# Nonadiabatic Berry phase in rotating systems

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The time-dependent Schrödinger equation for a particle in general rotating systems is solved analytically, and the exact solutions are used to study the nonadiabatic Berry phase. It is shown that the nonadiabatic Berry topological phase appears in general rotating systems for purely mechanical reasons and is observable. An alternative expression for the nonadiabatic Berry phase is given and discussed.

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The adiabatic Berry topological phase [1] has been studied extensively by both theoretical [2–7] and experimental [8–12] physicists in recent years, and much knowledge and deep insights have been obtained in connection with physical problems such as the Aharonov-Bohm effect [3], non-Abelian gauge theories [4], chiral anomalies [5] and fractional statistics [6], and the quantum Hall effect [7]. In a fundamental generalization of Berry's results, Aharonov and Anandan [13] have dropped the adiabatic condition and studied the Berry phase for any cyclic evolution by identifying the time integral of the expectation value of the Hamiltonian as the dynamical phase. A noncyclic and nonunitary generalization has also been studied [14]. The existence of the nonadiabatic Berry phase has been verified experimentally [15–17]. It turns out [13,18,19] that a nonadibatic Berry topological phase defined for any path $[0,T]\rightarrow H$ in the Hilbert space $H$ is the same for all paths in $H$ which project to a given closed curve in the projective Hilbert space $P$. This is a very important conclusion. However, generally speaking, such a definition is too general and formal to provide a tractable calculation of the nonadiabatic Berry phase, since it is related to the dynamical effect on the adiabatic Berry topological phase as a whole physical problem [19]. In other words, the concrete study of the nonadiabatic Berry topological phase depends on the specific structure of the Hamiltonian of the systems.

A general rotating system is a practical model that is of both theoretical [20–22] and experimental [23] interest, especially in its quantum-interference effects and related phenomenon. The analytical solution of a particle in general rotating systems can be obtained and the theoretical predictions can be tested by experiments. We have obtained the adiabatic solution of the rotating system and discussed a possible experiment to probe Berry topological phase in a previous publication [22]. In this paper we will solve the problem exactly and use it to study further the nonadiabatic effect on Berry's topological phase. Now consider a dynamical system with a Hamiltonian in an inertial frame $\mathbf{r}'=(x',y',z')$ given by

$$\hat{H}'=-\frac{\hbar^2}{2m}\nabla_{\mathbf{r}'}^2+U(r') , \tag{1}$$

with $U(r')$ a central-field potential. By means of a canonical transformation in the active sense, one can, from Eq. (1), obtain exactly a time-dependent Hamiltonian [20]

$$\hat{H}=\frac{\hbar^2}{2m}\nabla_{\mathbf{r}}^2+U(r)-\mathbf{w}(t)\cdot\hat{\mathbf{L}} , \tag{2}$$

in a reference frame $\mathbf{r}=(x,y,z)$ attached to a physical system that is being rotated at any varying angular velocity $\mathbf{w}(t)$ about a fixed center related to the inertial frame. In Eq. (2),

$$\hat{\mathbf{L}}=(\hat{L}_x,\hat{L}_y,\hat{L}_z) \tag{3}$$

is the angular momentum of the system. The equation of motion for a particle is

$$i\hbar\frac{\partial\Psi_l(t)}{\partial t}=\hat{H}(t)\Psi_l(t) . \tag{4}$$

Let

$$\Psi_l(t)=\exp\left[-\frac{i}{\hbar}E_l t\right]\exp\left[-\frac{i}{\hbar}\hat{L}_z\beta(t)\right]\times\exp\left[-\frac{i}{\hbar}\hat{L}_y\alpha(t)\right]R_l(r)\psi_l(t) , \tag{5}$$

then the equation of motion for $\psi_l(t)$ is

$$i\hbar\frac{\partial\psi_l(t)}{\partial t}=\hat{h}(t)\psi_l(t) . \tag{6}$$

In Eq. (5) the radial functions $R_l(r)$ are the eigenfunctions of $\hat{H}'$ with eigenvalues $E_l$ and in Eq. (6) the operator $\hat{h}(t)$ is

$$\hat{h}(t)=-\Omega(t)\hat{L}_z , \tag{7}$$

where

$$\Omega(t)=-\mathbf{w}(t)\cdot\boldsymbol{\lambda}(t)+\frac{d\beta(t)}{dt}\cos\alpha(t) , \tag{8}$$

with

$$\boldsymbol{\lambda}(t)=(\sin\alpha(t)\cos\beta(t),\ \sin\alpha(t)\sin\beta(t),\ \cos\alpha(t)) . \tag{9}$$

The time-dependent parameters $\alpha(t)$ and $\beta(t)$ are determined by

 

$$\frac{d\boldsymbol{\lambda}(t)}{dt}=\mathbf{w}(t)\times\boldsymbol{\lambda}(t)\ , \tag{10}$$

with the boundary condition

$$\boldsymbol{\lambda}(T)=\boldsymbol{\lambda}(0)\ , \tag{11}$$

where $T$ is the evolution period. The solutions of (4) and (6) are

$$\psi_l(t)=\exp\left[+\frac{i}{\hbar}\hat{L}_z\int_0^t\Omega(t')dt'\right]\psi_l(0)\ , \tag{12}$$

$$\Psi_l(t)=\hat{U}_l(t)R_l(r)\psi_l(0)\ , \tag{13}$$

with

$$\hat{U}_l(t)=\exp\left[-\frac{i}{\hbar}E_lt\right]\exp\left[-\frac{i}{\hbar}\hat{L}_z\beta(t)\right]\times\exp\left[-\frac{i}{\hbar}\hat{L}_y\alpha(t)\right]\times\exp\left[+\frac{i}{\hbar}\hat{L}_z\int_0^t\Omega(t')dt'\right]\ . \tag{14}$$

We now consider a cyclic solution which is related to Berry's topological phase. If the initial state is chosen as

$$\psi_{lm}(0)=Y_{lm}(0,\phi)\ , \tag{15}$$

then

$$\Psi_{lm}(t)=\exp\left[-\frac{i}{\hbar}\int_0^tE_{lm}(t')dt'\right]\times\sum_{m'}D^{(l)}_{m'm}(0,\alpha(t),0)\times\exp[-im'\beta(t)]R_l(r)Y_{lm'}(\theta,\phi)\ , \tag{16}$$

where spherical harmonics $Y_{lm}(\theta,\phi)$ are the eigenfunctions of the operator $\hat{L}_z$ with eigenvalues $m\hbar$, $D^{(l)}_{m',m}$ are the Wigner functions with rank $l$, and $E_{lm}(t)$ are given by

$$E_{lm}(t)=E_l-m\Omega(t)\hbar\ . \tag{17}$$

After one period, a nonadiabatic Berry's topological phase shift will be induced. Using Eq. (11) we have, from Eq. (16),

$$\Psi_{lm}(T)=\exp(-i\Phi_{lm})\Psi_{lm}(0)\ , \tag{18}$$

where the total phase shift is

$$\Phi_{lm}=\frac{1}{\hbar}\int_0^T\left[E_{lm}(t)+m\frac{d\beta(t)}{dt}\hbar\right]dt\ . \tag{19}$$

Following the definition given in Ref. [13], the dynamical phase is readily obtained using Eq. (16):

$$\frac{1}{\hbar}\int_0^T\langle\Psi_{lm}(t)|\hat{H}(t)|\Psi_{lm}(t)\rangle dt=\frac{1}{\hbar}\int_0^T\left(E_{lm}(t)+m\frac{d\beta(t)}{dt}\cos\alpha(t)\hbar\right)dt\ , \tag{20}$$

and the nonadiabatic Berry phase

$$\gamma_{lm}=-m\int_0^T\frac{d\beta(t)}{dt}[1-\cos\alpha(t)]dt\ . \tag{21}$$

We note from Eqs. (21) and (16) that $\gamma_{lm}$ is indeed a geometric phase associated with a curve in the (projective) Hilbert space, as claimed by Aharonov and Anandan [13]. It is interesting to note that Eq. (21) can be rewritten as

$$\gamma_{lm}=-m\Omega(C_\lambda)\ , \tag{22}$$

where $\Omega(C_\lambda)$ is the solid angle that $C_\lambda$ subtends at $\boldsymbol{\lambda}=0$. This equation indicates that the nonadiabatic Berry phase can also be regarded as a geometric property of a closed curve $C_\lambda$ in the $\boldsymbol{\lambda}$-parameter space. This result indeed is of general nature: For the dynamical systems with a time-dependent Hamiltonian of the form $\hat{H}(t)=\hat{H}(\mathbf{R}(t))$ with $\mathbf{R}(t)$ a set of parameters, removing the dynamical phase part, the evolution of the wave function $\Psi(t)$ is determined by a set of renormalized parameters $\boldsymbol{\lambda}(t)\equiv\boldsymbol{\lambda}(\mathbf{R}(t))$. In the adiabatic limit, $\boldsymbol{\lambda}(t)\cong\mathbf{R}(t)$, so that the adiabatic Berry phase can be regarded a geometric property of the $\mathbf{R}$-parameter space [1]. For our rotating systems, in this limit, i.e.,

$$\left|\frac{d}{dt}\left[\frac{\mathbf{w}}{|\mathbf{w}|}\right]\right|\ll|\mathbf{w}|\ , \tag{23}$$

we have

$$\boldsymbol{\lambda}\cong\frac{\mathbf{w}}{|\mathbf{w}|}\ , \tag{24}$$

and Eq. (22) becomes

$$\gamma_{lm}=-m\Omega(C_{\mathbf{w}})\ , \tag{25}$$

which recovers the adiabatic results [22]. In Eq. (25), $\Omega(C_{\mathbf{w}})$ is the solid angle that $C_{\mathbf{w}}$ subtends at $\mathbf{w}=0$.

Although it is straightforward to obtain Berry's topological phase from Eq. (21) [or Eq. (22)] in general nonadiabatic cases, it should be calculated numerically, since the solutions of Eqs. (9) and (10) should be obtained by numerical calculation. In what follows we proceed to illustrate analytically the above results through a simple example. Let

$$\mathbf{w}(t)=(w\sin\phi\cos(w_0t),w\sin\phi\sin(w_0t),w\cos\phi)\ . \tag{26}$$

From Eqs. (9) and (10) we get

$$\boldsymbol{\lambda}(t)=(\sin\alpha\cos(w_0t),\sin\alpha\sin(w_0t),\cos\alpha)\ , \tag{27}$$

where $\alpha$ is given by

$$\cos\alpha=\frac{w\cos\phi-w_0}{\sqrt{w^2-2ww_0\cos\phi+w_0^2}}\ , \tag{28}$$

and then, from (21),

$$\gamma_{lm}=-2m\pi(1-\cos\alpha)\ . \tag{29}$$

Obviously, Eq. (29) can be rewritten as $\gamma_{lm}=-m\Omega(C_\lambda)$.

Despite the differences between the situation considered here and the adiabatic cases [22], the nonadiabatic Berry phase is, in principle, observable in general rotating systems. Following the experimental program suggested in Ref. [22], we can extract the nonadiabatic Berry phase easily from the data by varying appropriately the initial state and identifying the time integral of expectation value of the Hamiltonian [Eq. (20)] as the dynamical phase.

In conclusion, the nonadiabatic Berry's topological phase in general rotating systems has been studied in detail. We have solved analytically the time-dependent Schrödinger equation for a particle in general rotating systems. It has been shown that the nonadiabatic Berry phase appears in general rotating systems purely due to mechanical effects and is observable. An alternative expression for the nonadiabatic Berry phase is given and discussed.

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