Squeezing and antisqueezing for a harmonic oscillator having a sudden change of mass

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Following a recent paper by De Brito and Baseia [Phys. Rev. A **40**, 4097 (1989)], the generation of squeezing is examined for a harmonic oscillator when a sudden change of mass takes place. The correspondence of such oscillators with the radiation field and some related results, such as modulation of squeezing, are also considered.

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The investigation of time-dependent Hamiltonians describing harmonic oscillators (HO), with time-dependent mass and/or time-dependent frequency, and their relations with coherent [1] and squeezed [2] states is a current matter found in several papers [3,4]. It is in this context that, say, we can mention the well-known parametric amplifier [5] as an example of a time-dependent Hamiltonian that generates squeezed states.

In its classical version [5], the parametric oscillator is described by the equation

$$\ddot{x}+\gamma\dot{x}+\Omega^2(t)x=0\ , \tag{1}$$

where $\Omega(t)=\omega_0(1+\epsilon\sin\omega_p t)^{1/2}$ is a time-dependent frequency, $\epsilon\ll 1$, and ω_p is a pump-wave frequency. The striking feature of the parametric amplifier is that, under certain conditions [6] (e.g., $\omega_p=2\omega_0$, $\epsilon=\gamma/\omega_0$), it is able to oscillate without damping.

The quantum-mechanical version of the parametric amplifier is described by the Hamiltonian (see, e.g., Yuen, Ref. [2])

$$\hat{H}(t)=f_1(t)\hat{a}^{\dagger}\hat{a}+f_2^*\hat{a}^2+f_2\hat{a}^{\dagger 2}+f_3^*\hat{a}+f_3\hat{a}^{\dagger}\ , \tag{2}$$

which is quadratic in terms of creation and annihilation operators $\hat{a}$ and $\hat{a}^{\dagger}$. As it is well known, this quantum-mechanical system is able to generate squeezed states.

Another system which is a somewhat reminiscent of the parametric oscillator is that described by the Bateman Hamiltonian [7]. In a previous paper [8], it was pointed out that the (time-dependent) Bateman Hamiltonian

$$\hat{H}_B(t)=e^{-2\lambda t}\frac{1}{2M_0}\hat{p}^2+e^{2\lambda t}\frac{M_0\omega^2}{2}\hat{q}^2 \tag{3}$$

belongs to the class of quadratic Hamiltonians, in terms of creation and annihilation operators $\hat{a}$ and $\hat{a}^{\dagger}$ that generate squeezed states of the electromagnetic field.

Here, we intend to pursue further this line and to examine the possibility of generating squeezed states when a sudden change of mass takes place. Starting from Ref. [8], we define creation and annihilation operators $\hat{b}(t)$ and $\hat{b}^{\dagger}(t)$ in terms of the usual canonical variables $\hat{q}$ and $\hat{p}$, through the canonical transformation [9]

$$\hat{b}(t)=\left[\frac{M(t)\omega}{2\hbar}\right]^{1/2}\hat{q}+i\left[\frac{1}{2\hbar M(t)\omega}\right]^{1/2}\hat{p} \tag{4a}$$

and

$$\hat{b}^{\dagger}(t)=\left[\frac{M(t)\omega}{2\hbar}\right]^{1/2}\hat{q}-i\left[\frac{1}{2\hbar M(t)\omega}\right]^{1/2}\hat{p}\ , \tag{4b}$$

where $M(t)=M_0\exp(2\lambda t)$, λ is a real parameter, and $[\hat{b},\hat{b}^{\dagger}]=1$.

Substituting $\hat{b}(t)$ and $\hat{b}^{\dagger}(t)$ in Eq. (1) of Ref. [8], namely, $\hat{H}(t)=\hbar\omega[\hat{b}^{\dagger}(t)\hat{b}(t)+\frac{1}{2}]$, we obtain the Bateman Hamiltonian in Eq. (3), which was rederived by many other authors [10].

At $t=0$ we have that Eq. (3) describes the well-known HO Hamiltonian

$$\hat{H}=\hbar\omega(\hat{a}^{\dagger}\hat{a}+\tfrac{1}{2}) \tag{5}$$

with

$$\hat{a}=\hat{b}(0)=\left[\frac{M_0\omega}{2\hbar}\right]^{1/2}\hat{q}+i\left[\frac{1}{2\hbar M_0\omega}\right]^{1/2}\hat{p}\ . \tag{6}$$

Using the commutation relations $[\hat{q},\hat{p}]=i\hbar$ and $[\hat{a},\hat{a}^{\dagger}]=1$, we can write Eq. (4) in terms of $\hat{a}$ and $\hat{a}^{\dagger}$ as

$$\hat{b}(t)=\mu(t)\hat{a}+\nu(t)\hat{a}^{\dagger}\ , \tag{7a}$$

$$\hat{b}^{\dagger}(t)=\nu(t)\hat{a}+\mu(t)\hat{a}^{\dagger}\ , \tag{7b}$$

where (with $\mu^2-\nu^2=1$)

$$\mu(t)=\tfrac{1}{2}[\sqrt{M(t)/M_0}+\sqrt{M_0/M(t)}]\ , \tag{8a}$$

$$\nu(t)=\tfrac{1}{2}[\sqrt{M(t)/M_0}-\sqrt{M_0/M(t)}]\ . \tag{8b}$$

At this point, instead of setting $M(t)=M_0\exp(2\lambda t)$ as above, which led to the Bateman Hamiltonian, we will assume that the mass of the system changes *suddenly* as $M_0 \to M_1$, at $t=t_1$. Accordingly, this can be analytically modeled by

$$M(t)=M_0\Theta(t_1-t)+M_1\Theta(t-t_1)\ , \tag{9}$$

where $\Theta(\tau)$ is the Heaviside step function [$\Theta(\tau)=1$ (0) if $\tau \geq 0$ ($\tau<0$)]. Now, it is known that a squeezed state $|\beta\rangle$ is a squeezed minimum-uncertainty state for a quadrature component of the field, $\hat{a}_1$ or $\hat{a}_2$ ($\hat{a}_1^\dagger=\hat{a}_1$, $\hat{a}_2^\dagger=\hat{a}_2$), defined by

$$\hat{a}=\hat{a}_1+i\hat{a}_2\ . \tag{10}$$

Following traditional calculations (see, e.g., Yuen, Ref. [2]) we can show that, for an eigenstate $|\beta\rangle$ of the annihilation operator $\hat{b}(t)$ [$\hat{b}(t)|\beta\rangle=\beta(t)|\beta\rangle$], we find with $\langle(\Delta\hat{o})\rangle^2=\langle\hat{o}^2\rangle-\langle\hat{o}\rangle^2$, for any operator $\hat{o}$,

$$(\Delta\hat{a}_1)^2=\tfrac{1}{4}(\mu-\nu)^2=\frac{1}{4}\frac{M(t)}{M_0}\ , \tag{11a}$$

$$(\Delta\hat{a}_2)^2=\tfrac{1}{4}(\mu+\nu)^2=\frac{1}{4}\frac{M_0}{M(t)}\ . \tag{11b}$$

Now, substituting Eq. (9) in Eq. (11), we find, for $t<t_1$,

$$(\Delta\hat{a}_1)^2=\tfrac{1}{4}\ , \tag{12a}$$

$$(\Delta\hat{a}_2)^2=\tfrac{1}{4}\ , \tag{12b}$$

$$(\Delta\hat{a}_1)(\Delta\hat{a}_2)=\tfrac{1}{4}\ . \tag{12c}$$

Hence, $|\beta(t)\rangle$ coincides with a coherent state $|\alpha\rangle$ ($\hat{a}|\alpha\rangle=\alpha|\alpha\rangle$), for $t<t_1$, as it should, since $\hat{b}(t_1)=a$ [cf. Eqs. (7)–(9)].

Next, for $t>t_1$, we find [cf. Eqs. (9) and (11)]

$$(\Delta\hat{a}_1)^2=\tfrac{1}{4}(1+\eta)\ , \tag{13a}$$

$$(\Delta\hat{a}_2)^2=\frac{1}{4}\frac{1}{(1+\eta)}\ , \tag{13b}$$

$$(\Delta\hat{a}_1)(\Delta\hat{a}_2)=\tfrac{1}{4}\ , \tag{13c}$$

with $\eta=\Delta M/M_0$ and $\Delta M=M_1-M_0$.

If $M_1<M_0$ then $\eta<0$ and the quadrature $\hat{a}_1$ ($\hat{a}_2$) is squeezed (enhanced); the reverse being true for $\hat{a}_1$ ($\hat{a}_2$) if $M_1>M_0$. Of course, if $\eta \ll 1$, the squeezing is very weak and becomes irrelevant. The foregoing results show that a sudden change of mass produces squeezing in one quadrature of the field, since the coherent states at masses different from the oscillating mass are squeezed states of the oscillator.

What about a physical interpretation? For a HO suffering from continuous changes of mass (as, e.g., in Ref. [8]) a paper by Colegrave and Abdala [4] has argued that the counterpart of such a system (exhibiting exponential evolution) is a single mode of an electromagnetic field in a Fabry-Perot cavity. Following this point of view we have that for a sudden change of mass, as assumed in this paper, the counterpart of such a HO would be a single mode of a (coherent) field inside an optical cavity having an electrooptical crystal used as a controlled gate (shutter) for Q switching a laser.

What would happen if we had a sudden change of mass followed by another sudden one? Namely, if instead of having the Eq. (9) yielding $M_0 \to M_1$ at $t=t_1$, we had

$$M(t)=M_0\Theta(t_1-t)+M_1\Theta(t-t_1)\Theta(t_2-t)+M_2\Theta(t-t_2)\ , \tag{14}$$

which yields $M_0\to M_1$ at $t=t_1$ and $M_1\to M_2$ at $t=t_2$. In this case, the connection between the quadrature, say, $\hat{a}_1=(\hat{a}+\hat{a}^\dagger)/2$ in terms of the operators $\hat{b}(t)$, $\hat{b}^\dagger(t)$, for $t_1<t<t_2$, given by

$$\hat{a}_1=(\mu-\nu)\hat{b}(t)+(\mu-\nu)\hat{b}^\dagger(t)\ , \tag{15}$$

with [cf. Eq. (8)]

$$\mu=\tfrac{1}{2}(\sqrt{M_1/M_0}+\sqrt{M_0/M_1})\ , \tag{16a}$$

$$\nu=\tfrac{1}{2}(\sqrt{M_1/M_0}-\sqrt{M_0/M_1})\ , \tag{16b}$$

should be written, in a second step, in terms of $\hat{b}'(t)$, $\hat{b}'^\dagger(t)$ for $t>t_2$. However, since

$$\hat{b}'(t)=\mu'\hat{b}(t)+\nu'\hat{b}^\dagger(t)\ , \tag{17a}$$

$$\hat{b}'^\dagger(t)=\nu'\hat{b}(t)+\mu'\hat{b}^\dagger(t)\ , \tag{17b}$$

with [cf. Eqs. (8) and (16)]

$$\mu'=\tfrac{1}{2}(\sqrt{M_2/M_1}+\sqrt{M_1/M_2})\ , \tag{18a}$$

$$\nu'=\tfrac{1}{2}(\sqrt{M_2/M_1}-\sqrt{M_1/M_2})\ , \tag{18b}$$

we obtain, from Eqs. (15)–(17).

$$\hat{a}_1=\tfrac{1}{2}(\hat{a}+\hat{a}^\dagger)=\tfrac{1}{2}(\mu-\nu)(\mu'-\nu')[\hat{b}'(t)+\hat{b}'^\dagger(t)]\ . \tag{19}$$

Following the same procedure, we find for the quadrature $\hat{a}_2$

$$\hat{a}_2=\frac{1}{2i}(\hat{a}-\hat{a}^\dagger)=\frac{1}{2i}\{(\mu+\nu)(\mu'+\nu')[\hat{b}'(t)-b'^\dagger(t)]\}\ . \tag{20}$$

Now, if one assumes that $M_2=M_0$, which means that the system returns to its original mass M_0, then we have [cf. Eqs. (16) and (18)] $\mu'=\mu$, $\nu'=-\nu$, and from Eqs. (19) and (20) we obtain (with $\mu^2-\nu^2=1$)

$$\hat{a}_1=\tfrac{1}{2}(\mu-\nu)(\mu+\nu)(\hat{b}'+\hat{b}'^\dagger)=\tfrac{1}{2}(\hat{b}'+\hat{b}'^\dagger) \tag{21a}$$

and

$$\hat{a}_2=\frac{1}{2i}(\hat{b}'-\hat{b}'^\dagger)\ . \tag{21b}$$

In this case for the final state $|\beta'\rangle$, which is an eigenstate of the operator $\hat{b}'$, for $t>t_2$, we obtain

$$(\Delta\hat{a}_1)^2=\tfrac{1}{4}\ , \tag{22a}$$

$$(\Delta\hat{a}_2)^2=\tfrac{1}{4}\ , \tag{22b}$$

$$(\Delta\hat{a}_1)(\Delta\hat{a}_2)=\tfrac{1}{4}\ , \tag{22c}$$

which shows that the system returns from the squeezed state $|\beta\rangle$ to a coherent state $|\beta'\rangle$, thus characterizing exactly the reciprocal of squeezing and, to be consistent with nomenclature, it should be called *antisqueezing*.

This does not mean that the final coherent state $|\beta'\rangle$ coincides with the initial coherent state $|\alpha\rangle$ and, of course, a point deserving a mention is that concerning the Berry phase for such an evolving system. In this context, a recent paper by Mizrahi [11] has studied the Berry phase (without assuming the adiabatic hypothesis) through the use of the invariants of Lewis and Riesenfeld [12] for time-dependent Hamiltonians as in the present case.

Since our model following Eq. (14) led to a reversible squeezing, its control through the mass change, when available, leads one to think of the possibility of obtaining *modulation of squeezing*. If such an effect is of practical or theoretical interest [13], then the results obtained from Eq. (14) turn out to also be a motivation in this direction.

As a final remark, we should mention that an alternative procedure to diagonalize our time-dependent Hamiltonian can be implemented by employing the transformation $(\hat{q},\hat{p})\rightarrow(\hat{Q}(t),\hat{P}(t))$ as [14]

$$\hat{Q}(t)=\left[\frac{M(t)}{M_0}\right]^{1/2}\hat{q}\ , \tag{23}$$

$$\hat{P}(t)=\left[\frac{M_0}{M(t)}\right]^{1/2}\hat{p}\ . \tag{24}$$

This leads to the time-independent Hamiltonian

$$\hat{H}(\hat{Q},\hat{P})=\frac{\hat{P}^{\,2}}{2M_0}+\tfrac{1}{2}M_0\omega_0^2\hat{Q}^{\,2} \tag{25}$$

with subsequent application of the Dirac diagonalization formalism in terms of a new annihilation operator $\hat{b}\,'\neq\hat{b}$ yielding, however, the same fluctuations as calculated in the present approach [cf. Eqs. (11)–(13) and (22)].

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