Transparent potential for the one-dimensional Dirac equation

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For the Dirac equation in one dimension, a transparent potential is constructed such that the transmission probability is unity at all energies. The potential is related to a soliton-type solution of the nonlinear Dirac equation with a specific type of nonlinearity that exhibits "supersymmetry."

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To begin, let us review the nonrelativistic version of the problem that we are going to examine. Consider the Schrödinger equation in one dimension

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+U(x)\right]\psi(x)=\varepsilon\psi(x) . \tag{1}$$

It is understood that $U(x)=0$ if $|x|$ is sufficiently large. If $U(x)$ is such that a wave of any shape incident on the potential, say, from the left, is transmitted to the right without reflection, then $U(x)$ is said to be reflectionless or transparent. If the incident wave is a plane wave, the transparency should hold for any wave number or energy. An elegant method for constructing transparent potentials was given a long time ago by Kay and Moses [1]. Later an intriguing relationship between the transparent potentials for the Schrödinger equation and the soliton solutions of nonlinear Schrödinger equations was found by Nogami and Warke [2]. They considered the multicomponent nonlinear Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}-\sum_{\nu=1}^{n}g_\nu\phi_\nu^2(x)\right]\phi_\mu(x)=\varepsilon_\mu\phi_\mu(x) , \tag{2}$$

where $g_\nu>0$ and n is the number of the components of the Schrödinger wave function. They showed exact one-to-one correspondence between the transparent potentials constructed by Kay and Moses and the potentials defined by

$$U(x)=-\sum_{\nu=1}^{n}g_\nu\phi_\nu^2(x) , \tag{3}$$

with $n=1,2,3,\ldots$. For each value of n, this $U(x)$ can be identified with one of the transparent potentials, and vice versa.

In the simplest case of $n=1$, we find

$$\phi(x)=(\kappa/2)^{1/2}\mathrm{sech}(\kappa x),\quad \varepsilon_1=-(\hbar^2/2m)\kappa^2 , \tag{4}$$

$$U(x)=-g\phi^2(x),\quad g=2\hbar^2\kappa/m . \tag{5}$$

When this $U(x)$ is used in Eq. (1), the transmission probability turns out to be unity for any incident energy [3,4]. Note that ε_1 of Eq. (4) is fixed for a given value of g, whereas ε of Eq. (1) for the transmission problem represents an arbitrary continuum energy. We do not know of any obvious intuitive reason why the soliton solutions of this specific type should be related to transparent potentials [5].

The purpose of this Brief Report is to point out a similar situation in a relativistic model. In this model the Schrödinger equation (1) is replaced with the Dirac equation in one space dimension

$$[\alpha p+\beta m+\beta S(x)+V(x)]\psi(x)=E\psi(x) , \tag{6}$$

where and henceforth $c=\hbar=1$ and $p=-id/dx$. The relativistic energy E includes the rest mass m, which we assume to be nonzero. The wave function ψ is a two-component spinor. The α and β are 2×2 Pauli matrices, which we choose to be $\alpha=\sigma_y$ and $\beta=\alpha_z$. For the potential $\beta S+V$, which we specify below, we solve Eq. (6) for the transmission problem.

The relativistic version of Eq. (2) (with $n=1$, for simplicity) is the following nonlinear Dirac equation:

$$\{\alpha p+\beta m-\beta g_s(\phi^\dagger\beta\phi)-g_v[\phi^\dagger\phi-\alpha(\phi^\dagger\alpha\phi)]\}\phi(x) =E_B\phi(x) . \tag{7}$$

This equation appears in the massive Thirring model, which in turn is related to the nonlinear σ model in one time and one space dimension [6]. Equation (7) has a soliton-type bound-state solution [6–8]. Since $\alpha=\sigma_y$, αp is real; this allows ϕ to be a real function of x such that $(\phi^\dagger\alpha\phi)=0$. The question that we now ask is: When used in Eq. (6), is the potential

$$\beta S+V=-\beta g_s(\phi^\dagger\beta\phi)-g_v\phi^\dagger\phi \tag{8}$$

transparent? The S is a Lorentz scalar and V is the zeroth component of a Lorentz vector. We will see that potential (8) is transparent if and only if $g_v=0$.

In solving Eq. (7) it is convenient to express the two components of ψ in terms of two real functions $\eta(x)$ and $\theta(x)$,

$$\phi(x)=\eta(x)\begin{bmatrix}\cos\theta(x)\\ \sin\theta(x)\end{bmatrix} . \tag{9}$$

Then $\phi^\dagger\phi=\eta^2$ and $\phi^\dagger\beta\phi=\eta^2\cos2\theta$. Equation (7) becomes

$$\frac{d\eta}{dx}=(m-g_s\eta^2\cos2\theta)\eta\sin2\theta , \tag{10}$$

$$\frac{d\theta}{dx}=(m-g_s\eta^2\cos2\theta)\cos2\theta-E_B-g_v\eta^2 . \tag{11}$$

The above two equations can be combined to yield

$$\eta^2[(2m - g_s\eta^2\cos 2\theta)\cos 2\theta - 2E_B - g_v\eta^2] = \text{const} . \quad (12)$$

We are interested in localized solutions such that $\eta(x) \to 0$ as $|x| \to \infty$; then the constant in Eq. (12) is zero. We also assume that $E_B \to m$ as the interaction is switched off. There is only one such solution that is given by

$$\tan\theta(x) = -\frac{\kappa \tanh(\kappa x)}{m + E_B}, \quad \kappa = (m^2 - E_B^2)^{1/2} , \quad (13)$$

$$\eta^2(x) = \frac{2\kappa^2[E_B + m\cosh(2\kappa x)]}{g_s[m + E_B\cosh(2\kappa x)]^2 + g_v[E_B + m\cosh(2\kappa x)]^2} . \quad (14)$$

Since Eq. (7) is nonlinear, the normalization of ϕ is nontrivial. We impose the condition

$$\int_{-\infty}^{\infty} dx\, \phi^\dagger\phi = \int_{-\infty}^{\infty} dx\, \eta^2 = 1 , \quad (15)$$

which leads to the energy eigenvalue

$$E_B = \frac{m\cos\zeta}{[1 + (g_s/g_v)\sin^2\zeta]^{1/2}}, \quad 2\zeta = [g_v(g_s + g_v)]^{1/2} . \quad (16)$$

If $g_v \to 0$, Eq. (16) is reduced to

$$E_B = m[1 + (g_s^2/4)]^{-1/2} . \quad (17)$$

The scalar potential $S = -g_s\eta^2\cos 2\theta$ in this special case becomes

$$S(x) = \frac{-2\kappa^2}{m + E_B\cosh(2\kappa x)} . \quad (18)$$

The above expressions for the solution have been transcribed from Ref. [7] by putting $N=2$ and rewriting g_S, g_V, λ_R, and ε_R as g_s, $-g_v$, κ, and E_B, respectively [9]. The E_B of Eqs. (16)–(18) is fixed for given values of g_s and g_v; this E_B should not be confused with E of Eq. (6), which represents the (arbitrary) energy of the incident particle.

Let us denote the upper and lower components of ψ by ψ_1 and ψ_2, respectively. Then Eq. (6) can be reduced to a Schrödinger-like equation by eliminating ψ_2 in favor of ψ_1. More conveniently, if we define χ by

$$\chi = \psi_1/\sqrt{D} , \quad D = m + E + S - V , \quad (19)$$

Eq. (6) becomes

$$\left[-\frac{1}{2m}\frac{d^2}{dx^2} + W \right]\chi = \varepsilon\chi, \quad \varepsilon = \frac{E^2 - m^2}{2m} , \quad (20)$$

where W is given by

$$W(x) = S + \frac{2EV + S^2 - V^2}{2m} + \frac{1}{8mD^2}\left[3\left[\frac{dD}{dX}\right]^2 - 2D\frac{d^2D}{dx^2} \right] . \quad (21)$$

It can be shown that $D(x)$ has no zero for any values of the g's; hence W of Eq. (21) is well defined [10].

Note that W is energy dependent. We can have a feel for the energy dependence from Table I, which shows the value of $W(0)$ for $E/m = 1.0$, 1.5, and 2.0. For the values of the g's, if we specify κ ($<m$), or equivalently E_B, and the ratio g_s/g_v, then g_s and g_v are determined. Let us arbitrarily assume that $\kappa^2/m^2 = 0.1$ or $E_B/m = 1.0513\ldots$. For the ratio g_s/g_v, we consider the following three cases.

Case A. Pure V: $g_s = 0, g_v \neq 0$.
Case B. Mixture of S and V: $g_s = g_v \neq 0$.
Case C. Pure S: $g_s \neq 0, g_v = 0$.

The $W(0)$ depends on E no matter how g_s and g_v are chosen. The energy dependence of W decreases from A to B, and it becomes very small in C, but it does not disappear. In fact, the energy dependence of W cannot be eliminated by any choice of functions $S(x)$ and $V(x)$. Potential C turns out to be exactly transparent, but it is different from any of those known transparent potentials (for the Schrödinger equation) that are energy independent.

We have numerically solved Eq. (6) with the potential of Eq. (8), and calculated the transmission probability. Actually we did this by means of Eq. (20). For the potential, we considered the three cases A, B, and C, as defined above. Figure 1 shows the transmission probability $|T|^2$ versus $(E-m)/m$ for the three potentials. Potential B is more transparent than A, and potential C is completely transparent. Although we show results only for $\kappa^2 = 0.1m^2$, we have confirmed that the transparency of potential C holds for any choice of the value of κ^2 ($<m^2$). Let us add that, for potential C, which is a Lorentz scalar, there is exact symmetry between positive- and negative-energy solutions of Eq. (6). Hence the transparency holds for negative energies ($E < -m$) as well. When $E < -m$, D of Eq. (19) becomes negative, but Eqs. (20) and (21) can still be used as such.

In the transmission problem with the Schrödinger equation, the transmission probability $|T|^2$ in general vanishes at threshold. The result that $|T|^2 = 1$ even at threshold implies that there is a bound state at threshold [11]. To be more precise, this bond state is what Aktosun and Newton called the "half-bound" state [12]. We found that potential C indeed has such a bound state. We first confirmed that potential $(1+\lambda)W(x, E=m)$, where $0 < \lambda \ll 1$, has two bound states, the ground state with $\varepsilon \simeq -\kappa^2/2m$ and the first excited state with $\varepsilon \simeq 0$. Then we examined how the two states vary as $\lambda \to 0$. Recall that $U(x)$ of Eq. (5) also has a bound state at threshold [4].

For the energy range shown in Fig. 1, the kinetic ener-

TABLE I. The energy dependence of $W(0)$, in units of m, of Eq. (21). Three sets A, B, and C of the values of the g's have been chosen. In all cases, $\kappa^2/m^2 = 0.1$.

$(E-m)/m$	0.0	0.5	1.0
$A(g_s=0, g_v\neq 0)$	-0.1054	-0.1572	-0.2088
$B(g_s=g_v\neq 0)$	-0.1027	-0.1283	-0.1540
$C(g_s\neq 0, g_v=0)$	-0.1	-0.0994	-0.0991

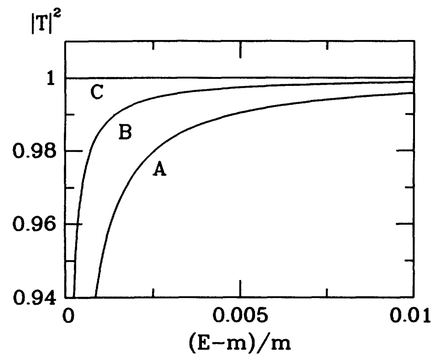

FIG. 1. The transmission probability $|T|^2$ vs $(E-m)/m$ for the three potentials A, B, and C, defined in the text. Potential C is exactly transparent, i.e., $|T|^2=1$ at all energies.

gy $E-m$ is less than $0.01m$, much smaller than the depth of the potential. The latter is about $0.1m$; see the column for $(E-m)/m=0$ of Table I. In this sense, potentials A and B are nearly transparent. At threshold, however, potentials A and B become completely opaque, i.e., $|T|^2=0$. For each of the potentials A and B, there are two bound states. Unlike potential C, however, the excited state near threshold has a finite (although very small) binding energy. If S and V, and hence κ, are all much smaller than m, and if $E\simeq m$, the relativistic system is reduced to its nonrelativistic counterpart; potentials A, B, and C are all reduced to the nonrelativistic potential $U(x)$ of Eq. (5), which is transparent. In this sense the opaqueness of potentials A and B is due to relativistic effects.

In the special case of $V=0$, the Dirac equation (6) obtains "supersymmetry" and can be reduced to the following two uncoupled equations [13]:

$$\left[-\frac{1}{2m}\frac{d^2}{dx^2}+U_\pm\right]\psi_\pm=\varepsilon\psi_\pm\ , \tag{22}$$

where $\psi_\pm=(\psi_1\pm\psi_2)/\sqrt{2}$, and

$$U_\pm(x)=S+\frac{1}{2m}\left[S^2\pm\frac{dS}{dx}\right]. \tag{23}$$

If we substitute S of Eq. (18), we find

$$U_\pm(x)=-\frac{2\kappa^2E_B[E_B+m\cosh(2\kappa x)\mp\kappa\sinh(2\kappa x)]}{m[m+E_B\cosh(2\kappa x)]^2}\ . \tag{24}$$

In contrast to W of Eq. (21), $U_\pm$ are independent of energy E. The $U_\pm$ are neither even nor odd with respect to x, but $U_+(x)=U_-(-x)$. It follows from time-reversal invariance that the transmission coefficient for U_+ is the same as that for U_-. If $\kappa/m\to0$ and $E_B\to m$, $U_\pm$ of Eq. (24) are both reduced to $U(x)$ of Eq. (5). We examined Eq. (22) numerically and confirmed the transparency of $U_\pm$ [14,15]. Equation (22) is of the form of the Schrödinger equation. Therefore one might expect that the transparent $U_\pm$ belong to the family of Kay and Moses's potentials. This does not seem to be the case, however. With the restriction that the transparent potential supports only one bound state of a finite binding energy, Kay and Moses's method uniquely leads to $U(x)$ of Eq. (5), except that the origin ($x=0$) can be chosen arbitrarily. The $U(x)$ of Eq. (5) and $U_\pm(x)$ of Eq. (24) are all transparent, energy independent, and have the common feature of two bound states, with $\varepsilon=-\kappa^2/2m$ and $\varepsilon=0$, respectively. The solution of the inverse scattering problem is not unique in this situation.

In summary, we found for the one-dimensional Dirac equation (6) that the pure Lorentz scalar potential $S=-g_s(\phi^\dagger\beta\phi)$ is transparent. Here, ϕ is the bound-state solution of the nonlinear Dirac equation (7) with $g_v=0$. An admixture of a Lorentz vector potential mars the exact transparency. We suspect that the problem of transparency is relevant to the question of whether the soliton-type solution of Eqs. (13)–(17) is entitled to be called a "soliton" [8]. It would be interesting to examine soliton-soliton collisions with the nonlinear Dirac equation. We can construct more complex solitonlike solutions in the Dirac version of the multicomponent Schrödinger equation ($n>1$). Such a generalization would also be interesting.

Note added in proof. Subsequent to completion of this work we realized that the $U_\pm(x)$ of Eq. (24) do in fact belong to the family of Kay and Moses's potentials. In this situation an analytical solution of the problem is feasible. We plan to present such an analysis in a forthcoming publication.

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[5] As an example of a more complex nonlinearity one can add a term like ϕ_v^4 to the potential in Eq. (2). For the one-component case ($n=1$), the potential then reads $-(g\phi^2+g'\phi^4)$. The Schrödinger equation for ϕ with this potential has a stationary-state solution. See Y. Nogami

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[8] Here we use the word "soliton-type" rather than "soliton" because we do not know, when two such soliton-type objects colide against each other, whether or not they emerge from the collision having the same shapes and velocities with which they entered.

[9] Equation (13) corresponds to Eq. (3.11) of Ref. [7]; the latter contains typographical errors that have been corrected here. In Ref. [7], a combination of attractive S and repulsive V was considered. In the present Brief Report we are assuming that S and V are both attractive. This is why we have replaced g_V of Ref. [7] with $-g_v$.

[10] Instead of eliminating ψ_2 in favor of ψ_1, we can eliminate ψ_1 and obtain another Schrödinger-like equation. The effective potential W' in this equation is obtained from W of Eq. (21) by reversing both of the signs of E and V. The W and W' are very different; nevertheless they lead to the same results.

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