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ARTICLES

Semiclassical chaos in quartic anharmonic oscillators

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Quartic anharmonic oscillators with dissipation and external force terms are studied in the semiclassical approximation. The equilibrium states are related to the quantum states by ratios of the semiclassical to quantum scales in length, action, and energy. The ratios give a measure of the departure of semiclassical from quantum invariants. In three cases semiclassical chaos is found to occur with the destruction of the equilibrium states and/or semiclassical invariants, but the quantum states are not destroyed. The fourth case is anomalous.

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I. INTRODUCTION

The possibility of quantum chaos has been investigated by quantization of classical maps [1] and by the introduction of Lyapunov exponents for quantum dissipative systems [2] and for quantum systems driven by external time-dependent terms [3]. A general procedure also follows from the Zaslavskii criterion, which requires the destruction of integral invariants including the energy states [4]. In this paper we consider the Zaslavskii criterion for quartic anharmonic oscillators with dissipation and external force terms. We follow a semiclassical Hamiltonian procedure that gives the canonically quantized energy ground state as well as the semiclassical Hamilton equations which yield the equilibrium states. It is shown that the equilibrium states are related to the ground state by the ratio of the quantum to semiclassical length scales and the ratio of quantum to semiclassical action. The quantum and semiclassical scales are explicitly stated and provide a measure of the departure of semiclassical from quantum invariants.

In three of the four cases considered semiclassical chaos is found to follow from the destruction of integral invariants and/or equilibrium states, but the quantum states are not destroyed. The fourth case is found to be anomalous.

II. BASIC EQUATIONS

We consider the Hamiltonians for four quartic anharmonic oscillators with dissipation and external force terms in the semiclassical approximation. The classical Hamiltonian with dissipation is not well defined, but the quantum Hamiltonian can be defined as a complex function in the semiclassical approximation. Semiclassical complex Hamiltonians with dissipation have been studied by several authors and procedures have been given for finding the decay time [5]. It has been shown by several investigators that Hamiltonians for the anharmonic oscillators can be represented as multinomial expansions in $z_k = x_k + iy_k$ defined on complex phase space and $z \rightarrow \partial/\partial z^*$ is the corresponding operator [6]. The connection between the Hilbert space $\mathcal{H}_n$ of the quantum-mechanical system defined on $(x_1, x_2, \ldots, x_n)$ and the function space γ_m defined on $z_k = ip_k + q_k$ has been studied by Bargmann [7] and the extended Hilbert space which admits complex Hamiltonians with dissipation has been studied by several investigators [5]. In this paper we utilize the oscillator variables $\sqrt{2m}\, z_k = \hbar p_k + im\omega x_k$.

We consider the quartic anharmonic oscillator Hamiltonians

$$H(p,x,t) = a^2 p^2 + V(x) + \tfrac{1}{2} im\omega\gamma x^2 - xf\cos(\Omega t) , \tag{1}$$

where

$$V(x) = \tfrac{1}{2}\epsilon_2 \alpha x^2 + \tfrac{1}{4}\epsilon_4 \beta x^4 , \tag{2}$$

$\epsilon_2 = \pm 1$ and $\epsilon_4 = \pm 1$. The dissipative term has been chosen so that Hamilton's equations give the Duffing equation and the three other second-order differential equations which follow with $\epsilon_2 = \pm 1$ and $\epsilon_4 = \pm 1$ from the Euler-Lagrange equation. The equivalence of the Euler-Lagrange equation and Hamilton's equations was proved by Mandelstam and Yourgrau [8].

Utilizing different procedures, three of the four cases have been considered in the literature [9–11]. Utilizing terminology from mechanical engineering, the four cases can be characterized as follows.

(i) Soft spring and positive stiffness [9]:

$$\epsilon_2 = +1, \quad \epsilon_4 = -1 . \tag{3}$$

(ii) Hard spring and negative stiffness [10]:

$$\epsilon_2 = -1, \quad \epsilon_4 = +1 . \tag{4}$$

(iii) Hard spring and positive stiffness [11]:

$$\epsilon_2 = +1, \quad \epsilon_4 = +1 . \tag{5}$$

(iv) Soft spring and negative stiffness:

$$\epsilon_2=-1,\quad \epsilon_4=-1\ . \tag{6}$$

We now consider the four cases and compare the results. We utilize the following equations [12]:

$$i\hbar\dot{z}=[z,H]=[z,z^*]\frac{\partial H}{\partial z^*}\ , \tag{7}$$

$$[z,z^*]=-2ab\ , \tag{8}$$

$$z\to[z,z^*]\frac{\partial}{\partial z^*}\ , \tag{9}$$

where $z=ap+ibx$, $H(z,z^*)=u+iv$, $a^2=\hbar^2/2m$, and $b^2=m\omega_a^2/2$. The normalized canonical variable $z=x+iy$ and the corresponding operator were introduced by Fock in 1928 and have been utilized by Bargmann and others [6] in studies of the complex rotation group and the harmonic and anharmonic oscillators, by Lanczos in a complex representation of Hamilton's equations [13], by Hioe, MacMillan, and Montroll in studies of the anharmonic oscillator [6], and by Strocchi [14] and Heslot [15] in complex representations of quantum mechanics.

Hamilton's equations (7) give the equations of motion

$$m\dot{x}=\hbar p-\frac{\omega\gamma}{\omega_a}mx\ , \tag{10}$$

$$-\hbar\dot{p}=\frac{\partial V}{\partial x}-f\cos(\Omega t)\ , \tag{11}$$

and

$$m\ddot{x}+\frac{\omega\gamma}{\omega_a}m\dot{x}+\frac{\partial V}{\partial x}=f\cos(\Omega t)\ , \tag{12}$$

which is the Duffing equation for $\epsilon_2=+1$ and $\epsilon_4=-1$, and no quantization ($\omega_a=\omega$).

We consider first the initial-value problem with $f=0$. Canonical quantization gives

$$[z,z^*]=-\hbar\omega_a\ , \tag{13}$$

and from

$$\frac{\partial^2 V}{\partial x^2}=-\frac{\partial\hbar\dot{p}}{\partial x}=-\frac{\partial}{\partial x}\left[m\ddot{x}-\frac{\omega\gamma}{\omega_a}m\dot{x}\right]=\text{const}\ , \tag{14}$$

which holds for periodic solutions, we obtain

$$\epsilon_2\alpha+3\epsilon_4\beta x_0^2=m\omega_a^2\ , \tag{15}$$

which gives

$$\frac{1}{\epsilon_4}\left[\frac{\omega_a^2}{\omega^2}-\epsilon_2\right]=\frac{3\beta}{\alpha}x_0^2\ . \tag{16}$$

The ratios of quantum to semiclassical scales in length, action, and energy are

$$\frac{x_0^2}{x_{\rm cl}^2}=\frac{3\beta}{\alpha}\frac{\hbar}{m\omega_a}=\frac{\hbar\omega_a}{\frac{\alpha^2}{3\beta}[\epsilon_2+\epsilon_4(x_0^2/x_{\rm cl}^2)]}=\eta\ , \tag{17}$$

where $x_{\rm cl}^2=\alpha/3\beta$, the semiclassical energy is the continued fraction from

$$\frac{\hbar\omega_a}{m\omega_a^2x_{\rm cl}^2}=\frac{\hbar\omega_a}{\alpha^2/3\beta}\left[\epsilon_2+\epsilon_4\frac{\hbar\omega_a}{\frac{\alpha^2}{3\beta}\left[\epsilon_2+\epsilon_4\frac{\hbar\omega_a}{\alpha^2/3\beta(\cdots)}\right]}\right]^{-1}, \tag{18}$$

and η is a measure of the departure of semiclassical from quantum invariants. Thus canonical quantization gives quantum to semiclassical scale ratios in length and action and energy. The semiclassical and quantum scales are as follows:

	Length	Energy	Action	
Quantum	$\frac{\hbar}{m\omega_a}$	$\hbar\omega_a$	h	(19)
Semiclassical	$\frac{\alpha}{3\beta}$	$\frac{\alpha^2}{3\beta}\left[1+\frac{x_0^2}{x_{\rm cl}^2}\right]$	$\frac{m\omega_a\alpha}{3\beta}$	(20)

The quantum to semiclassical ratios provide a measure of the departure of semiclassical from quantum invariants. Note that the semiclassical invariants $\sim\beta^{-1}$ and diverge as $\beta\to0$. The initial value problem with $f=0$ gives the constants of the motion

$$x'^2=-\frac{\epsilon_2}{\epsilon_4}\frac{\alpha}{\beta}\ , \tag{21}$$

$$\hbar p'=\frac{\omega\gamma}{\omega_a}mx'\ , \tag{22}$$

and

$$H(\alpha,\beta,\gamma)=-\left[\tfrac{1}{4}\epsilon_2+\tfrac{1}{2}\left[\frac{\gamma}{\omega_a}\right]^2+\tfrac{1}{2}i\frac{\gamma}{\omega}\right]\frac{\epsilon_2}{\epsilon_4}\frac{\alpha^2}{\beta}\ . \tag{23}$$

Thus the equilibrium states are related to the quantum states by the ratio of $\hbar$ to the semiclassical action

$$\frac{x'^2}{x_0^2}=-\frac{\epsilon_2}{\epsilon_4}\frac{\alpha}{\beta}\frac{m\omega_a}{\hbar}\ . \tag{24}$$

The external force term $xf\cos(\Omega t)$ is an interaction term coupled to the internal variables by x. It is equivalent to an oscillating string potential added to V. At $f>0$ the interactions change x' and p'. The new equilibrium states are found as follows: Hamilton's equations provide a solution for f in terms of the internal variables

$$f\cos(\Omega t)=\hbar\dot{p}+\frac{\partial V}{\partial x}\ . \tag{25}$$

Then a new Hamiltonian can be written in terms of the internal variables

$$H(p,x,\dot{p})=a^2p^2+V(x)+\tfrac{1}{2}im\omega\gamma x^2-x\left[\hbar\dot{p}+\frac{\partial V}{\partial x}\right] \tag{26}$$

and new Hamilton equations obtained. The new constants of the motion represent the initial values shifted by the external force and f can be written in terms of the new constants. The new Hamilton equations are

$$\omega_a \frac{\partial H}{\partial z^*} = -i\dot{z} = \omega_a \left[ap - \frac{\omega\gamma}{\omega_a}\sqrt{m/2}x - \frac{i}{2b}\left[\hbar\dot{p} + x\frac{\partial \hbar \dot{p}}{\partial x} + x\frac{\partial^2 V}{\partial x^2} \right] \right] . \tag{27}$$

The new constants of the motion are

$$x'^2 = -\frac{\epsilon_2}{\epsilon_4}\frac{\alpha}{3\beta} , \tag{28}$$

$$\hbar p' = \frac{\omega\gamma}{\omega_a} m x' , \tag{29}$$

and

$$f' = \tfrac{2}{3}\epsilon_2 \alpha x' \tag{30}$$

is the effective force in the internal equilibrium and the equilibrium energy is

$$H(\alpha,\beta,\gamma) = -\left[\tfrac{5}{12}\epsilon_2 + \tfrac{1}{2}\left[\frac{\gamma}{\omega_a}\right]^2 + \tfrac{1}{2}i\frac{\gamma}{\omega}\right] \frac{\epsilon_2}{\epsilon_4}\frac{\alpha^2}{3\beta} . \tag{31}$$

We see that the semiclassical scales and x', p', $H(\alpha,\beta,\gamma)$, and $x'f'$ are functions of β^{-1} so that the destruction of equilibrium states occurs as $\beta \to 0$ provided that the equilibrium states exist: $\epsilon_2/\epsilon_4 = -1$ and x' is real. Otherwise the equilibrium states are imaginary. The destruction of the semiclassical invariants in length, action, and energy also occurs as $\beta \to 0$ provided there is a positive definite solution of (16) and the invariants are positive definite. In the following section we consider the four cases (3)–(6).

III. THE CASES $\epsilon_2 = \pm 1$ AND $\epsilon_4 = \pm 1$

In this section we consider the four cases (3)–(6).

A. Soft spring and positive stiffness: $\epsilon_2 = +1$ and $\epsilon_4 = -1$

The Hamiltonian is

$$H(p,x,t) = a^2p^2 + \tfrac{1}{2}\alpha x^2 - \tfrac{1}{4}\beta x^4 + \tfrac{1}{2}im\omega\gamma x^2 - x f\cos(\Omega t) . \tag{32}$$

The equilibrium states for the initial-value problem with $f=0$ are (21)–(23) with $\epsilon_2/\epsilon_4 = -1$, and they are all positive definite. With the external force turned on the new Hamilton equations give

$$x'^2 = \frac{\alpha}{3\beta}, \quad \hbar p' = \frac{\omega\gamma}{\omega_a} m x' , \tag{33}$$

$$H(\alpha,\beta,\gamma) = \left[\tfrac{5}{12} + \tfrac{1}{2}\left[\frac{\gamma}{\omega_a}\right]^2 + \tfrac{1}{2}i\frac{\gamma}{\omega}\right]\frac{\alpha^2}{3\beta} , \tag{34}$$

$$x'f' = \frac{2}{3}\frac{\alpha^2}{\beta} , \tag{35}$$

and the second-order equation of motion

$$m\ddot{x} + \frac{\omega\gamma}{\omega_a} m\dot{x} + \alpha x - \beta x^3 = f\cos(\Omega t) , \tag{36}$$

which with no quantization ($\omega_a = \omega$) is the Duffing equation. Note that (36) is a semiclassical equation as is evident by setting $h=0$ in the Hamilton equations from which (36) follows. The quantization rule gives

$$1 - \frac{\omega_a^2}{\omega^2} = \frac{3\beta}{\alpha}\frac{\hbar}{m\omega_a} . \tag{37}$$

The ratios of quantum to semiclassical scales in action and energy are

$$\frac{x_0^2}{x_{\rm cl}^2} = \frac{3\beta}{\alpha}\frac{\hbar}{m\omega_a} = \frac{\alpha^2}{3\beta}\frac{\hbar\omega_a}{(1 - x_0^2/x_{\rm cl}^2)} = \eta . \tag{38}$$

These ratios with the quantization rule give cubics in η and ω_a with solutions which must be positive definite. The solution defines the quantization region if it exists in terms of a numerical constraint on the energy scale ratio $3\hbar\omega\beta/\alpha^2$.

The equilibrium states are functions of β^{-1} so that the destruction of equilibrium states occurs as $f \to \infty$ as $\beta \to 0$ and $\omega_a \to \omega$. The observed period doubling occurs as $\omega_a \to \omega$ [16]. The canonically quantized states $\hbar\omega_a \to \hbar\omega$ continuously as $f \to \infty$. The semiclassical scales are also functions of β^{-1} and are destroyed as $\beta \to 0$ provided there is a positive definite solution for ω_a and the scales are physical. Thus for case (i) semiclassical chaos occurs with the destruction of semiclassical equilibrium states and integral invariants if they exist, but the quantum states are not destroyed.

B. Hard spring and negative stiffness: $\epsilon_2 = -1$ and $\epsilon_4 = +1$

The Hamiltonian is

$$H(p,x,t) = a^2p^2 - \tfrac{1}{2}\alpha x^2 + \tfrac{1}{4}\beta x^4 + \tfrac{1}{2}im\omega\gamma x^2 - xf\cos(\Omega t) . \tag{39}$$

The quantization rule gives

$$\frac{\omega_a^2}{\omega^2} + 1 = \frac{3\beta}{\alpha}x_0^2 . \tag{40}$$

The ratios of quantum to semiclassical scales are

$$\frac{x_0^2}{x_{\rm cl}^2} = \frac{3\beta}{\alpha}\frac{\hbar}{m\omega_a} = \frac{\alpha^2}{3\beta}\frac{\hbar\omega_a}{(-1 + x_0^2/x_{\rm cl}^2)} = \eta , \tag{41}$$

and we find that physical scales exist for a positive real sum of the continued fraction or a positive real solution of the cubics in η or ω_a.

Hamilton's equations give the second-order equation of motion

$$m\ddot{x} + \frac{\omega\gamma}{\omega_a}m\dot{x} - \alpha x + \beta x^3 = f\cos(\Omega t) . \tag{42}$$

The constants of the motion with $f > 0$ are

$$x'^2=\frac{\alpha}{3\beta}\ , \tag{43}$$

$$\hbar p'=\frac{\omega\gamma}{\omega_a}mx'\ , \tag{44}$$

and

$$x'f'=-\frac{2}{3}\frac{\alpha^2}{\beta}\ , \tag{45}$$

and the equilibrium energy is

$$H(\alpha,\beta,\gamma)=\left[\tfrac{5}{12}+\tfrac{1}{2}\left[\frac{\gamma}{\omega_a}\right]^2+\tfrac{1}{2}i\frac{\gamma}{\omega}\right]\frac{\alpha^2}{3\beta}\ . \tag{46}$$

The equilibrium states are real and they are functions of β^{-1}. The semiclassical and quantum scales are positive definite. The same considerations hold as for case (i): as $f\rightarrow\infty$ as $\beta\rightarrow 0$ and $\omega\rightarrow\omega_a$, the semiclassical equilibrium states and invariants are destroyed, but the quantum states are not destroyed.

C. Hard spring and positive stiffness: $\epsilon_2=+1$ and $\epsilon_4=+1$

This case has been extensively studied utilizing various procedures [11]. It has been known since the work of Bender and Wu that there is a singularity as $\beta\rightarrow 0$ in the energy obtained from the Rayleigh-Schrödinger perturbation series. Subsequently Simon and others utilized a scaling procedure to find the energy spectrum in the semiclassical approximation. The Hamiltonian is

$$H(p,x,t)=a^2p^2+\tfrac{1}{2}\alpha x^2+\tfrac{1}{4}\beta x^4+\tfrac{1}{2}im\omega\gamma x^2-xf\cos(\Omega t)\ . \tag{47}$$

The quantization rule gives the scale ratios

$$\frac{\omega_a^2}{\omega^2}-1=\frac{3\beta}{\alpha}x_0^2=\frac{\hbar\omega_a}{\frac{\alpha^2}{3\beta}(1+x_0^2/x_{\rm cl}^2)}=\eta\ . \tag{48}$$

Hamilton's equations give the equation of motion

$$m\ddot{x}+\frac{\omega\gamma}{\omega_a}m\dot{x}+\alpha x+\beta x^3=f\cos(\Omega t)\ . \tag{49}$$

We consider first the initial-value problem with $f=0$. The constants of the motion from

$$x'^2=-\frac{\alpha}{\beta} \tag{50}$$

and

$$\hbar p'=\frac{\omega\gamma}{\omega_a}mx' \tag{51}$$

are imaginary and

$$H(\alpha,\beta,\gamma)=-\left[\tfrac{1}{4}+\tfrac{1}{2}\left[\frac{\gamma}{\omega_a}\right]^2+\tfrac{1}{2}i\frac{\gamma}{\omega}\right]\frac{\alpha^2}{\beta} \tag{52}$$

is less than 0 and there are no equilibrium states.

At $f>0$ the equilibrium energy is negative and there are no real x' or hp'. The ratios of semiclassical to quantum scales are positive definite and diverge as $\beta\rightarrow 0$. The quantum states $h\omega_a\rightarrow h\omega$ continuously. Thus $xf\rightarrow\infty$ as $\beta\rightarrow 0$ destroys the semiclassical invariants, but the quantum invariants are not destroyed.

D. Soft spring and negative stiffness: $\epsilon_2=-1$ and $\epsilon_4=-1$

The Hamiltonian is

$$H(p,x,t)=a^2p^2-\tfrac{1}{2}\alpha x^2-\tfrac{1}{4}\beta^4+\tfrac{1}{2}i\omega\gamma mx^2=xf\cos(\Omega t)\ . \tag{53}$$

The quantization rule gives

$$\frac{\omega_a^2}{\omega^2}+\frac{3\beta}{\alpha}\frac{\hbar}{m\omega_a}+1=0\ , \tag{54}$$

which does not have a positive definite solution for ω_a. Thus the ratios of the semiclassical to quantum scales are not positive definite. Hamilton's equations give the equation of motion

$$m\ddot{x}+\frac{\omega\gamma}{\omega_c}m\dot{x}-\alpha x-\beta x^3=f\cos(\Omega t)\ . \tag{55}$$

The initial value problem at $f=0$ gives imaginary constants of the motion from

$$x'^2=-\frac{\alpha}{\beta}\ , \tag{56}$$

$$\hbar p'=\frac{\omega\gamma}{\omega_a}mx'\ , \tag{57}$$

and

$$H(\alpha,\beta,\gamma)=-\left[\tfrac{1}{4}+\tfrac{1}{2}\left[\frac{\gamma}{\omega_a}\right]^2+\tfrac{1}{2}i\frac{\gamma}{\omega}\right]\frac{\alpha^2}{\beta} \tag{58}$$

is negative for all α,β,γ. At $f>0$ the equilibrium energy is less than 0 and there are no real x' or hp'. As $\beta\rightarrow 0$ the unphysical equilibrium is destroyed and $\omega_a^2/\omega^2\rightarrow -1$.

IV. SUMMARY

Three of the four anharmonic oscillators considered are found to approach semiclassical chaos as the external force term $x'f'(\beta^{-1})\rightarrow\infty$ as $\beta\rightarrow 0$. In the asymptote $\beta\rightarrow 0$ the equilibrium states are destroyed and/or the semiclassical scales in length, action, and energy are destroyed, but the quantum states if they exist change continuously as $h\omega_a\rightarrow h\omega$ and are not destroyed. In the fourth case with $\epsilon_2=-1=\epsilon_4$ the equilibrium states are unphysical and the semiclassical scales are anomalous and are destroyed as $\beta\rightarrow 0$. The quantum states if they exist are not destroyed.

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