Low-pressure relativistic electron flow

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Most theories of electron flow have involved laminar flows or special flows of a single-orbital type. A general theory of electron flow has been published [C. W. Mendel, Jr., D. B. Seidel, and S. A. Slutz, Phys. Fluids **26**, 3628 (1983)]. This theory can be used to make some general statements about these flows, but solutions to particular problems are complex and difficult. By constraining the general theory to flows where the electron pressure normal to the flow direction is small compared to the electromagnetic pressure, finite electron orbits can be taken into account in a general manner. An example of the utility of the theory under this assumption is given.

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I. INTRODUCTION

There have been many theories of magnetically insulated electron flows. Most make assumptions about the electron orbits to make the problem tractable. These theories have been valuable for comparison to experiment when line characteristics (i.e., line currents and voltages) are considered. The primary models of this type are the Brillouin theories [1–3], quasilaminar theories [4–6], and laminar flows with constant ratio of plasma frequency to gyrofrequency [7]. It is easily shown that the voltage-current characteristics of these flows must be the same within the accuracy of most experiments, because they depend largely upon the pressure balance across the flow [8]. For this reason these models are satisfactory for comparison to experiment when voltage and current characteristics are of interest.

A more general theory of these flows has been developed [9–12] which depends upon a description in terms of the density of electrons in (P,W) space, where P is the canonical momentum along the flow and W is the total energy of the electrons. In this theory the finite (i.e., nonzero) size of the electron orbits in the direction of the electric field (normal to the electron-flow direction) is taken into account. This theory also applies to three-dimensional flows if another canonical momentum component is added [10]. This description of magnetically insulated flows is a natural one for situations where the canonical momentum P and total energy W of the electrons change only slightly over a gyration, i.e., over a gyroperiod or gyrolength. When changes in space or time are slow, electrons move slowly in (P,W) space. In addition, there is an action variable that is adiabatically invariant and the electron dynamics can be described in terms of that variable. This assumption of gradual change applies to most experimental situations.

Ideally, this theory allows problems to be solved where the nonzero height of the electron orbits is important. For example, finite orbit height is important in wave dispersion. Because of the finite height of the orbits, a perturbation of an electron at one point in the electric-field direction causes the charge density to be perturbed over the entire extent of that electron's gyration. This results in dispersion relationships which involve integral equations [11,13]. With few exceptions, studies of oscillations in these flows have been done with laminar flows [14–20]. There has been some work done for finite orbits in special situations [13].

The difficulty with the general theory is that it is intractable for many problems. It has been used to write general integral equations for eigenmodes and to prove some general stability criteria in terms of the distribution of electrons in (P,W) space [11], and there are other generalizations that can be proved. Among the latter are statements about expected distributions and integral expressions for line parameters such as power-transmission efficiency. However, when a quantitative solution is desirable, problems arise. One reason for this is that the electron distribution is specified in terms of P and W, and it is possible, and indeed likely, that one will specify electrons that never show up in the solution because they have been put in a region of (P,W) space where they cannot possibly participate in the flow.

By specifying the distribution in terms of the aforementioned adiabatic parameter and another related parameter [11], this problem is overcome, but the mathematics is still quite complicated. There is, however, a simplifying assumption, namely, that the electron pressure is small, which is justified in many and perhaps most situations. The pressure we refer to herein is the yy component of the kinetic pressure tensor, where y is the dimension in the electric-field direction. This assumption will allow much simpler calculations and is the basis of the following theory. Nonzero electron pressure implies finite electron orbits and vice versa. However, since electron pressure is proportional to the square of orbital height, a low-pressure theory is a higher-order approximation than small-orbital-height theory. The effect of electron pressure on field profiles is shown as an example.

In Sec. II A we briefly review the general theory and discuss the two parameters mentioned above and their effectiveness for describing the distribution of electrons in (P,W) space. A number of fundamental properties of magnetically insulated flows are discussed in Sec. II B, and a simplifying field transformation is given in Sec. II C. Corresponding results from electromagnetic, relativistic particle-in-cell simulations suggest the validity of assuming that the electron pressure is small when compared to the electromagnetic pressure. The derivation of the low-pressure electron-flow theory is given in Sec. III, and Sec. III B provides an example.

II. RELATIONSHIP BETWEEN LOW-PRESSURE FLOW AND PREVIOUS THEORY

A. Review of general theory

In Refs. [9] and [10] the integrals for the charge and current densities in a magnetically insulated flow were derived. In the following the momentum P will be normalized to mc and the total energy W will be normalized to mc^2. Defining $E=-eE_y/mc^2$ (Fig. 1), $B=-eB_z/mc$, $A=eA_x/mc$, $\phi=eV/mc^2$, and $J(P,W)$ as the distribution function [9,10], then

$$E'=\int\int dP\,dW(1+\phi+W)Q^{-1/2}J(P,W),\quad \phi'=E\ ,$$
$$B'=-\int\int dP\,dW(A+P)Q^{-1/2}J(P,W),\quad A'=B\ , \tag{1}$$

where

$$Q=Q(y,P,W)=[1+\phi(y)+W]^2-1-[A(y)+P]^2\ . \tag{2}$$

The spatial variable y is not scaled; times are multiplied by the speed of light, and so they also have dimensions of meters. The primes in Eqs. (1) and in what follows will always refer to derivatives with respect to y.

The relevance of Q is that an electron at y with total energy W and canonical momentum P will have $|\gamma u_y|=Q^{1/2}$ if Q is non-negative. u_y is the ratio of the y component of velocity to the speed of light, and $\gamma=1/(1-u^2)^{1/2}$. Equation (2) is easily derived from the definitions of the canonical momentum $P=\gamma u_x-A$ and total energy $W=\gamma-1-\phi$, plus the relationship $(\gamma u)^2=(\gamma u_x)^2+(\gamma u_y)^2$. An electron with P,W cannot reach any y where $Q(y,P,W)$ is negative. Otherwise, $(\gamma u_x)^2$ would be larger than $(\gamma u)^2$.

We will write Maxwell's equations for one-dimensional time-independent problems, but the integral source terms in Eqs. (1) can be used in the full equations [10]. Occasionally, variations in time or along the flow will be mentioned. However, all field variations in time and in the x direction (Fig. 1) must be gradual. There is no restriction on variations in y.

The manipulations of Q is central to the low-pressure flow approximation. It is thus important to understand how it behaves. Figure 2 shows four Q functions from different regions of (P,W) space. Figure 2(a) is the typical situation. An electron with these P and W would oscillate with a sinusoidlike motion between the values y_- and y_+, which are the zeros of $Q(y,P,W)$ in y.

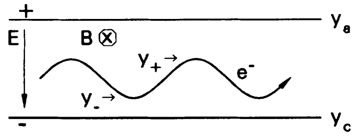

FIG. 1. Schematic of a magnetically insulated electron flow. A single orbit is shown. The minimum and maximum extents across the flow are given by y_- and y_+, respectively. The magnetic field is into the plane of the figure.

For the Q function shown in Fig. 2(b), the electrons at y_L are in stable straight-line orbits. Because Q is quadratic in y even when the maximum is at $Q=0$, electrons in these stable straight-line orbits have well-defined gyrolengths and gyrofrequencies. If perturbed, these electrons will have a sinusoidal orbit in the x,y plane. The $Q(y,P,W)$ shown in Fig. 2(c) has three zeros, but an electron can only stay in the system if it is between y_- and y_+, where Q is greater than zero. When the minimum in Q occurs at $Q=0$, y_- and y_* coincide [Fig. 2(d)]. In this situation the straight-line orbits at $y=y_L$ are unstable. If perturbed, these electrons move away from their original position exponentially.

Generally, flows will have only two zeros of Q, although there may be a small region of (P,W) space where there are three [10], as shown in Fig. 2(c). This occurs when the density is sufficiently large at some part of the line. In these situations the integrals in Eqs. (1) are between y_- and y_+ of Fig. 2(c).

We will write integrals such as those in Eqs. (1) without stating the limits of the integral, but always assuming that they are done over the correct portion of (P,W) space to include the distribution correctly, i.e., for $Q(y,P,W)>0$ between y_- and y_+.

The integrals in Eqs. (1) can be rewritten by integrating by parts:

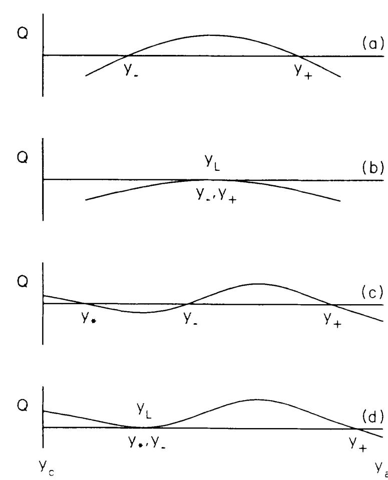

FIG. 2. Function of $Q(y,P,W)$ vs y for (a) a typical point in (P,W) space, (b) a point on the stable laminar line, (c) a point near the cathode line where there are three zero crossings, and (d) a point in the unstable laminar line. Above the $Q=0$ axis, $(\gamma u_y)^2=Q$. Electrons with P,W cannot reach any y where $Q(y,P,W)<0$. Any electron which has P,W that gives $Q(y,P,W)$ such as the case in (b) is in a straight-line orbit at $y=y_-=y_+$. It cannot be at any other value of y since Q would be less than zero. An electron with $Q(y,P,W)$ such as that in case (d) could be in a straight-line orbit at $y=y_-=y_+$, but since its y velocity would increase with distance from that position, the orbit would be unstable.

$$E' = -\int\int dP\, dW\, Q^{1/2}\frac{\partial J}{\partial W}, \quad \phi' = E \ , \tag{3a}$$

$$B' = +\int\int dP\, dW\, Q^{1/2}\frac{\partial J}{\partial P}, \quad A' = B \ . \tag{3b}$$

The boundary terms in these integrations are zero because either $Q=0$ or $J=0$ at the boundaries. These integrals are convenient because the only place where y appears in the integrand is in the Q function.

Multiplying Eq. (3a) by E and Eq. (3b) by B, subtracting, and integrating yields the pressure integral [9]

$$\frac{B^2-E^2}{2} + G = \frac{B_c^2}{2} + G_c \ , \qquad G(y) = \int\int dP\, dW\, J Q^{1/2} \ . \tag{4}$$

$G(y)$ is the yy component of the electron-pressure tensor. The subscript c refers to values at the cathode, and the cathode is assumed to be a space-charge-limited electron emitter so that $E_c=0$.

Figure 3 shows the region of (P,W) space which contains all electrons in the flow. The boundaries of this space consist of the anode line A, given by $Q(y_a,P,W)=0$, the cathode line C, given by $Q(y_c,P,W)=0$, and the laminar line L, given by $Q(y,P,W)=0$ and $dQ(y,P,W)/dy=0$ for all y between y_c and y_a. Electrons above the cathode line will hit the cathode in less than one gyration, and those above the anode line will hit the anode in less than one gyration. All electrons on the laminar line are in straight-line orbits, and the set of all of these electrons form a laminar flow. Using Eq. (2) for Q, then, from $Q=Q'=0$,

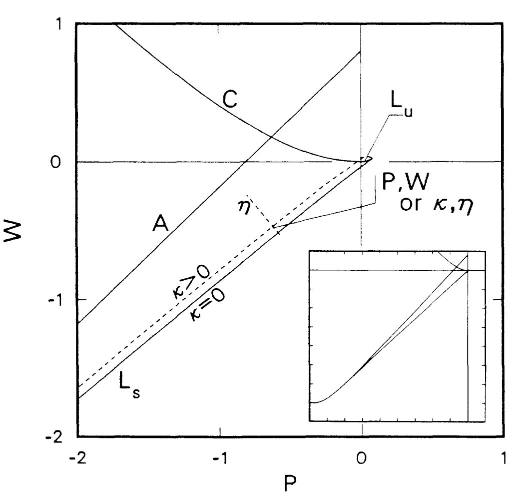

FIG. 3. Plot of (P,W) space typical of a magnetically insulated transmission line. The inset shows the entire allowed region. The cathode line (C), the anode line (A), and the laminar line (L) are shown. Also shown are two constant-κ lines, $\kappa=0$ and $\kappa=1/B_c$, and a constant-η line. Note the small section of unstable laminar line extending from the origin into the first quadrant. Positions in P,W space can also be specified in terms of $\kappa=H(P,W)$ and an orthogonal variable η.

$$P_L(y) = -A(y) + \frac{E(y)}{[B^2(y)-E^2(y)]^{1/2}} \ , \qquad W_L(y) = -1-\phi(y) + \frac{B(y)}{[B^2(y)-E^2(y)]^{1/2}} \ . \tag{5}$$

In Eqs. (5) and in what follows, an L subscript refers to values on the laminar line.

The laminar line L commonly has two parts, the stable part L_s and the unstable part L_u. These two parts can be seen in Fig. 3. On L_u the orbitals of the individual electrons are unstable because $\partial^2 Q/\partial y^2$ is positive [Fig. 2(d)].

There can be difficulty with specifying the distribution function J as a function of P,W as has already been mentioned. It is possible to specify electrons in a region of (P,W) space that is outside of the allowed region shown in Fig. 3. If this occurs, these electrons do not contribute to the integrals in Eqs. (1) or (3) because $Q<0$ for all y. It is better to specify the distribution using the function $H(P,W)$, given by

$$H(P,W) = \int_{y_-(P,W)}^{y_+(P,W)} dy\, Q^{1/2}(y,P,W) \ . \tag{6}$$

The limits of the integral are specified by $Q(y_+,P,W)=Q(y_-,P,W)=0$, and $Q(y,P,W)>0$ for $y_-<y<y_+$. H is also a function of x and t if the potentials are a function of x and t. $H(P,W)$ is an action integral of the electron motion.

In this theory the electrons have been replaced by a line of charge extending for the full gyroheight in the y direction. The charge density along this line varies with the same density distribution as the average of the gyrating electrons. $H(P,W,x,t)$ is the Hamiltonian of these lines of charge [10]. It can be shown [10] that H is adiabatically conserved along an electron path if spatial changes along the flow take place over many gyrolengths and temporal changes take place over many gyroperiods. Under the same restriction, the path of a line of charge is constrained by an equation of motion

$$H(P,W,x,t) = \kappa \ ,$$

where κ is a constant of motion. Henceforth we will refer to these lines of charge as electrons, but the theory is always within the above approximation. H is actually the Hamiltonian of the lines of charge, not the electrons.

The parameter κ has several interesting features [10,11]. An electron on L_s has $\kappa=0$. Figure 3 shows a $\kappa>0$ contour in the P,W plane. It is desirable to specify an electron by its value of κ and an orthogonal variable η. From the definition of κ,

$$d\kappa = \frac{\partial H}{\partial P}dP + \frac{\partial H}{\partial W}dW \ . \tag{7}$$

Therefore η can be specified by

$$\Theta(\eta)d\eta = -\frac{\partial H}{\partial W}dP + \frac{\partial H}{\partial P}dW \ . \tag{8}$$

The function $\Theta(\eta)$ is positive (or negative) definite. A convenient $\Theta(\eta)$ can be determined by requiring that $\eta=y$ when $\kappa=0$. This will be the $\Theta(\eta)$that we will use, and an expression for $\Theta(\eta)$ will be derived later. An electron on the laminar line has a straight-line orbit with $y=\eta$. The Jacobian of the transformation from P, W to κ, η then gives

$$dP\,dW = h^{-2}\Theta(\eta)d\kappa\,d\eta \ , \tag{9}$$

where

$$h^2 = \left[\frac{\partial H}{\partial P}\right]^2 + \left[\frac{\partial H}{\partial W}\right]^2 .$$

We will need the partial derivatives of P and W with respect to κ and η. From Eqs. (7)–(9) it is easily found that

$$\begin{aligned} h^2 dW &= \frac{\partial H}{\partial W}d\kappa + \frac{\partial H}{\partial P}\Theta(\eta)d\eta \ , \\ h^2 dP &= \frac{\partial H}{\partial P}d\kappa - \frac{\partial H}{\partial W}\Theta(\eta)d\eta \ . \end{aligned} \tag{10}$$

Therefore

$$\begin{aligned} &\frac{\partial W}{\partial \kappa} = h^{-2}\frac{\partial H}{\partial W}, \quad \frac{\partial P}{\partial \kappa} = h^{-2}\frac{\partial H}{\partial P} \ , \\ &\frac{\partial W}{\partial \eta} = h^{-2}\Theta(\eta)\frac{\partial H}{\partial P}, \quad \frac{\partial P}{\partial \eta} = -h^{-2}\Theta(\eta)\frac{\partial H}{\partial W} \ , \end{aligned} \tag{11}$$

Another function which will prove very useful is $\Gamma(\eta)$, defined by

$$\begin{aligned} \Gamma(\eta) &= -\frac{1}{2}\left.\frac{\partial^2 Q(y, P_L(\eta), W_L(\eta))}{\partial y^2}\right|_{y=\eta} \\ &= B^2(\eta) - E^2(\eta) - \frac{E'(\eta)B(\eta) - B'(\eta)E(\eta)}{[B^2(\eta)-E^2(\eta)]^{1/2}} \ , \end{aligned} \tag{12}$$

where we have used Eqs. (2) and (5). On the stable laminar line L_s, $\partial^2 Q/\partial y^2 < 0$ at $y=\eta$, and so $\Gamma > 0$. In contrast, on the unstable laminar line L_u, $\Gamma < 0$.

The relationship between the shape of $Q(y)$ and the stability of straight-line orbits can be understood by looking at Fig. 2. Figure 2(b) corresponds to an electron on L_s and at $y=y_- = y_+$. If that electron is instantly perturbed (i.e., it is given a change in velocity without a change in position), the parabola moves upward a bit to look more like Fig. 2(a). The electron is then moving in a small sinusoid, with y_- and y_+ close together on the y axis. In a situation such as Fig. 2(d), an electron on L_u is at $y=y_* = y_-$. If that electron is instantly perturbed, the Q curve also moves upward a bit, but it is still quadratic upward, so that the electron moves more and more rapidly as it moves away from its original position.

Note [Eq. (12)] that $\Gamma_c = B_c^2 - E_c'$ when $E_c = 0$. At the anode and in any vacuum region near the anode, $\Gamma = B_c^2 + 2G_c$. Typically, G_c is negligible, so that $\Gamma_a \simeq B_c^2$.

As an example of the utility of Γ, the laminar lines given by Eqs. (5) can be differentiated with respect to η to get

$$\begin{aligned} \frac{dP_L(\eta)}{d\eta} &= -\frac{B(\eta)\Gamma(\eta)}{B^2(\eta)-E^2(\eta)} \ , \\ \frac{dW_L(\eta)}{d\eta} &= -\frac{E(\eta)\Gamma(\eta)}{B^2(\eta)-E^2(\eta)} \ . \end{aligned} \tag{13}$$

Clearly, from Eq. (12), Γ can be negative if the electron density is high enough. In such regions the laminar orbits will be unstable ($Q=0$, $Q'=0$ will occur at a minimum in Q) and the laminar line will move up and to the right in (P, W) space as η increases [Eqs. (13)]. Although there may be regions in a flow where $\Gamma < 0$ (usually in a narrow band near the cathode), in what follows expressions involving $\Gamma^{1/2}$ will be used only in regions where $\Gamma \geq 0$.

By looking at the motion of electrons so near the cathode that $B \simeq B_c$ and $E' \simeq E_c' = B_c^2 - \Gamma_c$, it is found that electrons with zero total energy, corresponding to electrons at rest on the cathode, cannot leave the cathode [21] unless $\Gamma \leq 0$. This occurs because (in our normalized units) $u^2/2 = E_c' y^2/2$ (from energy conservation) and $u_x = B_c y$ (from canonical momentum conservation), so that if E_c' were not greater than B_c^2, u_x^2 would be greater than u^2 in a region near the cathode. From this and Eq. (12) we get Table I. Ordinarily, it is expected that electrons will be emitted from the cathode with $P, W=0$; i.e., they will enter (P, W) space at the origin. On the other hand, it can be argued [21] that electrons are near the laminar line, and since there are electrons at the origin, the stable laminar line must go through or near the origin. It is thus expected that $\Gamma \simeq 0$, $E' \simeq B_c^2$ in some region around the cathode. In mks units this electron charge density near the cathode is $e\epsilon_0 B_{zc}^2/m$, which corresponds to $10^{13} e^-/\mathrm{cm}^3$ at 1 T.

B. Connecting general flow to low-pressure flow

The general theory discussed in Ref. [10] covers most situations of interest, but is unwieldy when applied to most problems where a numerical solution is desired. In this paper it is shown that the theory can be simplified greatly by including only flows where the electrons are close to the $\kappa=0$ line. Restricting the theory to such flows needs to be justified.

If the charge density near the cathode is given approximately by the value above (i.e., $E_c' = B_c^2$), the unstable laminar line will be short, and since all electrons leave the cathode with $P=W=0$, they leave the cathode with small κ. Then, since κ should be conserved, all electrons should have small κ.

Time-dependent two-dimensional particle-in-cell simulations have been valuable tools for studying these flows. Figure 4 shows a simulation of a feed transition such as those found at the input to a magnetically insulated transmission line [22]. The simulation is essentially in an equilibrium at the time of Fig. 4. The cell size is 0.2 cm in the flow direction. The cell size is variable in the y

TABLE I. Determination of orbital stability and ability of electrons to leave cathode relative to E_c'/B_c^2 and Γ_c.

E_c'/B_c^2	Γ_c	Unstable orbits?	$P, W=0$ electrons can leave cathode?
>1	<0	yes	yes
$=1$	$=0$	no	yes
<1	>0	no	no

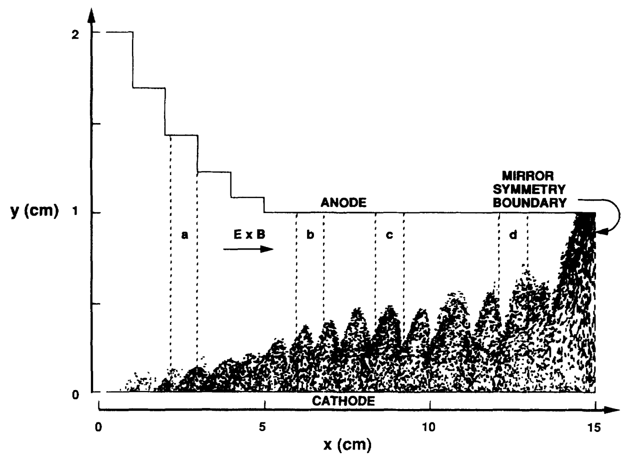

FIG. 4. Simulation line geometry with particles shown at a time when the flow extends over the entire simulation. The transmission line is 15 cm long and 1 cm high in the uniform section. The emission algorithm places "electrons" (actually charges of the order of 10^{10} electrons with the same mass-to-charge ratio as an electron) with $P=W=0$ at random locations in the cells next to the cathode.

direction, starting at 0.033 cm near the cathode and having 20 cells across the 1-cm gap between the cathode and anode. Four diagnostic locations along the line are denoted by the letters $a-d$. The wave structure seen is probably not an accurate portrayal of a real flow. Changes in the cathode emission algorithm that are unimportant within the accuracy of the code (due to cell-size limits) can eliminate this feature.

The function $\Gamma(y)$ for locations $a-d$ in Fig. 4 are shown in Fig. 5. In all cases $\Gamma(y)=B_c^2$ in a region near the anode, as expected. The finite cell size at the cathode forces Γ to go to a large negative value in the first cell near the cathode. This occurs because the finite cell size (and other numerical effects) leads to some spread in P, W for the electrons near the emission point of $P=0$, $W=0$. Since electrons can never be outside of the allowed region

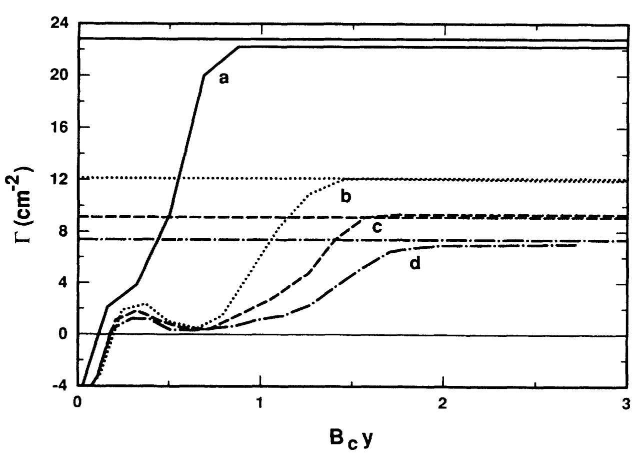

FIG. 5. Plots of $\Gamma(y)$ for the four regions shown in Fig. 4. The horizontal lines are the local values of B_c^2, and Γ approaches these lines near the anode, as expected.

on the laminar line side of that region, the laminar line must pass at least a small distance from the origin, as shown in Fig. 3. This requires at least a small unstable laminar line and, therefore, a region of negative Γ near the cathode (Table I).

The electrons for each location of Fig. 4 are shown in (P, W) space in Fig. 6. The cathode line is the same for all locations since the vector and scalar potentials are zero at the cathode. The scalar potential at the anode is also the same for all locations, but the vector potential at the anode decreases from location a to d because of decreased line height and increased electron flow. As a result, the anode line moves toward positive P, going from location a to d. The laminar line therefore must also change.

There are several interesting features in Fig. 6. The important feature for the theory to be presented here is that the electrons are always near the laminar line, which is to say that $\kappa=H(P,W,x,t)$ is small for the P, W values of all electrons. This is true despite the large structure of the flow seen in Fig. 4. The electrons at location d, which appear to be to the right of the laminar line, are an artifact due to the finite width of the diagnostic band from which these electrons are taken (five cells). Location d is close to the load, and fields vary more rapidly in the flow direction there. Another interesting feature is the presence of electrons above the cathode line. These will all be captured by the cathode in less than one gyration unless their P and W change in less than a gyration to get them below the cathode line.

In a uniform (constant gap) section of the line, the electron cloud as a unit must conserve energy and canonical momentum, except for energy and momentum that is transferred to the cathode by electron loss. This can be seen in Fig. 6(c), where the electrons above the cathode line are in the $P>0$, $W>0$ quadrant. As electrons spread along the laminar line, the cathode is heated and "feels" a force in the forward direction. The electron cloud is cooled and "feels" a drag in the backward direction. There is thus an evaporation of electrons to the cathode and a viscous drag on the electron cloud, which then requires further electron emission.

The extent of the electron spread along the laminar line in the simulations (i.e., the spread in η) is dependent upon the cell size near the cathode and the particular emission algorithm employed. This spread is determined by numerical effects. Experiments indicate that the spread in energy is less than 1 keV in some cases [23], whereas the spread is of the order of 100 keV in the simulations. Emission algorithms that add thermal spread in κ and η tend to reduce the numerically induced spread along the laminar line and reduce the wave structure seen in the envelope of the electron cloud (Fig. 4). Nevertheless, the simulation supports the value of a low-pressure-flow assumption since numerical effects should not reduce the spread in κ.

Figure 7 is a plot of the simulation electron density normalized to the critical value for cathode emission (Table I). The high density near the cathode is necessary because of cathode emission. Since the finite cell size is observed to cause excessive spreading of the electron cloud along the laminar line, it must also cause more electron recapture by the cathode and more subsequent reemission. For this reason the thickness of the layer of high density near the cathode in the simulations, which is the same as the layer of $\Gamma<0$ discussed earlier, depends upon the cell size. Outside this thin layer, the density rises with distance from the cathode, except in the case of the thin sheath in the very wide portion of the transmission line (location a). Nevertheless, Fig. 7 indicates that the density profile can be approximated by a uniform electron density of the proper width and equal to the critical value.

The tendency seen in simulations for electrons to be near the laminar line should also occur in real flows where conservation of κ is expected. We are of the opinion that electrons in experimental flows are near the laminar line just as they are in the above simulations.

C. Field transformation

A simple transformation from B,E simplifies the theory and makes the relationship with zero pressure more apparent. Defining two functions $g(y)$ and $f(y)$ by

$$B=g(y)\cosh f(y), \quad E=g(y)\sinh f(y), \quad g^2=B^2-E^2 ,$$

the pressure relationship [Eqs. (4)] and the differential Eqs. (3) become

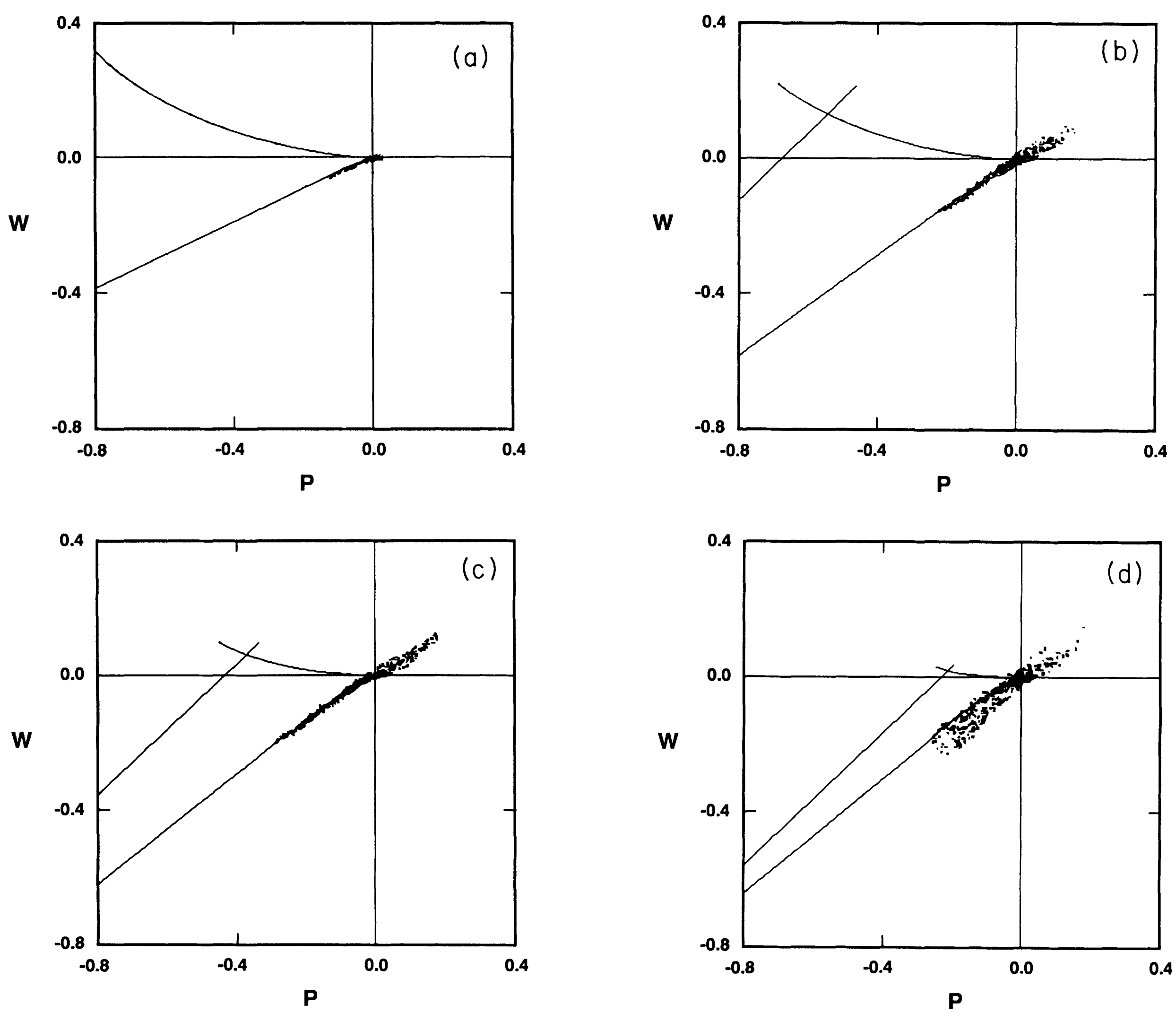

FIG. 6. P, W plots for the same simulation at the same four locations as in Figs. 4 and 5. The tendency for particles to lie along the laminar line is apparent. Particles above the cathode line will hit the cathode in less than one gyroperiod. Electrons to the right of the laminar line are an artifact of the finite size of the (1 cm wide) band containing the particles. This is noticeable primarily in (d), which is near the load (see Fig. 4) where the fields are varying appreciably over the diagnostic band.

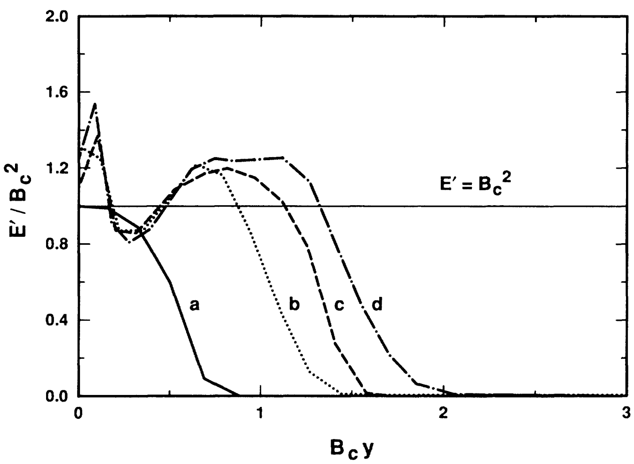

FIG. 7. Plots of the electron density E' normalized to the local critical value for cathode emission of B_c^2. In mks units this is $\rho/(\epsilon_0 e B_{zc}^2/m)$. The high value near the cathode must occur so that electrons can leave the cathode. This layer is comparable to a cell size in thickness. The density appears to be well approximated by a layer of the critical density of the proper thickness.

$$\begin{aligned}&\tfrac{1}{2}g^2+G=\tfrac{1}{2}B_c^2+G_c \ ,\\ &G(y)=\int\int dP\,dW\,JQ^{1/2} \ ,\\ &g(y)f'(y)=-\int\int dP\,dW\left[\cosh f\frac{\partial J}{\partial W}+\sinh f\frac{\partial J}{\partial P}\right]\\ &\qquad\times Q^{1/2}(y,\kappa,\eta) \ .\end{aligned} \tag{14}$$

In the limit of zero pressure ($G=0$), $g=B_c$ and

$$B=B_c\cosh f, \quad E=B_c\sinh f \ .$$

This is a general form for all planar laminar flows, since by differentiating both of these

$$\frac{B'}{E'}=\frac{E}{B}=\tanh f \ ,$$

which is to say that the electron drift velocity is equal to E/B at all points in the flow.

We again transform from P, W to κ,η, and so Eqs. (14) above become

$$\begin{aligned}&\tfrac{1}{2}g^2+G=\tfrac{1}{2}B_c^2+G_c \ ,\\ &G(y)=\int\int d\kappa\,d\eta\,h^{-2}\Theta JQ^{1/2}(y,\kappa,\eta) \ ,\\ &g(y)f'(y)=-\int\int d\kappa\,d\eta\,h^{-2}\Theta\left[\cosh f(y)\frac{\partial J}{\partial W}\right.\\ &\qquad\left.+\sinh f(y)\frac{\partial J}{\partial P}\right]Q^{1/2} \ .\end{aligned} \tag{15}$$

In expressions such as these, functions in the integrand will depend upon η unless dependence upon y is shown explicitly.

From Eqs. (5),

$$\begin{aligned}&P_L(\eta)=-A(\eta)+\sinh f(\eta) \ ,\\ &W_L(\eta)=-1-\phi(\eta)+\cosh f(\eta) \ ,\end{aligned} \tag{16}$$

for any flow. The function Γ [Eq. (12)] is easily found to be

$$\Gamma(\eta)=g(\eta)[g(\eta)-f'(\eta)] \ . \tag{17}$$

III. LOW-PRESSURE ELECTRON FLOW

There are several simplifications that can be made if it is assumed that the electrons have low values of κ, i.e., if they are near to the laminar line (Fig. 3). Since the electron pressure G is zero when the electrons are all on the laminar line, one approximation might be to assume $g=B_c$, and from Eqs. (15) above then

$$B_cf'(y)=-\int\int d\kappa\,d\eta\,h^{-2}\Theta\left[\cosh f(y)\frac{\partial J}{\partial W}+\sinh f(y)\frac{\partial J}{\partial P}\right]Q^{1/2} \ .$$

In this approximation the assumption is that electron pressure is negligible. The situation is comparable to the quasineutral model for plasmas where the electron and ion charge densities are equal, and yet there may be electric-field divergence. The example in Sec. III B will show that this is a satisfactory approximation in some cases. Under this approximation, then, we have a finite-orbit theory with zero electron pressure. The example in Sec. III B will indicate that the electron pressure is quite small, even for flows with y_+-y_- comparable to the thickness of the electron sheath.

Small pressure is related to small orbital height. This can be seen in the following way. The κ value of an electron is approximately given by [Eq. (6)]

$$\begin{aligned}\kappa=H&\sim(y_+-y_-)\frac{\gamma m}{mc}|v_y|\\ &\sim(y_+-y_-)\frac{\gamma m}{mc}\frac{eB_z}{\gamma m}(y_+-y_-)=\frac{eB_z}{mc}(y_+-y_-)^2 \ ,\end{aligned}$$

where $eB_z/\gamma m$ is the gyrofrequency. The average electron pressure $\langle G\rangle$ is

$$\begin{aligned}(y_a-y_c)\langle G\rangle&=\int_{y_c}^{y_a}G\,dy\\ &=\int\int dP\,dW\,JH\\ &\sim\int\int dP\,dW\,J\frac{eB_z}{mc}(y_+-y_-)^2 \ ,\end{aligned}$$

from Eqs. (4) and (6). Therefore, for a given electron-density profile, the importance of electron pressure is determined by orbital size. In the example in Sec. III B, it will be seen that orbital size can get fairly large before the low-pressure approximation is violated.

We now look at two related approximations that will allow us to calculate the function Θ from E, B, and Γ, and allow us to reduce the number of integrals in the f'

expression for many interesting distributions in (P,W) space. Since electrons are assumed to be close to the stable laminar line L_s, we will approximate

$$\left.\frac{\partial H}{\partial P}\right|_{\kappa,\eta} \simeq \left.\frac{\partial H}{\partial P}\right|_{0,\eta}, \text{ etc.};$$

i.e., we will use the derivatives on L_s at the same η. Using this approximation, Eqs. (15) become

$$\tfrac{1}{2}g^2+G=\tfrac{1}{2}B_c^2+G_c \ , \quad (18a)$$

$$G=\int d\eta\, h_L^{-2}\Theta \int d\kappa\, JQ^{1/2} \ , \quad (18b)$$

$$\begin{aligned} g(y)f'(y) &= -\int d\eta\, h_L^{-2}\Theta \int d\kappa \left[\cosh f(y)\frac{\partial J}{\partial W} + \sinh f(y)\frac{\partial J}{\partial P}\right] Q^{1/2} \\ &= -\int d\eta\, h_L^{-2}\Theta \left[\cosh f(y)\int d\kappa \frac{\partial J}{\partial W}Q^{1/2} + \sinh f(y)\int d\kappa \frac{\partial J}{\partial P}Q^{1/2}\right] . \end{aligned} \quad (18c)$$

We need to get $Q(y,\kappa,\eta)$, $h_L(\eta)$, and $\Theta(\eta)$ to replace the dependence upon P,W. This will be accomplished by expanding Q in a Taylor series and keeping the lowest-order terms. This is our second approximation. The result will be a parabolic function of y. The $Q(y,P,W)$ examples in Fig. 2 would be well approximated (when $Q>0$) by a parabola, except when near the unstable laminar line. On the unstable laminar line, one solution of $Q=0$ occurs at $y=y_*=y_-$ and the parabolic approximation is not valid [Fig. 2(d)]. It will be seen that the only time we need the parabolic approximation is on the stable laminar line, and so there will not be a problem.

A. Transformation from P,W to κ,η at low pressure

The spatial variable y only occurs in the source integrals [Eqs. (3)] in the function Q. Since small κ implies small y_+-y_-, we expand $Q(y,\kappa,\eta)$ in $\Delta y=y-\eta$ and κ, i.e.,

$$\begin{aligned} Q(y,\kappa,\eta) &= Q(\eta,0,\eta)+Q'(\eta,0,\eta)\Delta y + \tfrac{1}{2}Q''(\eta,0,\eta)\Delta y^2+\cdots \\ &\quad + \left.\frac{\partial Q}{\partial\kappa}\right|_{\eta,0,\eta}\kappa + \frac{1}{2}\left.\frac{\partial^2 Q}{\partial\kappa^2}\right|_{\eta,0,\eta}\kappa^2+\cdots \\ &\quad + \left.\frac{\partial Q'}{\partial\kappa}\right|_{\eta,0,\eta}\kappa\,\Delta y+\cdots . \end{aligned}$$

But $Q(\eta,0,\eta)=Q'(y,0,\eta)|_{y=\eta}=0$, and so the first two terms are both zero and can be dropped. Since κ is of order Δy^2, the $\kappa\cdot\Delta y$ term is of order Δy^3. We therefore keep the κ and Δy^2 terms and drop all others. Recalling Eq. (12), Q becomes

$$Q(y,\kappa,\eta)=\kappa\left.\frac{\partial Q}{\partial\kappa}\right|_{\eta,0,\eta} - \Gamma(\eta)\Delta y^2 \ , \quad (19)$$

where $(\partial Q/\partial\kappa)|_{\eta,0,\eta}$ is determined by putting Q into the equation for H [Eq. (6)] and setting $H=\kappa$. This yields

$$Q(y,\kappa,\eta)=\frac{2\Gamma^{1/2}(\eta)\kappa}{\pi} - \Gamma(\eta)\Delta y^2 \ . \quad (20)$$

This expansion could have instead been done by expanding ϕ and A in Δy and keeping terms up through Δy^2.

The reader will not be surprised that the derivatives of H with respect to P and W can now be found in the limit of $\kappa=0$. The calculations are somewhat tedious, and for that reason they have been put in the Appendix. Once these derivatives are known, $h_L(\eta)$, $\Theta(\eta)$, and the Jacobian on the laminar line are known, as are the relationships between derivatives of J with respect to P,W to those with respect to κ,η. Finally, in the Appendix, everything is put into Eqs. (18) to get the field equations. The result of this is

$$\left.\frac{\partial H}{\partial W}\right|_L = \frac{\pi B(\eta)}{g(\eta)\Gamma^{1/2}(\eta)} = \frac{\pi B_c \cosh f(\eta)}{g(\eta)\Gamma^{1/2}(\eta)} \ , \quad (21a)$$

$$\left.\frac{\partial H}{\partial P}\right|_L = -\frac{\pi E(\eta)}{g(\eta)\Gamma^{1/2}(\eta)} = -\frac{\pi B_c \sinh f(\eta)}{g(\eta)\Gamma^{1/2}(\eta)} \ , \quad (21b)$$

$$h_L^2=\frac{\pi^2(B^2+E^2)}{(B^2-E^2)\Gamma}=\frac{\pi^2\cosh(2f)}{\Gamma} \ , \quad (22)$$

$$\Theta=\frac{\pi\Gamma^{1/2}(B^2+E^2)}{(B^2-E)^{3/2}}=\frac{\pi\Gamma^{1/2}\cosh(2f)}{g} , \quad (23)$$

$$h_L^{-2}\Theta(\eta)=\frac{\Gamma^{3/2}}{\pi(B^2-E^2)^{1/2}}=\frac{\Gamma^{3/2}}{\pi g} \ . \quad (24)$$

It is also shown in the Appendix that the criterion $B_c\bar{\kappa}<1$, where $\bar{\kappa}$ is the mean κ of the electrons, is generally sufficient to assure good approximation. Often, this is a stronger criterion than necessary.

The derivatives of J with respect to P and W become

$$\begin{aligned} \frac{\partial J}{\partial W} &= \frac{\pi B}{(B^2-E^2)^{1/2}\Gamma^{1/2}}\frac{\partial J}{\partial\kappa} - \frac{(B^2-E^2)E}{(B^2+E^2)\Gamma}\frac{\partial J}{\partial\eta} \\ &= \frac{\pi\cosh f}{\Gamma^{1/2}}\frac{\partial J}{\partial\kappa} - \frac{g\sinh f}{\Gamma\cosh(2f)}\frac{\partial J}{\partial\eta} \end{aligned} \quad (25a)$$

and

$$\begin{aligned} \frac{\partial J}{\partial P} &= -\frac{\pi E}{(B^2-E^2)^{1/2}\Gamma^{1/2}}\frac{\partial J}{\partial\kappa} - \frac{(B^2-E^2)B}{(B^2+E^2)\Gamma}\frac{\partial J}{\partial\eta} \\ &= -\frac{\pi\sinh f}{\Gamma^{1/2}}\frac{\partial J}{\partial\kappa} - \frac{g\cosh f}{\Gamma\cosh(2f)}\frac{\partial J}{\partial\eta} . \end{aligned} \quad (25b)$$

The equations for G and f [Eqs. (18)] are then

$$\begin{aligned} G &= \int d\eta \frac{\Gamma^{3/2}}{\pi g}\int d\kappa\, JQ^{1/2} \\ &= \int d\eta \frac{2^{1/2}\Gamma^{7/4}}{\pi^{3/2}g}\left.K(\eta,\mu)\right|_{\mu=\pi\Gamma^{1/2}\Delta y^2/2} , \end{aligned} \quad (26)$$

$$g(y)f'(y)=-\int d\eta\left[\frac{\Gamma}{g}\cosh[f(y)-f]\int d\kappa\frac{\partial J}{\partial\kappa}Q^{1/2}-\int d\eta\frac{\Gamma^{1/2}}{\pi}\,\frac{\sinh[f(y)+f]}{\cosh(2f)}\int d\kappa\frac{\partial J}{\partial\eta}Q^{1/2}\right]$$
$$=-\int d\eta\frac{2^{1/2}\Gamma^{5/4}}{\pi^{1/2}g}\cosh[f(y)-f]\frac{\partial K(\eta,\mu)}{\partial\mu}\bigg|_{\mu=\pi\Gamma^{1/2}\Delta y^2/2}$$
$$+\int d\eta\frac{2^{1/2}\Gamma^{3/4}}{\pi^{3/2}}\,\frac{\sinh[f(y)+f]}{\cosh(2f)}\,\frac{\partial K(\eta,\mu)}{\partial\eta}\bigg|_{\mu=\pi\Gamma^{1/2}\Delta y^2/2}, \quad (27)$$

where

$$K(\eta,\mu)=\int_0^\infty d\nu\,\nu^{1/2}J(\nu+\mu,\eta)\ .$$

The integral expression for K is simple, and the integral can be done analytically for many distributions. Therefore many problems can be solved with a single integral over η.

Since we are only looking at small Δy, the hyperbolic cosine term in the first part of Eq. (27) is near unity and the sinh/cosh term in the second part is approximately $\tanh(2f)$, which is less than 1. The ratio of the remainder of the integrands is approximately

$$\frac{\Gamma^{1/2}}{g}\left|\frac{\partial J/\partial\kappa}{\partial J/\partial\eta}\right| .$$

Although $\Gamma^{1/2}/g$ is small near the cathode, so is $\tanh(2f)$. We therefore should be able to neglect the second term in many cases where the distribution varies slowly along the laminar line (i.e., with η) and quickly normal to the laminar line (i.e., with κ). This appears to be the case with the simulations mentioned above, although we do not know how much of this is due to numerical effects.

If the distribution $J(\kappa,\eta)$ has a spread in κ at the cathode, the integrals will have to include some electrons above the cathode line. Otherwise, the charge density would drop by one-half in a narrow interval near the cathode. By including the electrons above the cathode line, this will not occur. The problem will then include some electrons being emitted from the cathode with nonzero energy and a like number being recaptured by the cathode. This also occurs in simulations (Fig. 6). In the problem presented in Sec. III B, this will be handled by assigning $g(-y)=g(y)$, $f(-y)=-f(y)$, and integrating from $\eta=-\infty$.

B. Example: Equilibrium profiles

An example of the use of this theory shows the effect of orbit height on field profiles and electron pressure. Look at the particular distribution given by

$$J=\frac{2g^2(\eta)}{\pi\Gamma^{3/2}(\eta)\lambda^2(\eta)}f_0'(\eta)e^{-2\kappa/[\pi\Gamma^{1/2}(\eta)\lambda^2(\eta)]}, \quad (28)$$

where f_0' is the derivative of f_0. We will see that λ is the mean orbital half-height and that, in the limit of $\lambda=0$, $f(y)=f_0(y)$. Since $\lambda(\eta)$ is arbitrary thus far, the primary assumption is the exponential variation with κ. The reason for this formulation will soon be apparent. Distributions varying with κ in ways other than exponential (e.g., a square distribution) could be done as easily. We will assume that the variation of f_0 is slow enough to neglect the $\partial K/\partial\eta$ term in Eq. (27).

With this distribution function, typical electrons with η have κ of the order $\pi\Gamma^{1/2}(\eta)\lambda^2(\eta)/2$. Replacing κ by $\pi\Gamma^{1/2}\lambda^2/2$ in $Q(y,\kappa,\eta)=0$ [Eq. (20)] gives $|\Delta y|=\lambda$, and so the typical electron orbit will be twice this value in height. The functional behavior of $\lambda(\eta)$ determines the variation of the height of the typical orbits across the flow. We will restrict ourselves to constant λ, and so the mean orbital height will be independent of position. Putting this in Eqs. (26) and (27), the κ integrals can be done immediately. The lower limit of these integrals is 0 (i.e., on the laminar line), and the upper limit is taken to be $+\infty$ since we restrict ourselves to cases where electrons are close to the laminar line and, therefore, where the exponent in J in Eq. (28) is large on the anode line. Then

$$g^2=(B_c^2+2G_c)-2G\ ,$$
$$2G(y)=\int d\eta\lambda^2 g^2(\eta)[g(\eta)-f'(\eta)]f_0'\frac{1}{\pi^{1/2}\lambda}e^{-(\Delta y/\lambda)^2}, \quad (29)$$
$$g(y)f'(y)=\int d\eta\, g(\eta)f_0'(\eta)\cosh[f(y)-f(\eta)]\times\frac{1}{\pi^{1/2}\lambda}e^{-(\Delta y/\lambda)^2} .$$

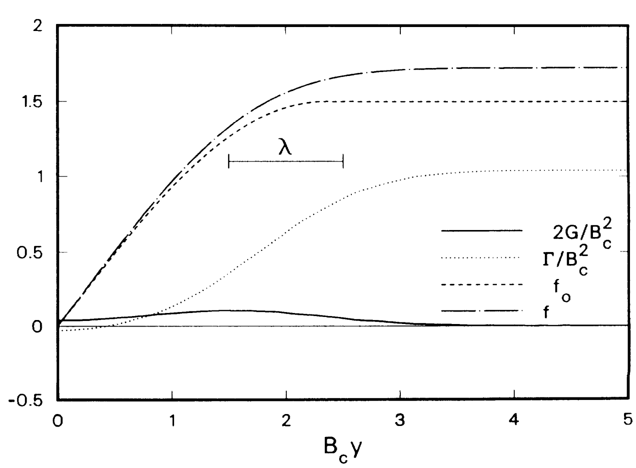

FIG. 8. Example. The function f, twice the electron pressure ($2G$), and the function Γ for a flow with $f=y_0\sin(y/y_0)$ and $y_0=1.5$. The scale height of the fields [e.g., $|B/(dB/dy)|$] is comparable to the height (i.e., y_+-y_-) of the typical orbit, yet the electron pressure is still small when compared to field pressure ($2G\ll B_c^2$). Also shown is the function f_0.

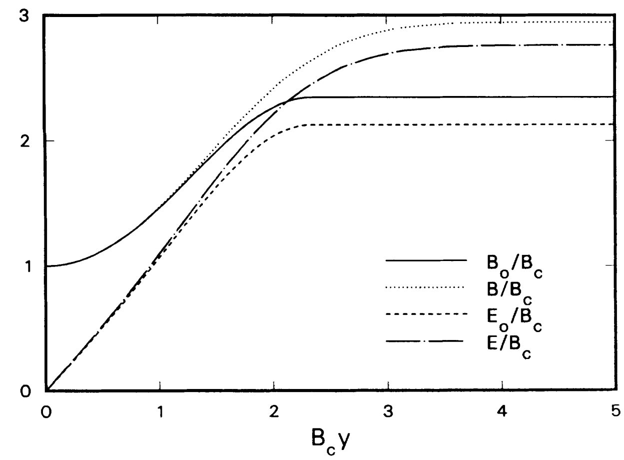

FIG. 9. Fields B, E normalized to B_c, for the example. Also shown are the fields B_0, E_0 for the case where all electrons have $\kappa=0$, i.e., for a laminar flow. The spreading of the charge and current densities due to nonzero gyroheight is evident.

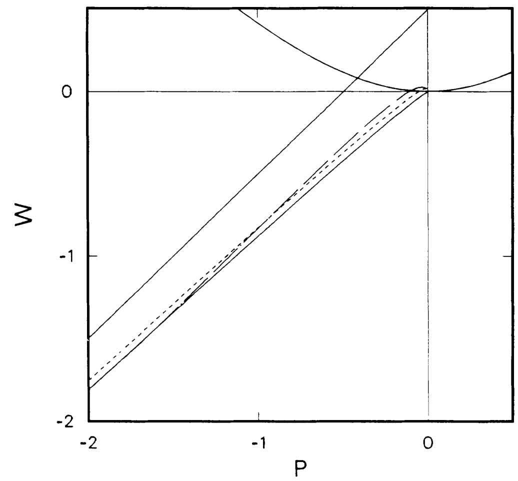

FIG. 11. (P, W) space showing the cathode, anode, and laminar lines for the flow of Figs. 8, 9, and 10. Also shown are the lines $B_c\kappa=1.0$ (dashed line) and a contour of constant low electron density in (P, W) space, $J(P, W)=\text{const}$ (dot-dashed line).

Note that if the exponential and factor before it are replaced by a Dirac δ function, the solution is

$$f=f_0,\quad g^2[1+\lambda^2(g-f_0')f_0']=B_c^2\ ,$$

and in the limit of small λ, $g=B_c$. That is to say, for very small λ the solution goes to the zero-pressure solution presented in Sec. II C.

Equations (29) for g^2, G, and f are integro-differential equations for f and g^2. Given a pressure $g(\eta)$, the equation for f can be solved as a series, using care to assure that the series converges. This solution can be put into the integral expression for G, which yields a new $g(\eta)$ to be put back into the equation for f. This was done for

$$f_0(y)=\begin{cases} y_0\sin(y/y_0), & y<\pi y_0/2 \\ y_0, & y>\pi y_0/2\ , \end{cases}$$

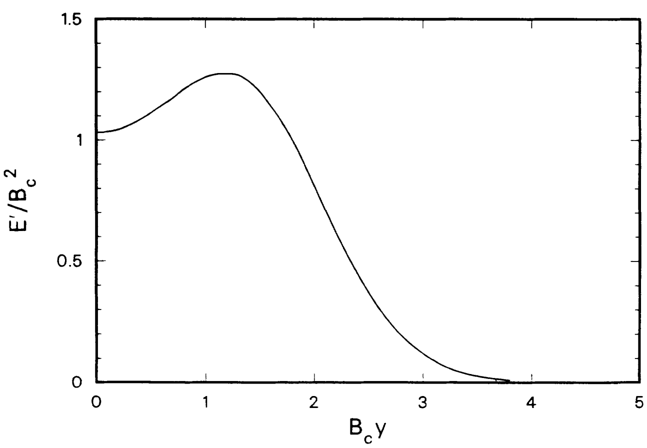

FIG. 10. Charge density for the flow in Figs. 8 and 9. There is continual emission and reabsorption at the cathode in the model, but since the electrons at the cathode are not cold (i.e., P and W are not zero for all emitted electrons), there is no thin, high-density layer.

for $y_0=1.5$. This function was chosen because it gives a similar charge distribution to the simulation data shown in Fig. 7.

The solution is shown in Figs. 8 and 9. The function f, twice the electron pressure ($2G_e$), and the function Γ are shown in Fig. 8. Note that the electron pressure never exceeds about 0.07 of the total pressure (which is $B_c^2/2$), even though the electron orbits are fairly large. Note also that the electron density extends about λ further across the transmission line than the $\lambda=0$ ($f=f_0$) case. The fact that Γ is varying on a scale of about 2λ indicates that the approximation for Q, etc., should be fairly good, even at this large λ. Figure 9 shows the normalized fields calculated from the data in Fig. 8, and the charge density is shown in Fig. 10. The charge-density profile is qualitatively similar to that seen in the simulation (Fig. 7). Figure 11 shows (P, W) space for the flow. Also shown are the lines $B_c\kappa=1.0$ and a contour of the electron density in (P, W) space at e^{-2} of the maximum value. The distribution in (P, W) space is also qualitatively similar to the simulation [Fig. 6(c)]. In the example the (normalized) anode scalar and vector potentials are 10.3 and 11.8, respectively.

IV. CONCLUSIONS

By specifying the distribution of electrons in (P, W) space as a function of κ, η and by assuming that electrons in relativistic electron flow are close to the laminar line, we have been able to simplify the general theory of magnetically insulated relativistic electron flows. Theoretical considerations and time-dependent particle-in-cell simulations offer some justification for the assumption. The resulting theory has made solutions practical for some problems of interest. This theory might be helpful in studying wave dispersion in these flows, flow characteristics, and the effect of finite electron orbits on enhancement in ion diodes.

A sample calculation has shown that finite orbits cause a fairly small change in the fields and that a finite-orbit zero-pressure flow is probably a good approximation for many situations. In situations such as the calculation of dispersion using the results of Ref. [11], a theory including finite orbit size is desirable, because a disturbance in the orbit at one value of y disturbs the electron density over the entire interval from y_- to y_+.

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APPENDIX

To do the integrals of H, we use the approximate function Q [Eq. (20)]. The zeros of Q are then

$$y_\pm=\eta+\Delta y_\pm,\quad \Delta y_\pm=\pm(2\kappa/\pi\Gamma^{1/2})^{1/2}\ .$$

In the limit of $\kappa=0$, the approximation for Q will give the correct value for the integrals over y, and therefore we can get the derivatives of H on the stable laminar line L_s, i.e., on $\kappa=0$. These are given by

$$\left.\frac{\partial H}{\partial W}\right|_L=\left[\int_{y_-}^{y_+}dy\frac{(1+\phi+W)}{Q^{1/2}}\right]_L$$
$$=\lim_{\kappa\to 0}\int_{\Delta y_-}^{\Delta y_+}d(\Delta y)\frac{\phi(y)-\phi(\eta)+B(\eta)[B^2(\eta)-E^2(\eta)]^{-1/2}}{[2\Gamma^{1/2}(\eta)\kappa/\pi-\Gamma(\eta)\Delta y^2]^{1/2}}\ ,$$

where Eqs. (5) have been used for P_L, and W_L. Changing variables to $\chi=(\pi\Gamma^{1/2}/2\kappa)^{1/2}\Delta y$,

$$\left.\frac{\partial H}{\partial W}\right|_L=\frac{B(\eta)}{[B^2(\eta)-E^2(\eta)]^{1/2}\Gamma^{1/2}(\eta)}\int_{-1}^{1}\frac{d\chi}{(1-\chi^2)^{1/2}}$$
$$=\frac{\pi B(\eta)}{[B^2(\eta)-E^2(\eta)]^{1/2}\Gamma^{1/2}(\eta)}$$
$$=\frac{\pi B(\eta)}{g(\eta)\Gamma^{1/2}(\eta)}\ . \quad \text{(A1a)}$$

Likewise,

$$\left.\frac{\partial H}{\partial P}\right|_L=-\frac{\pi E(\eta)}{[B^2(\eta)-E^2(\eta)]^{1/2}\Gamma^{1/2}(\eta)}$$
$$=-\frac{\pi E(\eta)}{g(\eta)\Gamma^{1/2}(\eta)}\ . \quad \text{(A1b)}$$

Substituting these into Eq. (9),

$$h_L^2=\frac{\pi^2(B^2+E^2)}{(B^2-E^2)\Gamma}=\frac{\pi^2\cosh(2f)}{\Gamma}\ . \quad \text{(A2)}$$

If there is an unstable laminar line L_u, these expressions are only valid where $\Gamma\geq 0$, i.e., on L_s. We can now derive $\Theta(\eta)$. From Eq. (8),

$$\Theta=\left.\frac{\partial H}{\partial P}\right|_L\left.\frac{dW}{d\eta}\right|_L-\left.\frac{\partial H}{\partial W}\right|_L\left.\frac{dP}{d\eta}\right|_L\ .$$

Substituting Eqs. (13) and (21),

$$\Theta=\frac{-\pi E}{(B^2-E^2)^{1/2}\Gamma^{1/2}}\frac{(-E\Gamma)}{B^2-E^2}-\frac{\pi B}{(B^2-E^2)^{1/2}\Gamma^{1/2}}\frac{(-B\Gamma)}{B^2-E^2}$$
$$=\frac{\pi\Gamma^{1/2}(B^2+E^2)}{(B^2-E^2)^{3/2}}=\frac{\pi\Gamma^{1/2}\cosh(2f)}{g}\ . \quad \text{(A3)}$$

Using the Jacobian for the transformation from P,W to κ,η on L_s [see Eq. (9)],

$$h_L^{-2}\Theta(\eta)=\frac{\Gamma^{3/2}}{\pi(B^2-E^2)^{1/2}}=\frac{\Gamma^{3/2}}{\pi g}\ . \quad \text{(A4)}$$

The higher derivatives of H on L_s can be calculated in a similar way, except that the derivatives of the limits of the integrals (i.e., y_+,y_-) must be taken into account, and the process must be done as a limit, to avoid divergences. The derivatives all exist, and, for example,

$$\left.\frac{\partial^2 H}{\partial W^2}\right|_L=-\left.\frac{\partial^2 H}{\partial P^2}\right|_L=\int_{y_-}^{y_+}\frac{dy}{Q^{1/2}}=\frac{\pi}{\Gamma^{1/2}}\ ,$$
$$\left.\frac{\partial^2 H}{\partial W\,\partial P}\right|_L=0\ ,$$

from which it can be shown that

$$\left.\frac{dh^2}{d\kappa}\right|_L=-\frac{2\pi}{\Gamma^{1/2}\cosh(2f)}=-\frac{2\Gamma^{1/2}}{\pi\cosh^2(2f)}h_L^2\ .$$

By expanding h^2 in a Taylor series in κ and using the above in the first term, we can estimate the accuracy of the assumption of $h^2=h_L^2$. In flows with electrons only, $\Gamma^{1/2}\leq B_c$. Since Γ is small near the cathode and $\cosh(2f)$ is large near the anode, $B_c\bar{\kappa}<1$, where $\bar{\kappa}$ is the mean κ of the electrons, is generally sufficient to assure a good approximation. It can also be shown that $Q(y)$ given by Eq. (20) is a good approximation if the same inequality is satisfied. Often, this is a stronger criterion than necessary.

Since the electron distribution in (P,W) space is going to be stated as a function of κ,η, we need to convert the derivatives of J in Eqs. (18) to derivatives with respect to κ and η. Using Eqs. (7) and (8),

$$\frac{\partial J}{\partial W}=\frac{\partial J}{\partial \kappa}\frac{\partial \kappa}{\partial W}+\frac{\partial J}{\partial \eta}\frac{\partial \eta}{\partial W}=\frac{\partial J}{\partial \kappa}\frac{\partial H}{\partial W}+\frac{\partial J}{\partial \eta}\Theta^{-1}\frac{\partial H}{\partial P},$$

$$\frac{\partial J}{\partial P}=\frac{\partial J}{\partial \kappa}\frac{\partial \kappa}{\partial P}+\frac{\partial J}{\partial \eta}\frac{\partial \eta}{\partial P}=\frac{\partial J}{\partial H}\frac{\partial H}{\partial P}-\frac{\partial J}{\partial \eta}\Theta^{-1}\frac{\partial H}{\partial W}.$$

Consistent with the small-κ approximation, we use the derivatives of H on L_s:

$$\frac{\partial J}{\partial W}=\frac{\pi B}{(B^2-E^2)^{1/2}\Gamma^{1/2}}\frac{\partial J}{\partial \kappa}+\frac{(B^2-E^2)^{3/2}}{\pi\Gamma^{1/2}(B^2+E^2)}\frac{(-\pi E)}{(B^2-E^2)^{1/2}\Gamma^{1/2}}\frac{\partial J}{\partial \eta}=\frac{\pi B}{(B^2-E^2)^{1/2}\Gamma^{1/2}}\frac{\partial J}{\partial \kappa}-\frac{(B^2-E^2)B}{(B^2+E^2)\Gamma}\frac{\partial J}{\partial \eta}=\frac{\pi\cosh f}{\Gamma^{1/2}}\frac{\partial J}{\partial \kappa}-\frac{g\sinh f}{\Gamma\cosh(2f)}\frac{\partial J}{\partial \eta}\tag{A5a}$$

and

$$\frac{\partial J}{\partial P}=-\frac{\pi E}{(B^2-E^2)^{1/2}\Gamma^{1/2}}\frac{\partial J}{\partial \kappa}-\frac{(B^2-E^2)B}{(B^2+E^2)\Gamma}\frac{\partial J}{\partial \eta}=-\frac{\pi\sinh f}{\Gamma^{1/2}}\frac{\partial J}{\partial \kappa}-\frac{g\cosh f}{\Gamma\cosh(2f)}\frac{\partial J}{\partial \eta}.\tag{A5b}$$

Using the Jacobian on the laminar line [Eq. (A4)] and Eqs. (18), the electron pressure G is given by

$$G=\int\int d\kappa\, d\eta\frac{\Gamma^{3/2}}{\pi g}\left[\frac{2\Gamma^{1/2}\kappa}{\pi}-\Gamma\,\Delta y^2\right]^{1/2}J(\kappa,\eta)=\int d\eta\frac{2^{1/2}\Gamma^{7/4}}{\pi^{3/2}g}\int d\kappa\left[\kappa-\frac{\pi\Gamma^{1/2}}{2}\Delta y^2\right]^{1/2}J(\kappa,\eta)=\int d\eta\frac{2^{1/2}\Gamma^{7/4}}{\pi^{3/2}g}K(\eta,\mu)\Big|_{\mu=\pi\Gamma^{1/2}\Delta y^2/2},$$
$$K(\eta,\mu)=\int_0^\infty d\nu\,\nu^{1/2}J(\nu+\mu,\eta).\tag{A6}$$

By substituting Eqs. (20), (A2), (A3), (A5), and (A6) into Eq. (18c), we get an equation for f:

$$g(y)f'(y)=-\int d\eta\frac{\Gamma}{g}\cosh[f(y)-f)]\int d\kappa\frac{\partial J}{\partial \kappa}\left[\frac{2\Gamma^{1/2}\kappa}{\pi}-\Gamma\,\Delta y^2\right]^{1/2}+\frac{1}{2\pi}\int d\eta\,\Gamma^{1/2}\frac{\sinh[f(y)+f]}{\cosh(2f)}\int d\kappa\frac{\partial J}{\partial \eta}\left[\frac{2\Gamma^{1/2}\kappa}{\pi}-\Gamma\,\Delta y^2\right]^{1/2}$$
$$=-\int d\eta\frac{2^{1/2}\Gamma^{5/4}}{\pi^{1/2}g}\cosh[f(y)-f]\frac{\partial K(\eta,\mu)}{\partial \mu}\Big|_{\mu=\pi\Gamma^{1/2}\Delta y^2/2}+\int d\eta\frac{\Gamma^{3/4}}{2^{1/2}\pi^{3/2}}\frac{\sinh[f(y)+f]}{\cosh(2f)}\frac{\partial K(\eta,\mu)}{\partial \eta}\Big|_{\mu=\pi\Gamma^{1/2}\Delta y^2/2}.\tag{A7}$$

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