, and the dc beam has $f = 1$.

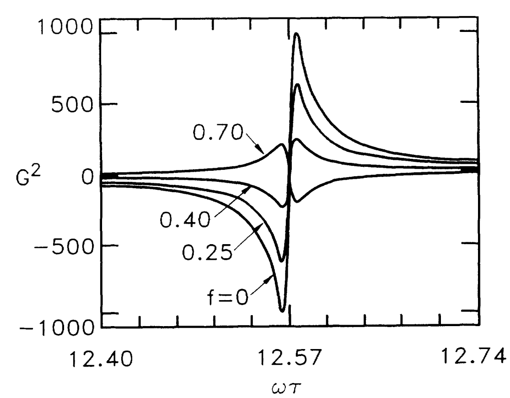

FIG. 5. Growth factor $G^2(0,0;\kappa=0)$ vs $\omega\tau$ in the vicinity of the resonance of p occurring at $\omega\tau = 4\pi\left(1+\frac{1}{2Q}\right)$ for uniform bunches of finite length in the presence of a single deflecting mode. Plots are shown for several values of filling factor f assuming $Q = 1000$.

tive. Equation (5.5) pertains to a single deflecting mode, but an arbitrary wake function is composed of single-mode wake functions, and $\tilde{w}_m$ will therefore also be negative for large $|m|$. Thus ΔG^2 will generally be positive with short bunches, and a slight lengthening of δ-function bunches will always lead to an increase of BBU. However, as the bunches are lengthened further, the growth factor may or may not continue to increase, a circumstance which we saw in the examples of Figs. 4 and 5.

C. Bunch distortion and transverse beam size

Throughout this paper we have been concerned with beams which are energetically monochromatic, having no transverse size, and therefore having zero emittance. With these model beams, although BBU will distort the bunches, the emittance will remain zero. For real beams with finite emittance, however, bunch distortion will lead to emittance growth which will be in some sense commensurate to the distortion. Consequently, this BBU-induced emittance growth is quantitatively related to the amount of bunch distortion and the corresponding increase in transverse beam size, and therefore it is of interest to calculate the transverse beam size.

As was done previously, we consider a misaligned beam which enters the accelerator parallel to the axis with a constant offset. After steady state has been reached, the time dependence of the beam displacement at position σ is

$$x(\sigma,\zeta) = \sum_{k=-\infty}^{+\infty} x_k(\sigma) e^{ik(2\pi/\omega\tau)\zeta} \ , \tag{5.26}$$

with

$$x_0(0) = x_0 \ , \quad x_{k\neq 0}(0) = 0 \ . \tag{5.27}$$

A mean offset $\overline{x}(\sigma)$ can be calculated by averaging the offset weighted by the charge distribution over an rf period in the manner

$$\overline{x}(\sigma) = \frac{1}{\omega\tau}\int_{-\omega\tau/2}^{\omega\tau/2} x(\sigma,\zeta)F(\zeta)\,d\zeta = \sum_{k=-\infty}^{+\infty} x_k(\sigma) F_{-k} \ . \tag{5.28}$$

From the mean offset, a mean-square transverse beam size $\overline{d^2}(\sigma)$ can be defined:

$$\begin{aligned}\overline{d^2}(\sigma) &\equiv \overline{[x(\sigma,\zeta)-\overline{x}(\sigma)]^2} \\ &= \frac{1}{\omega\tau}\int_{-\omega\tau/2}^{\omega\tau/2} [x(\sigma,\zeta)-\overline{x}(\sigma)]^2 F(\zeta)\,d\zeta \\ &= \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} x_k(\sigma)x_l(\sigma)\left[F_{-(k+l)} - F_{-k}F_{-l}\right] \ .\end{aligned} \tag{5.29}$$

Close to the accelerator entrance ($\sigma \ll 1$), where the time dependence of the displacement is given by Eq. (5.17), the mean-square beam size is

$$\overline{d^2}(\sigma) \simeq x_0^2 \frac{\varepsilon^2\sigma^4}{4} \sum_{k,l\neq 0} \tilde{w}_k \tilde{w}_l F_k F_l \left[F_{-(k+l)} - F_{-k}F_{-l}\right] \ . \tag{5.30}$$

The focusing strength does not appear in this expression, and the bunch distortion and transverse beam size are therefore initially independent of the amount of focusing. This will be discussed further in Sec. V D below. We note that $\overline{d^2}(\sigma) = 0$ for a beam of δ-function bunches, as expected, and that a time-independent offset at the accelerator entrance results in $\overline{d^2}(\sigma) = 0$ for the dc beam as well. Bunches of finite length will, of course, result in a nonzero mean-square beam size.

D. Steady-state BBU simulations

Steady-state BBU of a coasting beam comprised of bunches of finite length and arbitrary charge distribution may be calculated numerically with the aid of the recurrence relation, Eq. (5.14). To provide an example, we again consider a misaligned beam of uniform bunches. The recurrence relation is then initialized with the aid of Eq. (5.16). The Fourier coefficients of the form factor for the beam current are calculated from Eq. (5.22), and they are

$$F_m = \frac{\omega\tau}{m\pi\alpha}\sin\left(\frac{m\pi\alpha}{\omega\tau}\right) \ . \tag{5.31}$$

We set $Q = 1000$ and $\varepsilon = 0.2$, and we choose $\omega\tau = 10$ for two reasons. First, it is far away from the resonances of p and therefore represents a typical case of practical interest to the designer of a cw accelerator. Second, according to Figure 4, the growth factor $G^2(0,0;\kappa=0)$ in this case changes sign when the filling factor $f \simeq 0.1$. The centroids of shorter bunches are stable with respect to BBU and are deflected toward the linac axis, and the centroids of longer bunches are unstable and are deflected away from the linac axis. This choice of $\omega\tau$ therefore leads to a qualitative change in the transverse dynamics with only modest bunch lengthening and is accordingly an interesting case to study.

Steady-state BBU in the absence of focusing ($\kappa = 0$) is depicted in Fig. 6. The bunch centroids are deflected in the expected manner, yet the bunches become increasingly distorted as they propagate down the accelerator. As we discussed earlier, the distortion results in nonzero transverse beam size and degraded beam quality. Moreover, portions of each bunch can be deflected to displacements exceeding the initial displacement even when the bunch centroids are deflected toward the axis. The qualitative shape of the deformed bunch, i.e., whether it is concave or convex with respect to the accelerator axis at a given location is also a function of the bunch length. In addition, the displacement of the longitudinal halo can change markedly with bunch length. Figure 6 shows the halo extending beyond the bunches themselves when the bunches are short, but as the bunches are lengthened, the halo extends to transverse displacements which are less than the bunch displacement.

The effect of focusing in suppressing steady-state BBU of both the bunches and the halo is quantified in Fig. 7.

Here, the bunch length is fixed such that $f=0.2$. When a small amount of focusing is applied, the bunches and the halo are shifted closer to the accelerator axis with very little change of shape. As the focusing strength is increased, the bunch deformation remains relatively unaffected at small values of σ, which is consistent with the result of Eq. (5.30). However, the effect of increased focusing is pronounced at large values of σ; the bunches can be constrained to transverse displacements lying within their initial offset, and the bunch distortion can likewise be suppressed. This holds true for the halo as well, i.e., for sufficiently strong focusing, $|x(\sigma,\zeta)| \lesssim x_0$.

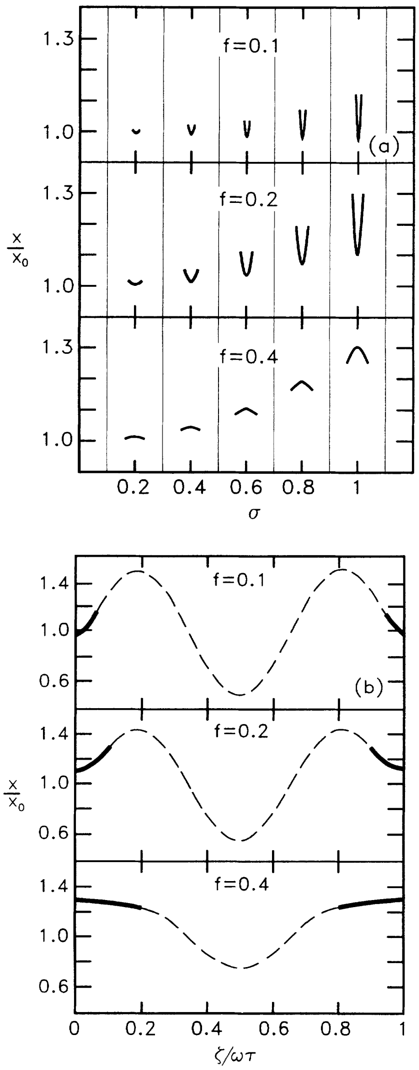

FIG. 6. (a) Steady-state transverse displacement of bunches vs position σ along the linac and (b) longitudinal halo at $\sigma=1$ vs $\zeta/\omega\tau$ for filling factors $f=0.1$, 0.2, and 0.4, and with $\kappa=0$, $\omega\tau=10$, $Q=1000$, and $\varepsilon=0.2$. Transverse displacement, plotted as the ordinate, is normalized with respect to the initial displacement. The linac entrance corresponds to $\sigma=0$, and the linac exit corresponds to $\sigma=1$. In (a), the bunch shape at selected values of σ is plotted on a scale such that a bunch with $f=1$ would span two divisions of the abscissa. In (b), the thick lines depict the bunches themselves, and the dashed lines represent the halo. The bunch centroids are located at $\zeta/\omega\tau=0,1$.

VI. BBU WITH RANDOM PARAMETERS

A. Steady-state solution with random initial conditions

1. Direct-current beam

The general solution for the transverse displacement $\xi(\sigma,\zeta)$ of a dc beam at location σ and time ζ as a function

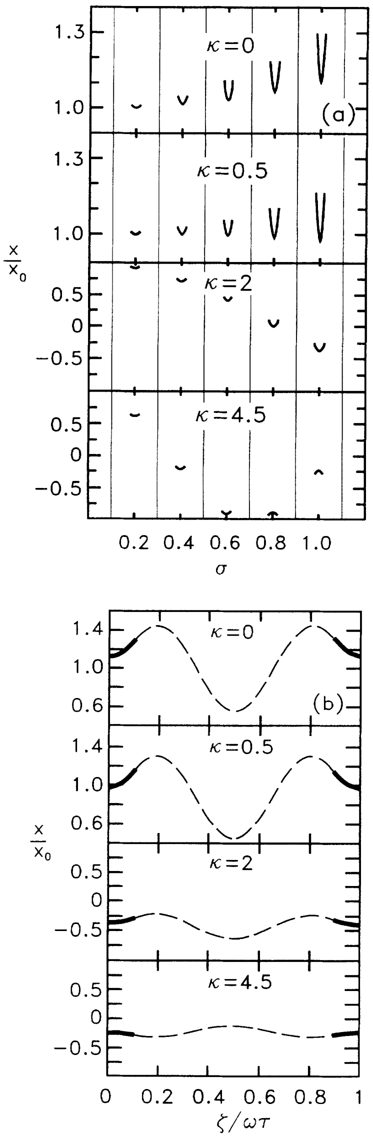

FIG. 7. (a) Steady-state transverse displacement of bunches vs position σ along the linac and (b) longitudinal halo at $\sigma=1$ vs $\zeta/\omega\tau$ for $\kappa=0$, 0.5, 2, and 4.5 and with $f=0.2$, $\omega\tau=10$, $Q=1000$, and $\varepsilon=0.2$. Transverse displacement, plotted as the ordinate, is normalized with respect to the initial displacement. The linac entrance corresponds to $\sigma=0$, and the linac exit corresponds to $\sigma=1$. In (a), the bunch shape at selected values of σ is plotted on a scale such that a bunch with $f=1$ would span two divisions of the abscissa. In (b), the thick lines depict the bunches themselves, and the dashed lines represent the halo. The bunch centroids are located at $\zeta/\omega\tau=0,1$. Note that there is a change of scale of the ordinate in the lower two subplots relative to the upper two subplots in both (a) and (b).

of the offset $\xi(0,\zeta)$ and angle $\xi'(0,\zeta)$ at the accelerator entrance was given in Eq. (3.6) as

$$\xi(\sigma,\zeta) = \int_0^\infty c(\sigma,\mu)\xi(0,\zeta-\mu)\,d\mu + \int_0^\infty s(\sigma,\mu)\xi'(0,\zeta-\mu)\,d\mu \,. \tag{6.1}$$

The displacement $\xi(\sigma,\zeta)$ can be interpreted as the sum of the outputs of two linear systems of impulse response functions $c(\sigma,\zeta)$ and $s(\sigma,\zeta)$ driven by the inputs $\xi(0,\zeta)$ and $\xi'(0,\zeta)$. We assume that $\xi(0,\zeta)$ and $\xi'(0,\zeta)$ are stationary random processes with means $\overline{\xi(0,\zeta)} = \overline{\xi'(0,\zeta)} = 0$, and with autocorrelation functions $\overline{\xi(0,\zeta)\xi(0,\zeta+\eta)} = R_\xi(0,\eta)$ and $\overline{\xi'(0,\zeta)\xi'(0,\zeta+\eta)} = R_{\xi'}(0,\eta)$. We also assume that $\xi(0,\eta)$ and $\xi'(0,\eta)$ are not cross correlated, although the analysis can easily be extended to include nonzero cross correlation.

In terms of the spectral densities of $\xi(0,\eta)$ and $\xi'(0,\eta)$,

$$S_\xi(0,Z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_\xi(0,\eta)e^{-i\eta Z}\,d\eta \,, \qquad S_{\xi'}(0,Z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{\xi'}(0,\eta)e^{-i\eta Z}\,d\eta \,, \tag{6.2}$$

the spectral density of $\xi(\sigma,\zeta)$ is given by

$$S_\xi(\sigma,Z) = S_\xi(0,Z)\,|\mathcal{C}[\lambda(\sigma,Z)]|^2 + S_{\xi'}(0,Z)\left|\frac{\mathcal{S}[\lambda(\sigma,Z)]}{\lambda(0,Z)}\right|^2 . \tag{6.3}$$

The mean-square transverse displacement along the accelerator is then given by

$$\overline{\xi^2(\sigma,\zeta)} = R_\xi(\sigma,0) = \int_{-\infty}^{+\infty} dZ\, S_\xi(\sigma,Z) = \int_{-\infty}^{+\infty} dZ\left(S_\xi(0,Z)\,|\mathcal{C}[\lambda(\sigma,Z)]|^2 + S_{\xi'}(0,Z)\left|\frac{\mathcal{S}[\lambda(\sigma,Z)]}{\lambda(0,Z)}\right|^2\right) . \tag{6.4}$$

For the remainder of this section, and without loss of generality, we will assume a coasting beam entering the accelerator parallel to the axis. The mean-square displacement is then

$$\overline{x^2}(\sigma) = \int_{-\infty}^{+\infty} dZ\, S_x(0,Z)\,|\cos[\lambda(Z)\sigma]|^2 \ . \tag{6.5}$$

Analogous to our earlier definition, we introduce a growth factor

$$G^2(\sigma;\kappa) \equiv \frac{1}{2\varepsilon\overline{x^2}(\sigma)}\frac{d^2[\overline{x^2}(\sigma)]}{d\sigma^2} \ . \tag{6.6}$$

The extra factor of 1/2 has been included since the growth factor is defined in terms of the mean-square displacement and not the rms value. Since the initial conditions now include many frequencies each having different growth rates, the growth factor will vary along the accelerator, and its value calculated close to the origin may not accurately characterize the BBU further along the accelerator. Nevertheless, the growth factor near the origin, where it can be calculated easily, still gives an indication of the stability of the beam.

Close to the accelerator entrance the growth factor is

$$\begin{aligned} G^2(0;\kappa) = -\frac{1}{2\varepsilon x_0^2}\int_{-\infty}^{+\infty} dZ\, S_x(0,Z)\,[\lambda^2(Z)+\lambda^{*2}(Z)] &= -\frac{1}{2\varepsilon x_0^2}\int_{-\infty}^{+\infty} dZ\, S_x(0,Z)[2\kappa^2 - \varepsilon\tilde{w}(Z) - \varepsilon\tilde{w}^*(Z)] \\ &= -\frac{\kappa^2}{\varepsilon} + \frac{1}{2x_0^2}\int_{-\infty}^{+\infty} dZ\, S_x(0,Z)\,[\tilde{w}(Z)+\tilde{w}^*(Z)] \\ &= -\frac{\kappa^2}{\varepsilon} + \frac{1}{x_0^2}\int_0^{+\infty} d\zeta\, w(\zeta)R_x(0,\zeta) \,, \end{aligned} \tag{6.7}$$

where

$$x_0^2 \equiv \overline{x^2}(0) = R_x(0,0) \ . \tag{6.8}$$

If we assume a single deflecting mode

$$w_1(\zeta) = u(\zeta)e^{-\zeta/2Q}\sin\zeta \tag{6.9}$$

and a correlation function for the displacement of the beam at the entrance of the accelerator

$$R_x(0,\zeta) = x_0^2 e^{-|\zeta|/2Q'}\cos\left(\frac{\omega'}{\omega}\zeta\right) \,, \tag{6.10}$$

then the growth factor is

$$G^2(0;\kappa) = -\frac{\kappa^2}{\varepsilon} + \frac{1}{2}\left(\frac{1+\frac{\omega'}{\omega}}{\left(\frac{1}{2Q}+\frac{1}{2Q'}\right)^2 + \left(1+\frac{\omega'}{\omega}\right)^2} + \frac{1-\frac{\omega'}{\omega}}{\left(\frac{1}{2Q}+\frac{1}{2Q'}\right)^2 + \left(1-\frac{\omega'}{\omega}\right)^2} \right) . \quad (6.11)$$

As can be seen, a resonance will occur if the characteristic frequency ω' of the correlation function is close to the frequency ω of the deflecting mode, and, in the absence of sufficiently strong focusing, the beam will experience strong deflecting forces away from or toward the axis if $\omega' < \omega$ or $\omega' > \omega$, respectively.

2. δ-function bunches

The general solution for the transverse displacement $\xi_M(\sigma)$ of bunch M at location σ as a function of the offsets $\xi_m(0)$ and angles $\xi'_m(0)$ of the previous bunches at the accelerator entrance was given in Eq. (4.21). This equation can be rewritten as

$$\xi_M(\sigma) = \sum_{m=-\infty}^{+\infty} \xi_{M-m}(0)c_m(\sigma) + \sum_{m=-\infty}^{+\infty} \xi'_{M-m}(0)s_m(\sigma) , \quad (6.12)$$

where

$$c_m(\sigma) = \frac{\omega\tau}{2\pi}\int_{-\pi/\omega\tau}^{\pi/\omega\tau} dZ\, e^{im\omega\tau Z}\mathcal{C}[\Lambda(\sigma,Z)] ,$$
$$s_m(\sigma) = \frac{\omega\tau}{2\pi}\int_{-\pi/\omega\tau}^{\pi/\omega\tau} dZ\, e^{im\omega\tau Z}\frac{\mathcal{S}[\Lambda(\sigma,Z)]}{\Lambda(0,Z)} . \quad (6.13)$$

The functions $c_m(\sigma)$ and $s_m(\sigma)$ are the Fourier components of $\mathcal{C}[\Lambda(\sigma,Z)]$ and $\frac{\mathcal{S}[\Lambda(\sigma,Z)]}{\Lambda(0,Z)}$ which, because of causality, satisfy $c_{m<0}(\sigma) = s_{m<0}(\sigma) = 0$. We assume that $\xi_k(0)$ and $\xi'_k(0)$ are stationary random sequences with means $\overline{\xi_k(0)} = 0$ and $\overline{\xi'_k(0)} = 0$, with autocorrelation functions $\overline{\xi_k(0)\xi_{k+n}(0)} = R_\xi(0,n)$ and $\overline{\xi'_k(0)\xi'_{k+n}(0)} = R_{\xi'}(0,n)$, and with cross-correlation function $\overline{\xi_k(0)\xi'_{k+n}(0)} = 0$. Equation (6.12) represents a σ-dependent family of linear transformations which generate new stationary random sequences $\xi_M(\sigma)$ from the initial sequences $\xi_m(0)$ and $\xi'_m(0)$.

In terms of the spectral densities of the sequences $\xi_k(\sigma)$ and $\xi'_k(\sigma)$, defined as the z transforms of the correlation functions

$$S_\xi(\sigma,z) = \sum_{k=-\infty}^{+\infty} R_\xi(\sigma,k)z^{-k} ,$$
$$S_{\xi'}(\sigma,z) = \sum_{k=-\infty}^{+\infty} R_{\xi'}(\sigma,k)z^{-k} , \quad (6.14)$$

the spectral density at σ is related to the spectral density at $\sigma = 0$ by

$$S_\xi(\sigma,z) = C(\sigma,z^{-1})S_\xi(0,z)C(\sigma,z) + S(\sigma,z^{-1})S_{\xi'}(0,z)S(\sigma,z) , \quad (6.15)$$

where

$$C(\sigma,z) = \sum_{k=-\infty}^{+\infty} c_k(\sigma)z^{-k} ,$$
$$S(\sigma,z) = \sum_{k=-\infty}^{+\infty} s_k(\sigma)z^{-k} \quad (6.16)$$

are the transfer functions associated with the linear transformations of Eq. (6.12). Using the inverse of the z transform, the correlation function at σ can be found from the spectral density by

$$R_\xi(\sigma,k) = \frac{1}{2\pi i}\oint S_\xi(\sigma,z)z^k\, z^{-1}dz , \quad (6.17)$$

where the path of integration is a circle centered at the origin which encloses all of the singularities of the integrand. The mean-square displacement is then

$$\overline{\xi_M^2}(\sigma) = R_\xi(\sigma,0) = \frac{1}{2\pi i}\oint dz\, z^{-1}[C(\sigma,z^{-1})S_\xi(0,z)C(\sigma,z) + S(\sigma,z^{-1})S_{\xi'}(0,z)S(\sigma,z)] . \quad (6.18)$$

For the remainder of this section, and without loss of generality, we will assume a coasting beam entering the accelerator parallel to the axis. If we make the change of variable $z = e^{i\omega\tau Z}$, then

$$C(\sigma,Z) = \sum_{k=-\infty}^{+\infty} c_k(\sigma)e^{-ik\omega\tau Z} = \cos[\Lambda(Z)\sigma] , \quad (6.19)$$

and the mean-square displacement is

$$\overline{x_M^2}(\sigma) = \frac{\omega\tau}{2\pi}\int_{-\pi/\omega\tau}^{\pi/\omega\tau} dZ\left(|\cos[\Lambda(Z)\sigma]|^2 \times \sum_{k=-\infty}^{+\infty} R_x(0,k)e^{-ik\omega\tau Z}\right) . \quad (6.20)$$

Using the same definition of the growth factor as in the previous section, we have

$$G^2(0;\kappa) = -\frac{1}{2\varepsilon x_0^2}\frac{\omega\tau}{2\pi}\int_{-\pi/\omega\tau}^{\pi/\omega\tau} dZ\left([\Lambda^2(Z)+\Lambda^{*2}(Z)]\sum_{k=-\infty}^{+\infty} R_x(0,k)e^{-ik\omega\tau Z}\right) . \tag{6.21}$$

Upon substituting

$$\Lambda^2(Z) = \kappa^2 - \varepsilon\tilde{W}(Z) = \kappa^2 - \varepsilon\sum_{k=-\infty}^{+\infty}\tilde{w}\left(Z+\frac{2\pi k}{\omega\tau}\right) = \kappa^2 - \varepsilon\omega\tau\sum_{k=0}^{+\infty} w(k\omega\tau)e^{-ik\omega\tau Z} , \tag{6.22}$$

the growth factor becomes

$$\begin{aligned}G^2(0;\kappa) &= -\frac{1}{2\varepsilon x_0^2}\frac{\omega\tau}{2\pi}\int_{-\pi/\omega\tau}^{\pi/\omega\tau} dZ\left(2\kappa^2 - \varepsilon\,\omega\tau\sum_{k=0}^{+\infty} w(k\omega\tau)(e^{-ik\omega\tau Z}+e^{+ik\omega\tau Z})\right)\sum_{l=-\infty}^{+\infty} R_x(0,l)e^{-il\omega\tau Z} \\ &= -\frac{\kappa^2}{\varepsilon} + \frac{\omega\tau}{x_0^2}\sum_{k=0}^{+\infty} w(k\omega\tau)R_x(0,k) .\end{aligned} \tag{6.23}$$

If the bunches are uncorrelated, the only nonzero value of the correlation function is for $k = 0$. However, since $w(0) = 0$, the growth factor for a beam of uncorrelated δ-function bunches is $-\kappa^2/\varepsilon$, and the steady-state behavior of the beam is unaffected by BBU. Such a beam will nevertheless exhibit nonzero transient BBU [23, 24]. If, on the other hand, we assume a single deflecting mode and a correlation function for the bunch displacement at the accelerator entrance of the form

$$R_x(0,k) = x_0^2 \exp\left(-|k|\frac{\omega\tau}{2Q'}\right)\cos k\omega'\tau , \tag{6.24}$$

then the growth factor is

$$G^2(0;\kappa) = -\frac{\kappa^2}{\varepsilon} + \frac{\omega\tau}{2}\left(\frac{1}{4}\frac{\sin[(\omega+\omega')\tau]}{\sinh^2\left[\frac{\omega\tau}{4}\left(\frac{1}{Q}+\frac{1}{Q'}\right)\right]+\sin^2\left(\frac{(\omega+\omega')\tau}{2}\right)} + \frac{1}{4}\frac{\sin[(\omega-\omega')\tau]}{\sinh^2\left[\frac{\omega\tau}{4}\left(\frac{1}{Q}+\frac{1}{Q'}\right)\right]+\sin^2\left(\frac{(\omega-\omega')\tau}{2}\right)}\right) . \tag{6.25}$$

This growth factor exhibits the same resonance properties as the function p of Eq. (4.14). At the resonances $|\omega \pm \omega'|\tau = 2n\pi\left[1 \pm \frac{1}{2}\left(\frac{1}{Q}+\frac{1}{Q'}\right)\right]$, the terms in brackets reach their extrema $\pm\frac{1}{2n\pi}\frac{Q'Q}{Q'+Q}$. In the absence of sufficiently strong focusing the beam will experience strong deflecting forces away from or toward the axis according to whether G^2 is positive or negative, respectively.

B. Distribution of deflecting-mode frequencies

In the preceding sections we investigated BBU assuming the deflecting-mode frequency and wake function were precisely known and constant along the linac. We found that BBU was strongly dependent on the parameter $\omega\tau$ where $\omega/2\pi$ is the deflecting-mode frequency and $1/\tau$ is the bunch frequency. For example, the steady-state BBU of a beam of δ-function bunches depends on the function $p(\omega\tau, Q)$ defined in Eq. (4.14) which exhibits sharp, narrow resonances.

In a linac, the frequency of the accelerating mode of all the cavities will need to be tuned to precisely the same value; however, it is unrealistic to expect the frequencies of the deflecting modes to be identical. Thus it is meaningful to ask about the probable BBU of a beam in a linac where the deflecting-mode frequency follows a probability distribution.

We shall treat in detail the case of steady-state BBU, making the same assumptions as in Sec. V A, and Eq. (5.3) will be our point of departure for this investigation. Because the deflecting-mode frequency now depends on σ, $\tilde{w}_m$ depends on both σ and ω, and BBU is governed by the system of differential equations

$$\left(\frac{d^2}{d\sigma^2}+\kappa^2\right)x_m(\sigma) = \varepsilon\tilde{w}_m(\sigma,\omega)\sum_{k=-\infty}^{+\infty} F_k\, x_{m-k}(\sigma) . \tag{6.26}$$

Once again we characterize BBU at any point along the accelerator by a growth factor which is most easily calculated close to the accelerator entrance where the beam has not been deflected substantially from its original offset. With these assumptions we have $x_0(\sigma) \simeq x_0$ and $x_m(\sigma) \ll x_0$ for $m \neq 0$, and the solutions are

$$x_0(\sigma) = x_0\left(1 - \frac{\kappa^2\sigma^2}{2} + \varepsilon F_0\int_0^\sigma d\sigma_1\int_0^{\sigma_1} d\sigma_2\,\tilde{w}_0(\sigma_2,\omega)\right) , \tag{6.27}$$

$$x_m(\sigma) = x_0\varepsilon F_m\int_0^\sigma d\sigma_1\int_0^{\sigma_1} d\sigma_2\,\tilde{w}_m(\sigma_2,\omega) .$$

The displacement of the center of the bunch is

$$x(\sigma,0)=\sum_{m=-\infty}^{+\infty} x_m(\sigma)$$
$$=x_0\left(1-\frac{\kappa^2\sigma^2}{2}+\varepsilon\sum_{m=-\infty}^{+\infty}F_m\int_0^{\sigma}d\sigma_1\int_0^{\sigma_1}d\sigma_2\,\tilde{w}_m(\sigma_2,\omega)\right),\tag{6.28}$$

and the growth factor of the bunch centroid is

$$G^2(\sigma,0;\kappa,\omega)=\frac{1}{\varepsilon x_0}\frac{d^2}{d\sigma^2}x(\sigma,0)=\Omega^2(\sigma,\omega)-\frac{\kappa^2}{\varepsilon},\tag{6.29}$$

where

$$\Omega^2(\sigma,\omega)=\sum_{m=-\infty}^{+\infty}F_m\tilde{w}_m(\sigma,\omega).\tag{6.30}$$

We now assume that we have an ensemble of accelerators and that, at σ, the deflecting-mode frequency has a probability density $\tilde{f}(\sigma,\omega)$. Since $\Omega^2(\sigma,\omega)$ is a linear function of the quantities $\tilde{w}_m(\sigma,\omega)$, its expectation value is given by

$$\langle\Omega^2(\sigma)\rangle=\sum_{m=-\infty}^{+\infty}F_m\langle\tilde{w}_m(\sigma)\rangle$$
$$=\sum_{m=-\infty}^{+\infty}F_m\int_{-\infty}^{+\infty}\tilde{w}_m(\sigma,\omega)\tilde{f}(\sigma,\omega)\,d\omega.\tag{6.31}$$

Previously we used ω, the frequency of the deflecting mode, as a normalizing parameter in defining the dimensionless time ζ. Since ω is now a random variable, it is not suitable anymore as a normalizing variable, and we will use instead a frequency ω_0 which can be, for example, the mean of the distribution of ω. The wake functions will then be explicit functions of ω and will be of the form $w(\sigma,\frac{\omega}{\omega_0}\zeta)$.

The expectation values of the coefficients $\tilde{w}_m(\sigma,\omega)$ are

$$\langle\tilde{w}_m(\sigma)\rangle=\int_{-\infty}^{+\infty}\tilde{w}_m(\sigma,\omega)\tilde{f}(\sigma,\omega)\,d\omega$$
$$=\int_{-\infty}^{+\infty}d\zeta e^{-im\frac{2\pi}{\omega_0\tau}}\int_{-\infty}^{+\infty}w\left(\sigma,\frac{\omega}{\omega_0}\zeta\right)\tilde{f}(\sigma,\omega)\,d\omega.\tag{6.32}$$

If we define an "effective" wake function $\hat{w}(\sigma,\zeta)$ as

$$\hat{w}(\sigma,\zeta)\equiv\int_{-\infty}^{+\infty}w\left(\sigma,\frac{\omega}{\omega_0}\zeta\right)\tilde{f}(\sigma,\omega)d\omega,\tag{6.33}$$

we see that $\langle\tilde{w}_m(\sigma)\rangle$ is simply the value of the Fourier transform of $\hat{w}(\sigma,\zeta)$ at $m2\pi/\omega_0\tau$. Thus

$$\langle\Omega^2(\sigma)\rangle=\sum_{m=-\infty}^{+\infty}\langle\tilde{w}_m(\sigma)\rangle F_m=\int_0^{+\infty}\hat{w}(\sigma,\zeta)F(-\zeta)\,d\zeta.\tag{6.34}$$

If we compare this result with Eq. (5.19), we see that all the results obtained previously for steady-state BBU with arbitrary wake functions and charge distributions within the bunches are retained in the case of a distribution of deflecting-mode frequencies through the use of an effective wake function $\hat{w}(\sigma,\zeta)$.

To provide an example, we calculate the effective wake function for the case of a single deflecting mode and a Lorentzian distribution of frequencies. For simplicity we omit the σ dependence in the calculation. The wake function and the frequency distribution are

$$w\left(\frac{\omega}{\omega_0}\zeta\right)=u(\zeta)\exp\left(-\left|\frac{\omega}{\omega_0}\zeta\right|\frac{1}{2Q}\right)\sin\left(\frac{\omega}{\omega_0}\zeta\right),$$
$$\tilde{f}(\omega)=\frac{1}{\pi}\frac{\Delta\omega}{(\omega-\omega_0)^2+\Delta\omega^2}.\tag{6.35}$$

Since the frequency distribution will always be narrowly peaked around ω_0 (because $\Delta\omega\ll\omega_0$), we can replace the exponential decay in the wake function by $e^{-\zeta/2Q}$. The effective wake function is then

$$\hat{w}(\zeta)=u(\zeta)\frac{\Delta\omega}{\pi}\int_{-\infty}^{+\infty}e^{-\zeta/2Q}\sin\left(\frac{\omega}{\omega_0}\zeta\right)\times\frac{d\omega}{(\omega-\omega_0)^2+\Delta\omega^2}$$
$$=u(\zeta)\exp\left[-\zeta\left(\frac{1}{2Q}+\frac{\Delta\omega}{\omega_0}\right)\right]\sin\zeta.\tag{6.36}$$

Thus we see that in the case of a single deflecting mode, the effective wake function is identical to the original wake function but with a shorter decay time. The effect of a frequency distribution is to broaden the real bandwidth of the deflecting mode by the width of the Lorentzian distribution.

The modification to the wake function of a single deflecting mode caused by a frequency spread can be generalized to other probability densities for the deflecting-mode frequency. The effective wake function for a single deflecting mode is

$$\hat{w}(\zeta)=u(\zeta)\int_{-\infty}^{+\infty}\exp\left(-\left|\frac{\omega}{\omega_0}\zeta\right|\frac{1}{2Q}\right)\sin\left(\frac{\omega}{\omega_0}\zeta\right)\tilde{f}(\omega)\,d\omega.\tag{6.37}$$

Since the probability density is narrowly peaked around ω_0, we can take the exponential out of the integral, and

$$\hat{w}(\zeta)=u(\zeta)e^{-\zeta/2Q}\int_{-\infty}^{+\infty}\sin\left(\frac{\omega}{\omega_0}\zeta\right)\tilde{f}(\omega)\,d\omega.\tag{6.38}$$

By introducing a new variable $Z\equiv(\omega/\omega_0)-1$ and the probability density in Z space $\tilde{g}(Z)\equiv\omega_0\tilde{f}(\omega_0(Z+1))$, the effective wake function becomes

$$\hat{w}(\zeta)=u(\zeta)e^{-\zeta/2Q}\int_{-\infty}^{+\infty}\sin[(Z+1)\zeta]\,\tilde{g}(Z)\,dZ.\tag{6.39}$$

If the probability density $\tilde{f}(\omega)$ is symmetric around ω_0, then $\tilde{g}(Z)$ is an even function of Z, and

$$\hat{w}(\zeta) = u(\zeta)e^{-\zeta/2Q}\sin\zeta \int_{-\infty}^{+\infty} e^{iZ\zeta}\,\tilde{g}(Z)\,dZ\ . \quad (6.40)$$

Written in terms of $g(\zeta)$, the function whose Fourier transform is $\tilde{g}(Z)$, the effective wake function is

$$\hat{w}(\zeta) = 2\pi\, g(\zeta)\, w(\zeta)\ . \quad (6.41)$$

Thus, in the case of a symmetric probability density, the effective wake function is simply the original wake function multiplied by the inverse Fourier transform of the probability density of the deflecting-mode frequency.

We now list the modifying factors to the wake function for three common probability densities.

(i) Uniform:

$$\tilde{f}(\omega) = \begin{cases} \dfrac{1}{\Delta\omega} & \text{for } |\omega-\omega_0| < \dfrac{\Delta\omega}{2} \\ 0 & \text{for } |\omega-\omega_0| > \dfrac{\Delta\omega}{2}, \end{cases}$$
$$\tilde{g}(Z) = \begin{cases} \dfrac{\omega_0}{\Delta\omega} & \text{for } |Z| < \dfrac{\Delta\omega}{2\omega_0} \\ 0 & \text{for } |Z| > \dfrac{\Delta\omega}{2\omega_0}, \end{cases} \quad (6.42)$$
$$2\pi g(\zeta) = \frac{\sin\left(\dfrac{\Delta\omega}{2\omega_0}\zeta\right)}{\left(\dfrac{\Delta\omega}{2\omega_0}\zeta\right)}\ .$$

(ii) Lorentzian:

$$\tilde{f}(\omega) = \frac{1}{\pi}\frac{\Delta\omega}{(\omega-\omega_0)^2+\Delta\omega^2}\ ,$$
$$\tilde{g}(Z) = \frac{1}{\pi}\frac{\Delta\omega}{\omega_0}\frac{1}{Z^2+\left(\dfrac{\Delta\omega}{\omega_0}\right)^2}\ , \quad (6.43)$$
$$2\pi g(\zeta) = \exp\left(-\frac{\Delta\omega}{\omega_0}|\zeta|\right)\ .$$

(iii) Gaussian:

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}\Delta\omega}\exp\left[-\frac{1}{2}\left(\frac{\omega-\omega_0}{\Delta\omega}\right)^2\right]\ ,$$
$$\tilde{g}(Z) = \frac{1}{\sqrt{2\pi}}\frac{\omega_0}{\Delta\omega}\exp\left[-\frac{1}{2}\left(\frac{\omega_0}{\Delta\omega}\right)^2 Z^2\right]\ , \quad (6.44)$$
$$2\pi g(\zeta) = \exp\left[-\frac{1}{2}\left(\frac{\Delta\omega}{\omega_0}\right)^2\zeta^2\right]\ .$$

In the case of deflecting modes with infinite Q, the decay of the effective wake function will be governed by the modifying function $g(\zeta)$ and thus will depend strongly on the assumed probability density of the deflecting-mode frequency. For example, for large ζ, the effective wake function will decay as $1/\zeta$ for a uniform density, as $e^{-a\zeta}$ for a Lorentzian density, and as $e^{-a\zeta^2}$ for a Gaussian density.

Although we have defined an effective wake function in the context of steady-state BBU, it can also be used to study transient BBU provided that the transverse displacement does not change significantly over distances comparable to the correlation length of the deflecting-mode frequency. For example, if the frequencies of the deflecting modes in successive cavities are uncorrelated, this is equivalent to the assumption that the continuum approximation is valid. Under these conditions, one might expect that, for deflecting modes with infinite Q, transient BBU for large ζ would be related to the decay rate of the function $g(\zeta)$. Thus the rate of approach to the steady state would depend on the assumed probability density of the deflecting-mode frequency. Colombant and Lau [25,26] find an algebraic decay of the form $1/\zeta$ for large ζ; however, this result is a consequence of their assumption of a uniform probability density where $g(\zeta)$ also decays as $1/\zeta$ as seen in Eq. (6.42). Under different assumptions, the decay would be different. For example, if the probability density is assumed to be Lorentzian, one finds an exponential decay similar to the one obeyed by $g(\zeta)$, which is like that produced by a finite Q. Gluckstern, Neri, and Cooper [16,24,27] have also investigated transient BBU for a beam of δ-function bunches in terms of the rms frequency spread. Introducing a distribution of deflecting-mode frequencies into a linac has been considered as an additional cure of BBU in the next-generation linear collider [28] and in free-electron lasers [29].

VII. CONCLUSIONS

Both the transient and steady-state dynamics of cumulative beam breakup in linear accelerators with periodic beam current have been characterized using a formalism based on Fourier analysis of the equation of transverse motion. This formalism is universally applicable, being useful for BBU investigations over the entire spectrum of linac applications. In particular, it includes arbitrary velocity and acceleration of the beam, and thus is not restricted to relativistic beams.

Prior to this work, it was already known that transient BBU of a direct-current beam could be classified in terms of two dimensionless parameters involving the BBU coupling strength and the focusing strength. For a beam of δ-function bunches, a third dimensionless parameter needs to be included which is the ratio of the deflecting-mode frequency and the bunch frequency. This third parameter enriches the solution space for transient BBU, and we have calculated and classified the BBU both near to and far from the zero crossings of the wake function where the deflecting-mode frequency is an integer multiple of the bunch frequency. We also showed how the solution space of the beam of δ-function bunches maps onto the solution space of the dc beam in the appropriate limit.

Methods for suppressing transient BBU include damping the deflecting wake fields, tuning the wake fields so that the bunches are located near their zero crossings, applying strong transverse focusing, and introducing a distribution of deflecting-mode frequencies. The relative effectiveness of these methods depends on the three dimensionless parameters characterizing the linac under consideration. The decay of transient BBU in the presence of a distribution of deflecting-mode frequencies can depend strongly on the assumed distribution function, particularly if the wake fields in the constituent cavities decay slowly.

Steady-state BBU, which is of principal concern in cw linacs, was characterized in terms of the BBU coupling strength, the transverse focusing strength, the ratio of deflecting-mode and bunch frequencies, the bunch length, and the charge distribution within each bunch. The latter two parameters are of special importance in low-velocity beams. For a misaligned beam of δ-function bunches, steady-state BBU is nearly independent of the effective Q of the wake fields except near the zero crossings of the wake function where the deflecting-mode frequency is an even-integer multiple of the bunch frequency. Strongly peaked resonances occur in these regions which can be suppressed by lowering the effective Q of the wake fields, and they are generally less pronounced with bunches of finite length, the exception corresponding to extremely short, but still finite, bunches. The deflecting fields tend to distort bunches of finite length, leading to emittance growth and degraded beam quality. Unbunched particles comprising a diffuse longitudinal halo between the bunches respond to the deflecting fields but do not excite them. These particles can be deflected more than the bunches themselves. If allowed to impinge on the accelerating structures, they will contribute to activation over the long-term operation of the linac. Bunch distortion and halo displacement are both accentuated near the resonances. Steady-state BBU can also be considerable in the presence of random initial conditions of the bunches at the linac entrance. For example, with a beam of δ-function bunches, a resonance occurs if a harmonic of the characteristic frequency of the correlation function for the bunch displacement at the linac entrance is close to the characteristic frequency of the wake function, and strong deflecting fields may then be generated. Sufficiently strong transverse focusing will suppress all of these potential problems associated with steady-state BBU. Analytic formulas were derived for each case to use in estimating the required focusing strength.

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APPENDIX A: FOURIER-TRANSFORMED EQUATION OF TRANSVERSE MOTION

To derive the Fourier-analyzed BBU equation with periodic current, we first use Eq. (2.6) to write Eq. (2.5) in the form (with the σ dependence suppressed)

$$\mathcal{D}_\sigma^2 x(\zeta) = \varepsilon \sum_{k=-\infty}^{+\infty} F_k \int_{-\infty}^{\zeta} d\zeta'\, w(\zeta-\zeta') e^{ik(2\pi/\omega\tau)\zeta'} x(\zeta') = \varepsilon \sum_{k=-\infty}^{+\infty} F_k \int_0^{+\infty} d\zeta_1\, w(\zeta_1) e^{ik(2\pi/\omega\tau)(\zeta-\zeta_1)} x(\zeta-\zeta_1) , \tag{A1}$$

where the second equality follows after the change of variables $\zeta_1 = \zeta - \zeta'$, and $\mathcal{D}_\sigma^2$ is the differential operator in Eq. (2.5). Upon applying the Fourier transform to Eq. (A1), it becomes

$$\mathcal{D}_\sigma^2 \tilde{x}(Z) = \varepsilon \sum_{k=-\infty}^{+\infty} F_k \int_{-\infty}^{+\infty} d\zeta\, e^{-iZ\zeta} \int_0^{+\infty} d\zeta_1\, w(\zeta_1) e^{ik(2\pi/\omega\tau)(\zeta-\zeta_1)} x(\zeta-\zeta_1) . \tag{A2}$$

The right-hand side is a sum of Fourier transforms of the convolutions of $w(\zeta)$ and $e^{ik(2\pi/\omega\tau)\zeta}x(\zeta)$, and Eq. (2.8) therefore follows by inspection.

APPENDIX B: TRANSVERSE DISPLACEMENT OF THE dc BEAM

The transverse displacement of the dc beam is calculated from Eq. (3.14). Upon changing the integration variable in Eq. (3.14) from Z to θ as indicated in Eq. (3.15), $\xi(\sigma,\theta)$ is given by

$$\xi(\sigma,\zeta) = \frac{1}{2\pi} \int_0^{\zeta} d\mu\, e^{-\mu/2Q} \int_{-\infty+i/2Q}^{+\infty+i/2Q} d\theta\, e^{-i\mu\theta} \mathcal{F}_\mu[\lambda(\sigma,\theta)] \tag{B1}$$

where

$$\mathcal{F}_\mu[\lambda(\sigma,\theta)] \equiv \xi(0,\zeta-\mu)\mathcal{C}[\lambda(\sigma,\theta)] + \xi'(0,\zeta-\mu)\frac{\mathcal{S}[\lambda(\sigma,\theta)]}{\lambda(0,\theta)} , \tag{B2}$$

and $\lambda(\sigma,\theta)$ is given by Eq. (3.16). The poles of $\lambda(\sigma,\theta)$

are identical to the poles of $\tilde{w}(\theta)$ which, by causality, all lie on or below the real axis.

We now show with the aid of the closed contour of Fig. 8 that the integral over θ in Eq. (B1) is equivalent to an integral along a line displaced infinitesimally above the uppermost poles of $\tilde{w}(\theta)$. This line is taken to be the real axis. The contour of Fig. 8 contains no poles; therefore

$$\oint d\theta\, e^{-i\mu\theta}\mathcal{F}_\mu[\lambda(\sigma,\theta)] = 0 \,. \tag{B3}$$

Since $\lambda(|\theta| \to \infty) = K(\sigma)$, the line integral along leg A is

$$\mathcal{I}_A = \mathcal{F}_\mu[K] \lim_{x\to\infty} \int_x^{x+i/2Q} d\theta\, e^{-i\mu\theta} \,, \tag{B4}$$

and the line integral along leg B is

$$\mathcal{I}_B = \mathcal{F}_\mu[K] \lim_{x\to\infty} \int_{-x+i/2Q}^{-x} d\theta\, e^{-i\mu\theta} \,. \tag{B5}$$

Adding the contributions from these two line integrals gives

$$\mathcal{I}_A + \mathcal{I}_B = 2\mathcal{F}_\mu[K]\left(e^{\mu/2Q} - 1\right) \lim_{x\to\infty}\left(\frac{\sin x\mu}{\mu}\right) \,. \tag{B6}$$

Upon inserting this result into Eq. (B1), using the identity

$$\lim_{x\to\infty}\left(\frac{\sin x\mu}{\mu}\right) = \pi\delta(\mu) \,, \tag{B7}$$

and integrating over μ, we see that the contributions from legs A and B cancel:

$$\mathcal{I}_A + \mathcal{I}_B = 0 \,. \tag{B8}$$

Therefore, in view of Eq. (B3), the integral along the line $[-\infty + \frac{i}{2Q}, +\infty + \frac{i}{2Q}]$ is equal to the integral along the real axis excluding the poles lying on the real axis. This is the result expressed in Eq. (3.17).

APPENDIX C: TRANSIENT BBU OF THE dc BEAM

The asymptotic behavior of the transient displacement of a coasting dc beam in the presence of a single deflecting mode is calculated here. The method of steepest descent is used, and it is expected to be valid when BBU is substantial. The calculations of this appendix are restricted to the "misaligned beam," i.e., solution class II. If the beam induces an "impulse excitation" of the wake field, solution class I is needed. The corresponding results are generated in the same manner and can be inferred by inspection of the results developed here.

If BBU is large, then Eq. (3.21) can be written approximately in the form

$$\xi(\sigma,\zeta) - \xi(\sigma,\infty) \simeq 2\,\mathrm{Re}\left(\int d\theta\, g(\theta)\, e^{f(\theta)}\right) \,, \tag{C1}$$

where $g(\theta)$ is a slowly varying function of θ, and $f(\theta)$ has saddle points at which $\partial f/\partial\theta \equiv f'(\theta) = 0$. Choosing the saddle point θ_s for which the real part of $f(\theta)$ is largest, and anticipating that $\mathrm{Re}[f(\theta_s)] \gg 1$, we deform the contour of Fig. 8 so that the line integral passes through θ_s along the trajectory of steepest descent. This procedure results in the following approximate closed-form solution:

$$\begin{aligned}\int d\theta\, g(\theta)e^{f(\theta)} &\simeq \int d\theta\, g(\theta)\exp[f(\theta_s) + \tfrac{1}{2}f''(\theta_s)(\theta-\theta_s)^2] \\ &\simeq g(\theta_s)e^{f(\theta_s)}\int d\theta\, \exp[\tfrac{1}{2}f''(\theta_s)(\theta-\theta_s)^2] \\ &\simeq \sqrt{\frac{2\pi}{-f''(\theta_s)}}\,g(\theta_s)e^{f(\theta_s)} \,.\end{aligned} \tag{C2}$$

Considering now the case of the dc beam in the presence of a single deflecting mode, we write Eq. (3.21) in the form

$$x(\sigma,\zeta) - x(\sigma,\infty) = -\frac{e^{-\zeta/2Q}}{4\pi}\int_{-\infty}^{+\infty} d\theta \frac{1}{i\theta + \dfrac{1}{2Q}} e^{F(\sigma,\theta)} \,, \tag{C3}$$

where

$$F(\sigma,\theta) \equiv -i[\zeta\theta + \lambda_1(\theta)\sigma] + \ln\left(x_0 + \frac{ix_0'}{\lambda_1(\theta)}\right) \,. \tag{C4}$$

To find the saddle points, we set $F'(\theta) = 0$:

$$\frac{\partial F}{\partial\theta} = -i\zeta + i\frac{\varepsilon\theta\tilde{w}_1^2(\theta)\sigma}{\lambda_1(\theta)} \times\left(1 + \frac{1}{\sigma\lambda_1(\theta)}\frac{x_0'}{x_0\lambda_1(\theta) + ix_0'}\right) = 0 \,. \tag{C5}$$

Defining the parameters ψ, s_1, and s_2 according to Eq. (3.29), we have

$$(\psi-1)^4 + s_1(\psi-1)^3 - s_2^2\psi\left(1 + \frac{1}{\sigma\lambda_1(\psi)}\frac{x_0'}{x_0\lambda_1(\psi) + ix_0'}\right)^2 = 0 \,. \tag{C6}$$

If the term involving λ_1 is small, i.e.,

$$\sigma|\lambda_1(\psi)| \gg 1 \,, \tag{C7}$$

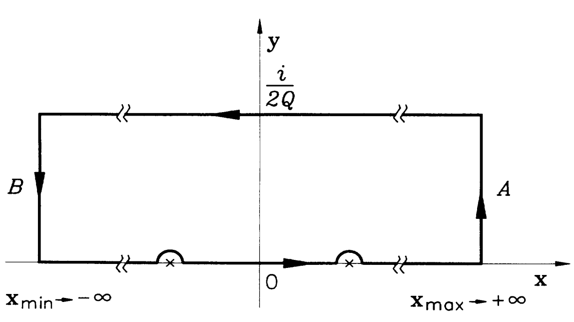

FIG. 8. Contour for calculating the displacement $\xi(\sigma,\zeta)$ of the dc beam.

then the transient displacement is found from Eq. (3.24) with $f(\theta)$ and $g(\theta)$ defined in Eqs. (3.25) and (3.27), respectively. We shall assume the inequality is satisfied, and check our assumption after evaluating the saddle points.

Returning to Eq. (3.24), we now have

$$f(\sigma,\psi) = -i\left[\zeta\sqrt{\psi} + \lambda_1(\psi)\sigma\right] , \tag{C8}$$

where ψ is governed by the quartic equation, Eq. (3.28), and

$$\lambda_1(\psi) = \frac{\zeta}{\sigma}\frac{s_2^2}{s_1}\frac{\sqrt{\psi}}{(\psi-1)^2} . \tag{C9}$$

Upon evaluating $e^{f(\sigma,\psi)}$, we find exponential growth $e^{\Gamma\zeta}$ in which Γ is given by Eq. (3.30) using the root of Eq. (3.28) corresponding to the largest growth rate. We also find, letting $f''(\psi) \equiv \partial^2 f(\theta_s = \sqrt{\psi})/\partial\theta^2$,

$$f''(\sigma,\psi) = \frac{i\zeta}{\sqrt{\psi}}\left[1 - \frac{\psi}{\psi-1}\left(3 + \frac{(\psi-1)^4}{s_2^2\psi}\right)\right] \tag{C10}$$

and

$$g(\psi) = -\frac{e^{-\zeta/2Q}}{4\pi\left(i\sqrt{\psi}+\dfrac{1}{2Q}\right)}\left(x_0 + i\frac{\sigma}{\zeta}\frac{s_1}{s_2^2}\frac{(\psi-1)^2}{\sqrt{\psi}}x_0'\right). \tag{C11}$$

The transverse displacement is calculated from Eq. (C2) using the results of Eqs. (C8), (C10), and (C11) in the manner

$$x(\sigma,\zeta) - x(\sigma,\infty) \simeq 2\,\mathrm{Re}\left(\sqrt{\frac{2\pi}{-f''(\sigma,\psi)}}g(\psi)e^{f(\sigma,\psi)}\right) \tag{C12}$$

Upon substituting the result of Eq. (C9) into Eq. (C7), we find the inequality of Eq. (C7) is valid provided the following inequalities hold in their respective domains (cf. Sec. III C):

$$\begin{aligned} &E_A \gg 1 , \quad \text{domain } A \\ &\kappa\sigma \gg 1 , \quad \text{domains } B \text{ and } C \\ &E_D \gg 1 , \quad \text{domain } D. \end{aligned} \tag{C13}$$

The inequalities associated with domains A and D are generally true when the transient BBU is pronounced, as is also required for the validity of the method of steepest descent. The inequality associated with domains B and C is consistent with the assumption of strong focusing which characterizes these domains. These considerations justify using the inequality of Eq. (C7).

APPENDIX D: TRANSVERSE MOTION WITH δ-FUNCTION BUNCHES

This appendix concerns the solution of the equation of transverse motion Eq. (4.1) and develops the result of Eq. (4.6).

With the definition

$$\tilde{\Xi}(\sigma,Z) \equiv \sum_{k=-\infty}^{+\infty}\tilde{\xi}\left(\sigma, Z - k\frac{2\pi}{\omega\tau}\right) , \tag{D1}$$

and using $F_k = 1$ for all k, Eq. (2.11) becomes

$$\left(\frac{\partial^2}{\partial\sigma^2} + K^2\right)\tilde{\xi}(\sigma,Z) = \varepsilon\,\tilde{w}(Z)\,\tilde{\Xi}(\sigma,Z) . \tag{D2}$$

We note that $\tilde{\Xi}(\sigma,Z)$ is periodic with period $2\pi/\omega\tau$, as are $\tilde{W}(Z)$ and $\Lambda(\sigma,Z)$ given in Eqs. (4.2) and (4.3), respectively, i.e.,

$$\tilde{\Xi}(\sigma,Z) = \tilde{\Xi}\left(\sigma, Z - n\frac{2\pi}{\omega\tau}\right) , \tag{D3}$$

where n is an integer. Accordingly, we have

$$\left(\frac{\partial^2}{\partial\sigma^2} + K^2\right)\tilde{\xi}\left(\sigma, Z - n\frac{2\pi}{\omega\tau}\right) = \varepsilon\tilde{w}\left(Z - n\frac{2\pi}{\omega\tau}\right)\tilde{\Xi}(\sigma,Z) , \tag{D4}$$

and upon summing over all integers n, the result is a homogeneous differential equation for $\tilde{\Xi}(\sigma,Z)$:

$$\left(\frac{\partial^2}{\partial\sigma^2} + \Lambda^2(\sigma,Z)\right)\tilde{\Xi}(\sigma,Z) = 0 . \tag{D5}$$

In the WKBJ approximation, the solution of Eq. (D5) is

$$\tilde{\Xi}(\sigma,Z) = A(Z)\mathcal{C}[\Lambda(\sigma,Z)] + B(Z)\frac{\mathcal{S}[\Lambda(\sigma,Z)]}{\Lambda(0,Z)} , \tag{D6}$$

where

$$A(Z) = \tilde{\Xi}(0,Z) , \quad B(Z) = \frac{\partial\tilde{\Xi}(0,Z)}{\partial\sigma} . \tag{D7}$$

The solution of Eq. (D2) is a superposition of the general solution of the corresponding homogeneous equation and the particular solution of the inhomogeneous equation found on substituting the result of Eq. (D6) into Eq. (D2). In the WKBJ approximation, the solution is

$$\tilde{\xi}(\sigma,Z) = \alpha(Z)\mathcal{C}[K(\sigma)] + \beta(Z)\frac{\mathcal{S}[K(\sigma)]}{K(0)} + \frac{\tilde{w}(Z)}{\tilde{W}(Z)}\tilde{\Xi}(\sigma,Z) . \tag{D8}$$

The last term is the particular solution of Eq. (D2) as can be verified by direct substitution. From Eqs. (D6) – (D8), we find

$$\begin{aligned} \alpha(Z) &= \tilde{\xi}(0,Z) - \frac{\tilde{w}(Z)}{\tilde{W}(Z)}A(Z) , \\ \beta(Z) &= \frac{\partial\tilde{\xi}(0,Z)}{\partial\sigma} - \frac{\tilde{w}(Z)}{\tilde{W}(Z)}B(Z) , \end{aligned} \tag{D9}$$

and

$$\tilde{\xi}(\sigma, Z) = \tilde{\xi}(0, Z)\mathcal{C}[K(\sigma)] + \frac{\partial\tilde{\xi}(0, Z)}{\partial\sigma}\frac{\mathcal{S}[K(\sigma)]}{K(0)} + \frac{\tilde{w}(Z)}{\tilde{W}(Z)}\left[A(Z)\Big(\mathcal{C}[\Lambda(\sigma, Z)] - \mathcal{C}[K(\sigma)]\Big) + B(Z)\left(\frac{\mathcal{S}[\Lambda(\sigma, Z)]}{\Lambda(0, Z)} - \frac{\mathcal{S}[K(\sigma)]}{K(0)}\right)\right] . \tag{D10}$$

Equation (4.4) is simply Eq. (D10) after Fourier transforming back from Z to ζ.

It remains to determine the functions $A(Z)$ and $B(Z)$ from the initial conditions. Using the identity

$$\sum_{k=-\infty}^{+\infty} e^{ik(2\pi/\omega\tau)\zeta} = \omega\tau \sum_{k=-\infty}^{+\infty} \delta(\zeta - k\omega\tau) , \tag{D11}$$

we find from Eqs. (D7) and (D1) that

$$A(Z) = \tilde{\Xi}(0, Z) = \sum_{k=-\infty}^{+\infty} \tilde{\xi}\left(0, Z - k\frac{2\pi}{\omega\tau}\right)$$
$$= \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i[Z-k(2\pi/\omega\tau)]\zeta}\xi(0,\zeta)\, d\zeta$$
$$= \omega\tau \sum_{k=-\infty}^{+\infty} \xi(0, k\omega\tau)e^{-ik\omega\tau Z} . \tag{D12}$$

A similar calculation gives

$$B(Z) = \frac{\partial\tilde{\Xi}(0, Z)}{\partial\sigma} = \omega\tau \sum_{k=-\infty}^{+\infty} \xi'(0, k\omega\tau)e^{-ik\omega\tau Z} . \tag{D13}$$

These are the results presented in Eq. (4.5).

The integral over Z in Eq. (4.4) can be decomposed into a sum of integrals over the interval $\omega\tau/2\pi$ centered on the bunches. Denoting the integrand by $f(Z)$, we have

$$\int_{-\infty}^{+\infty} dZ\, f(Z) = \sum_{k=-\infty}^{+\infty} \int_{(k-1/2)(2\pi/\omega\tau)}^{(k+1/2)(2\pi/\omega\tau)} dZ\, f(Z)$$
$$= \sum_{k=-\infty}^{+\infty} \int_{-\pi/\omega\tau}^{\pi/\omega\tau} dZ\, f(Z + k2\pi/\omega\tau) . \tag{D14}$$

Because $A(Z)$, $B(Z)$, $\tilde{W}(Z)$, and $\Lambda(\sigma, Z)$ all have period $2\pi/\omega\tau$, Eq. (4.6) follows at once.

APPENDIX E: TRANSVERSE DISPLACEMENT OF δ-FUNCTION BUNCHES

The transverse displacement of a beam of δ-function bunches is calculated from Eq. (4.21) by taking $\xi_k(0) = \xi'_k(0) = 0$ for $k < 0$. Upon changing the integration variable in Eq. (4.21) from Z to θ as indicated in Eq. (4.22), $\xi_M(\sigma)$ is given by

$$\xi_M(\sigma) = \frac{1}{2\pi}\sum_{m=0}^{M} e^{-m\omega\tau/2Q} \times \int_{-\pi+i\omega\tau/2Q}^{\pi+i\omega\tau/2Q} d\theta\, e^{-im\theta}\mathcal{F}_m[\Lambda(\sigma,\theta)] , \tag{E1}$$

where

$$\mathcal{F}_m[\Lambda(\sigma,\theta)] \equiv \xi_{M-m}(0)\mathcal{C}[\Lambda(\sigma,\theta)] + \xi'_{M-m}(0)\frac{\mathcal{S}[\Lambda(\sigma,\theta)]}{\Lambda(0,\theta)} , \tag{E2}$$

and $\Lambda(\sigma,\theta)$ is given by Eq. (4.23). The poles of $\Lambda(\sigma,\theta)$ are identical to the poles of $\tilde{W}(\theta)$ which, by causality, all lie on or below the real axis.

We now show with the aid of the closed contour of Fig. 9 that the integral in Eq. (E1) is equivalent to an integral along a line displaced infinitesimally above the uppermost poles of $\tilde{W}(\theta)$. This line is taken to be the real axis. The contour of Fig. 9 contains no poles; therefore

$$\oint d\theta\, e^{-im\theta}\mathcal{F}_m[\Lambda(\sigma,\theta)] = 0 . \tag{E3}$$

The integral along the line $[\pi, \pi+i\frac{\omega\tau}{2Q}]$ cancels the integral along the line $[-\pi+i\frac{\omega\tau}{2Q}, -\pi]$ because, as can be inferred from Eq. (4.2), the integrand is periodic in θ with period 2π. Therefore, in view of Eq. (E3), the integral along the line $[-\pi+i\frac{\omega\tau}{2Q}, \pi+i\frac{\omega\tau}{2Q}]$ is equal to the integral along the line $[-\pi, \pi]$ excluding the poles lying on this line. This is the result expressed in Eq. (4.24).

APPENDIX F: TRANSIENT BBU OF THE δ-FUNCTION BEAM

The asymptotic behavior of the transient displacement of a coasting beam comprised of δ-function bunches in the presence of a single deflecting mode is calculated here us-

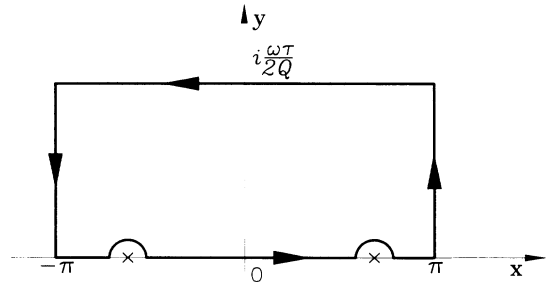

FIG. 9. Contour for calculating the displacement $\xi_M(\sigma)$ of the beam of δ-function bunches.

ing the method of steepest descent. The calculations of this appendix are restricted to the misaligned beam, i.e., solution class II. If the beam induces an "impulse excitation" of the wake field, solution class I is needed. The corresponding results are generated in the same manner and can be inferred by inspection of the results developed here. The introductory remarks of Appendix C through Eq. (C2) also apply here with respect to Eq. (4.28) and the contour of Fig. 9.

In parallel with the calculations of Appendix C, we write Eq. (4.28) in the form

$$x_M(\sigma) - x_\infty(\sigma) = -\frac{e^{-M\omega\tau/2Q}}{4\pi}\int_{-\pi}^{\pi} d\theta\, e^{F(\sigma,\theta)} , \tag{F1}$$

$$F(\sigma,\theta) \equiv -i[M\theta + \Lambda_1(\theta)\sigma] + \ln\left[\left(x_0 + \frac{ix_0'}{\Lambda_1(\theta)}\right)\frac{2e^{-\omega\tau/2Q}}{e^{i\theta} - e^{-\omega\tau/2Q}}\right] . \tag{F2}$$

Accordingly,

$$\frac{\partial F}{\partial\theta} = -iM\left(1 + \frac{e^{i\theta}}{M(e^{i\theta} - e^{-\omega\tau/2Q})}\right) + i\frac{\varepsilon\sigma\tilde{W}_1^2(\theta)\sin\theta}{\omega\tau\Lambda_1(\theta)\sin\omega\tau}\left(1 + \frac{1}{\sigma\Lambda_1(\theta)}\frac{x_0'}{x_0\Lambda_1(\theta) + ix_0'}\right) . \tag{F3}$$

To find the saddle points, we set $F'(\theta) = 0$. Defining the function δ and the parameters ψ, s_1, and s_2 according to Eqs. (4.36) and (4.37), respectively, we have

$$\delta^4 + s_1\delta^3\left(1 + \frac{e^{i\omega\tau\sqrt{\psi}}}{M(e^{i\omega\tau\sqrt{\psi}} - e^{-\omega\tau/2Q})}\right)^2 - s_2^2\left(\frac{\sin\left(\omega\tau\sqrt{\psi}\right)}{\sin\omega\tau}\right)^2 \times\left(1 + \frac{1}{\sigma\Lambda_1(\psi)}\frac{x_0'}{x_0\Lambda_1(\psi) + ix_0'}\right)^2 = 0 , \tag{F4}$$

where

$$\cos\left(\omega\tau\sqrt{\psi}\right) = \cos\omega\tau - \frac{\omega\tau}{2}\delta\sin\omega\tau , \tag{F5}$$

$$\sin\left(\omega\tau\sqrt{\psi}\right) = \sin\omega\tau\left[1 + (\omega\tau\cot\omega\tau)\delta - \left(\frac{\omega\tau}{2}\right)^2\delta^2\right]^{1/2} . \tag{F6}$$

If the terms involving M and Λ_1 are both small, then the transient displacement is found from Eq. (4.31) with $f(\theta)$ and $g(\theta)$ defined in Eqs. (4.32) and (4.34), respectively. The term involving M is small when

$$M \gg \left|\frac{e^{i\omega\tau\sqrt{\psi}}}{e^{i\omega\tau\sqrt{\psi}} - e^{-\omega\tau/2Q}}\right| . \tag{F7}$$

The term involving Λ_1 is small when

$$\sigma|\Lambda_1(\psi)| \gg 1 . \tag{F8}$$

We shall assume that these inequalities are satisfied, and then check our assumption after evaluating the saddle points.

Returning to Eq. (4.31), we now have

$$f(\sigma,\psi) = -i[M\omega\tau\sqrt{\psi} + \Lambda_1(\psi)\sigma] , \tag{F9}$$

where ψ is governed by the quartic equation, Eq. (4.35), through $\delta(\psi)$ given in Eq. (4.36), and

$$\Lambda_1(\psi) = \frac{M\omega\tau}{\sigma}\frac{s_2^2/s_1}{\delta^2}\frac{\sin\left(\omega\tau\sqrt{\psi}\right)}{\sin\omega\tau} . \tag{F10}$$

Upon evaluating $e^{f(\sigma,\psi)}$, we find exponential growth $e^{\Gamma M\omega\tau}$ in which Γ is given by Eq. (4.38) using the root of Eq. (4.35) corresponding to the largest growth rate. We also find, letting $f''(\psi) \equiv \partial^2 f(\theta_s = \omega\tau\sqrt{\psi})/\partial\theta^2$,

$$f''(\sigma,\psi) = iM\left\{\cot\left(\omega\tau\sqrt{\psi}\right) - \frac{1}{\delta}\frac{\sin\left(\omega\tau\sqrt{\psi}\right)}{\omega\tau\sin\omega\tau}\times\left[3 + \left(\frac{\sin\omega\tau}{\sin\left(\omega\tau\sqrt{\psi}\right)}\right)^2\frac{\delta^4}{s_2^2}\right]\right\} \tag{F11}$$

and

$$g(\psi) = -\frac{e^{-(M+1)(\omega\tau/2Q)}}{4\pi\left(e^{i\omega\tau\sqrt{\psi}} - e^{-\omega\tau/2Q}\right)} \times\left(x_0 + i\frac{\sigma}{M\omega\tau}\frac{s_1}{s_2^2}\frac{\sin\omega\tau}{\sin\left(\omega\tau\sqrt{\psi}\right)}\delta^2 x_0'\right) . \tag{F12}$$

The transverse displacement is calculated from Eq. (C2) using the results of Eqs. (F9), (F11), and (F12) in the manner

$$x_M(\sigma) - x_\infty(\sigma) \simeq 2\,\mathrm{Re}\left(\sqrt{\frac{2\pi}{-f''(\sigma,\psi)}}g(\psi)e^{f(\sigma,\psi)}\right) . \tag{F13}$$

The inequalities of Eqs. (F7) and (F8) must be checked before applying these results. We shall consider each domain of Sec. IV C separately. For $\psi \simeq 1$ (domains A and B), the inequality of Eq. (F7) reduces to

$$M \gg e^{\omega\tau/2Q}\sqrt{p^2 + q^2} , \quad \text{domains } A \text{ and } B \tag{F14}$$

where p and q are the resonance functions defined in Eqs. (4.14) and (4.15), respectively. For $\cos\left(\omega\tau\sqrt{\psi}\right) \simeq \cos\omega\tau$ and $\sin\left(\omega\tau\sqrt{\psi}\right) \simeq \omega\tau\sqrt{\psi} \ll 1$ (domains C and D), the inequality of Eq. (F7) reduces to

$$M \gg \left[\left(\cos\omega\tau - e^{-\omega\tau/2Q} - \frac{\sqrt{3}}{2}\frac{E_C}{M}|\cos\omega\tau|\right)^2 + \left(\frac{1}{2}\frac{E_C}{M}\cos\omega\tau\right)^2\right]^{-1/2} , \quad \text{domain } C , \tag{F15}$$

$$M \gg \left|\cos\omega\tau - e^{-\omega\tau/2Q} - \frac{E_D}{M}|\cos\omega\tau|\right|^{-1}, \quad \text{domain } D\,. \tag{F16}$$

Because generally $\omega\tau/2Q \ll 1$, these conditions are hardest to satisfy in the vicinity of the zero crossings of the wake function where $\omega\tau = 2n\pi$ and $\cos\omega\tau \simeq 1$, but they are comparatively easy to satisfy in the vicinity of the zero crossings where $\omega\tau = (2n+1)\pi$ and $\cos\omega\tau \simeq -1$. For $|e^{i\omega\tau\sqrt{\psi}}| \gg 1$ (domains E and F), the inequality reduces to

$$M \gg 1\,, \quad \text{domains } E \text{ and } F. \tag{F17}$$

This condition generally holds when the transient BBU is pronounced, as is also required for the validity of the method of steepest descent. Upon substituting the result of Eq. (F10) into Eq. (F8), we find the inequality of Eq. (F8) is valid provided the following inequalities hold in their respective domains:

$$\begin{aligned} &E_A \gg 1\,, \quad \text{domain } A \\ &\kappa\sigma \gg 1\,, \quad \text{domains } B,\ C, \text{ and } E \\ &E_D \gg 1\,, \quad \text{domain } D \\ &M \gg 1\,, \quad \text{domain } F. \end{aligned} \tag{F18}$$

The inequalities associated with domains A, D, and F are generally true when the transient BBU is pronounced, as is also required for the validity of the method of steepest descent. The inequality associated with domains B, C, and E is consistent with the assumption of strong focusing which characterizes these domains. These considerations justify using the inequality of Eq. (F8).

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