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Wigner distribution function of a simple optical system: An extended-phase-space approach

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We construct the Wigner distribution function for a coupled boson-fermion system. This function is regarded as a certain "superfield" defined on the extended phase space, the points of which are labeled by the anticommuting canonical c variables together with the ordinary ones. In this approach, the variables of a fermion and a boson are treated on a completely equal footing, and the phase-space representation is fully realized. We apply the formalism to the kinetic theory of the optical Dicke model with a two-level atom, and show how systematically a set of the generalized Fokker-Planck equations, which describes the effective dynamics of the radiation field, is derived from a single superfieldlike equation.

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About 60 years ago, it was found by Wigner [1] that the quantum expectation value of an observable can be represented as the statistical average of a corresponding classical physical quantity with respect to a certain distribution function on phase space. Since then, a great many works have been done on the phase-space representations from both fundamental and practical aspects of quantum theory. Various types of the representations and a class of phase-space distribution functions have been proposed in the literature [2]. Among others, Wigner's original quasiprobability distribution function (WDF) is most celebrated and is widely used in semiclassical studies of quantum systems.

One of the most remarkable features in the phase-space representation theories is that they do not contain any kinds of q-number quantities (i.e., matrices or operators) but do deal only with c-number variables.

On the other hand, in spite of the fact that a large variety of microscopic systems in nature are composed of interacting bosons and fermions, studies of phase-space representations seem to have been done mainly on the purely bosonic systems [3]. This might be due to the fact that fermions are described by the algebra of anticommutation relations, and, therefore, their corresponding classical phase space cannot be labeled naively by the ordinary real or complex canonical variables. A typical example is found in quantum-optical systems with two-level atoms. Klenner, Doucha, and Weis [4] proposed a simple and useful method, called the Wigner matrix formalism, in order to investigate the phase-space dynamics of the radiation field in the Dicke model [5]. Their distribution function contains a linear combination of the Pauli matrices describing the two-level atom. Therefore, due to such a matrix structure, the atom is not represented by c-number variables and the phase-space representation is not fully realized in their approach.

In a recent paper [6], we formulated the WDF of a second-quantized fermion by introducing the extraordinary phase space. The coordinate points of this phase space are labeled by a pair of anticommuting canonical c variables [7] (Grassmann variables, or G variables, for brevity). The fermionic analog of the Wigner operator was given and the Weyl correspondence was established. At the same time, we found some qualitative differences between bosons and fermions in their statistical phase-space descriptions.

In this Brief Report, we generalize our previous discussion to coupled boson-fermion systems, and show that the full realization of the phase-space representation is possible for optical models with two-level atoms. The constructed WDF may be regarded as a certain "superfield" [8] in a sense that the extended phase space, on which that function is defined, is now labeled by the fermionic G variables as well as the ordinary bosonic ones. We apply this formalism to the kinetic theory of the Dicke model. The atom variables are expressed in terms of a pair of fermionic creation and annihilation operators accompanied by a real Clifford c number. We find that inclusion of such a c number in the Hamiltonian requires slight modifications of the original formalism for a second-quantized fermion. We also show how systematically a set of the generalized Fokker-Planck equations, describing the effective phase-space dynamics of the radiation field, is derived from a single superfieldlike equation for the total WDF.

First, we recapitulate the basics of the phase-space representation of a second-quantized fermion proposed in Ref. [6].

With the system density operator $\hat{\rho}_f$, the WDF of the fermion is written as

$$W_f(\beta,\beta^*)=\mathrm{Tr}_g[\hat{\rho}_f\hat{\Delta}_f(\beta,\beta^*)] . \tag{1}$$

$\hat{\Delta}_f(\beta,\beta^*)$ is the fermionic analog of the ordinary Wigner operator [9], and is defined by the G Fourier transform of the G variant unitary displacement operator $\hat{D}(\zeta)=\exp(\hat{b}^\dagger\zeta-\zeta^*\hat{b})$:

$$\hat{\Delta}_f(\beta,\beta^*)=\int d^2\zeta \exp[-(\beta^*\zeta-\zeta^*\beta)]\hat{D}(\zeta) . \tag{2}$$

Here, $\hat{b}^\dagger$ and $\hat{b}$ are, respectively, the creation and annihilation operators of a fermion obeying the anticommutation relations $\{\hat{b},\hat{b}^\dagger\}=1$, $\{\hat{b}^\dagger,\hat{b}^\dagger\}=\{\hat{b},\hat{b}\}=0$. β^* and β are the corresponding classical phase-space G variables. The Berezin integrations over G numbers [7] are normalized as follows:

$$\int d^2\zeta\zeta\zeta^*=1, \quad \int d^2\zeta\zeta=\int d^2\zeta\zeta^*=0, \quad \int d^2\zeta=0 , \tag{3}$$

where $d^2\zeta\equiv d\zeta^* d\zeta$. The graded trace operation in Eq. (1) is defined by

$$\mathrm{Tr}_g(\hat{A})=\sum_{n=0,1}(-1)^n\langle n|\hat{A}|n\rangle=\int d^2\zeta\langle\zeta|\hat{A}|\zeta\rangle . \tag{4}$$

$\{|0\rangle,|1\rangle=\hat{b}^\dagger|0\rangle\}$ denotes the Fock basis, and $|\zeta\rangle\equiv\hat{D}(\zeta)|0\rangle$ is the normalized fermionic coherent state satisfying

$$\hat{b}|\zeta\rangle=\zeta|\zeta\rangle, \quad \int d^2\zeta|\zeta\rangle\langle\zeta|=1 . \tag{5}$$

The operation (4) has the following cyclicity property:

$$\mathrm{Tr}_g(\hat{A}\hat{B})=\pm\mathrm{Tr}_g(\hat{B}\hat{A}) , \tag{6}$$

provided that the sign $-(+)$ is taken from the case that both $\hat{A}$ and $\hat{B}$ are odd fermionic operators (otherwise). [Note, however, the discussions below Eq. (19).]

The Wigner operator (2) has the properties

$$\mathrm{Tr}_g[\hat{\Delta}_f(\beta,\beta^*)\hat{\Delta}_f(\beta',\beta'^*)]=\delta^{(2)}(\beta-\beta') , \tag{7}$$

$$\mathrm{Tr}[\hat{\Delta}_f(\beta,\beta^*)\hat{\Delta}_f(\beta',\beta'^*)]=\tfrac{1}{2}\exp(2\beta\beta'^*+2\beta^*\beta') , \tag{8}$$

with the G δ function defined by $\delta^{(2)}(\beta-\beta')\equiv(\beta-\beta')(\beta^*-\beta'^*)$. Equation (7) shows that $\hat{\Delta}_f$ forms a complete set in the space of relevant operators. For example, one can verify that Eq. (1) is inverted as

$$\hat{\rho}_f=\int d^2\beta\, W_f(\beta,\beta^*)\hat{\Delta}_f(\beta,\beta^*) . \tag{9}$$

It is important to see that there exists the Weyl correspondence [10] also in fermion theory:

$$A(\beta,\beta^*)=\mathrm{Tr}_g[\hat{A}_w(\hat{b},\hat{b}^\dagger)\hat{\Delta}_f(\beta,\beta^*)] , \tag{10a}$$

$$\hat{A}_w(\hat{b},\hat{b}^\dagger)=\int d^2\beta\, A(\beta,\beta^*)\hat{\Delta}_f(\beta,\beta^*) , \tag{10b}$$

where $\hat{A}_w(\hat{b},\hat{b}^\dagger)$ and $A(\beta,\beta^*)$ are a Weyl-ordered operator and its corresponding physical quantity in phase space, respectively. We note that these correspondence relations can be established *not with the standard trace operation but with the graded trace operation* (4).

From Eqs. (8) and (9), the quantum expectation value is found to be represented by the phase-space average as

$$\langle\hat{A}_w\rangle\equiv\mathrm{Tr}(\hat{\rho}_f\hat{A}_w)=\int d^2\beta\, W_f(\beta,\beta^*)\tilde{A}(\beta,\beta^*) . \tag{11}$$

$\tilde{A}(\beta,\beta^*)$ is not directly equal to $A(\beta,\beta^*)$, but is given by its G Fourier transform

$$\tilde{A}(\beta,\beta^*)=\tfrac{1}{2}\int d^2\zeta\exp[2(\beta^*\zeta-\zeta^*\beta)]A(\zeta,\zeta^*) . \tag{12}$$

This result should be compared with boson theory where the expectation value is directly connected to the average of the phase-space quantity with respect to the WDF. Thus we see one of the essential differences between fermions and bosons in their phase-space representations.

Introduction of a bosonic degree of freedom is straightforward. The total WDF is now written as

$$W(\alpha,\alpha^*,\beta,\beta^*)=\mathrm{Tr}_g[\hat{\rho}\hat{\Delta}_b(\alpha,\alpha^*)\hat{\Delta}_f(\beta,\beta^*)] , \tag{13}$$

where $\hat{\Delta}_b(\alpha,\alpha^*)$ is the ordinary bosonic Wigner operator [9]

$$\hat{\Delta}_b(\alpha,\alpha^*)=\frac{1}{\pi^2}\int d^2z\exp[z^*(\alpha-\hat{a})-z(\alpha^*-\hat{a}^\dagger)] , \tag{14}$$

with $d^2z\equiv d(\mathrm{Re}z)d(\mathrm{Im}z)$. α^* and α are the complex classical phase-space variables that correspond to the boson creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ obeying the commutation relations $[\hat{a},\hat{a}^\dagger]=1$, $[\hat{a},\hat{a}]=[\hat{a}^\dagger,\hat{a}^\dagger]=0$. In Eq. (13), the standard trace operation is understood for the boson part.

The WDF (13) is defined on the extended phase space labeled by the fermionic G variables as well as the ordinary bosonic variables, and, therefore, one may regard it as a certain "superfield" [8].

For the later discussion, we present here some of the basic formulas of the operator correspondence relations:

$$\hat{a}^\dagger\hat{\Delta}_b=\hat{\Delta}_b\left[\alpha^*+\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\alpha}\right], \quad \hat{a}\hat{\Delta}_b=\hat{\Delta}_b\left[\alpha-\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\alpha^*}\right] , \tag{15a}$$

$$\hat{\Delta}_b\hat{a}^\dagger=\left[\alpha^*-\frac{1}{2}\frac{\overrightarrow{\partial}}{\partial\alpha}\right]\hat{\Delta}_b, \quad \hat{\Delta}_b\hat{a}=\left[\alpha+\frac{1}{2}\frac{\overrightarrow{\partial}}{\partial\alpha^*}\right]\hat{\Delta}_b ,$$

$$\hat{b}^\dagger\hat{\Delta}_f=\hat{\Delta}_f\left[\beta^*+\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\beta}\right], \quad \hat{b}\hat{\Delta}_f=\hat{\Delta}_f\left[\beta+\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial\beta^*}\right] , \tag{15b}$$

$$\hat{\Delta}_f\hat{b}^\dagger=\left[\beta^*+\frac{1}{2}\frac{\overrightarrow{\partial}}{\partial\beta}\right]\hat{\Delta}_f, \quad \hat{\Delta}_f\hat{b}=\left[\beta+\frac{1}{2}\frac{\overrightarrow{\partial}}{\partial\beta^*}\right]\hat{\Delta}_f .$$

In Eqs. (15b), the operations $\overrightarrow{\partial}/\partial\beta$ and $\overleftarrow{\partial}/\partial\beta$ with respect to the G variables denote, respectively, the left and right differentiations [7].

Next, we examine this formalism for the kinetic theory of a simple optical system. Let us discuss it by employing the Dicke model [5] of a two-level atom interacting with a monochromatic radiation field. The system Hamiltonian is given by

$$\hat{H}=\Omega\hat{a}^\dagger\hat{a}+\tfrac{1}{2}\omega\hat{\sigma}_z+g(\hat{a}^\dagger+\hat{a})\hat{\sigma}_x \ . \qquad (16)$$

The creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ and the Pauli matrices $\hat{\sigma}$ denote the radiation field with frequency Ω and the atom of level distance ω, respectively. g is a coupling constant.

To apply our formalism to the system described by Eq. (16), we must contrive to transfer the Pauli-matrix representation to the second-quantized oscillator representation for the atom variables. A simple method may be the substitution $\hat{\sigma}_+\to\hat{b}^\dagger$, $\hat{\sigma}_-\to\hat{b}$, $\hat{\sigma}_z\to 2\hat{b}^\dagger\hat{b}-1$ $[\hat{\sigma}_\pm\equiv\frac{1}{2}(\hat{\sigma}_x\pm i\hat{\sigma}_y)]$. This substitution preserves the Pauli-matrix algebra as long as $\hat{b}^\dagger$ and $\hat{b}$ satisfy the usual anticommutation relations. In this case, however, there appears the sum of statistically bosonic and fermionic quantities in the Hamiltonian. [Here the term "bosonic (fermionic) quantity" means that it commutes (anticommutes) with G odd numbers, e.g., β's in the preceding discussion.]

Fortunately, this difficulty can be overcome simply by introducing a real Clifford c number "c" [11] in the following way:

$$\hat{\sigma}_+\to c\hat{b}^\dagger, \quad \hat{\sigma}_-\to\hat{b}c, \quad \hat{\sigma}_z\to 2\hat{b}^\dagger\hat{b}-1 \ , \qquad (17)$$

provided that this c number satisfies the properties $c^2=1$, $c\hat{b}+\hat{b}c=c\hat{b}^\dagger+\hat{b}^\dagger c=0$, $c\beta+\beta c=c\beta^*+\beta^* c=0$. Under this substitution, the Pauli-matrix algebra is again kept unchanged, and the Hamiltonian becomes bosonic as desired.

Thus, instead of the form (16), we employ

$$\hat{H}=\Omega\hat{a}^\dagger\hat{a}+\omega(\hat{b}^\dagger\hat{b}-\tfrac{1}{2})+g(\hat{a}^\dagger+\hat{a})(c\hat{b}^\dagger+\hat{b}c) \ . \qquad (18)$$

The Clifford c number c is also contained in the system density operator, the WDF, and the physical quantities concerning the atom, in general. Clearly, the operators $c\hat{b}^\dagger$ and $\hat{b}c$ are not even fermionic operators and, similarly, $c\beta^*$ and βc are not G even quantities, since

$$\mathrm{Tr}_g[(\hat{b}c)(c\hat{b}^\dagger)]=-\mathrm{Tr}_g[(c\hat{b}^\dagger)(\hat{b}c)] \ , \qquad (19a)$$

$$(\beta c)(c\beta^*)=-(c\beta^*)(\beta c) \ . \qquad (19b)$$

Therefore, in calculations with c, one has to pay attention to the following points: (i) the cyclicity property of the graded trace operation in Eq. (6) does not hold if *both* $\hat{A}$ and $\hat{B}$ include c irreducibly, and (ii) the order of the product in the integrand in Eq. (11) must be fixed as $W\tilde{A}$.

The time evolution of the WDF defined by Eq. (13) may be carried by the system density operator $\hat{\rho}$. In the Schrödinger picture, it is determined by the von Neumann equation

$$i\frac{\partial\hat{\rho}}{\partial t}=[\hat{H},\hat{\rho}] \ , \qquad (20)$$

and, accordingly,

$$\frac{\partial W}{\partial t}=-i\,\mathrm{Tr}_g([\hat{H},\hat{\rho}]\hat{\Delta}_b\hat{\Delta}_f)$$
$$=-i\,\mathrm{Tr}_g(\hat{\Delta}_b\hat{\Delta}_f\hat{H}\hat{\rho}-\hat{\rho}\hat{H}\hat{\Delta}_b\hat{\Delta}_f) \ . \qquad (21)$$

In the second equality, the cyclicity property of the graded trace operation has been used. This is indeed feasible, since $\hat{\Delta}_f$ is an even fermionic operator. [See the above point (i).]

By virtue of the operator correspondence relations (15), we know that there always exists the operator $\vec{\mathrm{L}}$ satisfying

$$\hat{\Delta}_b\hat{\Delta}_f\hat{H}(\hat{a},\hat{a}^\dagger,\hat{b},\hat{b}^\dagger)=-i\vec{\mathrm{L}}\left[\alpha,\alpha^*,\frac{\vec{\partial}}{\partial\alpha},\frac{\vec{\partial}}{\partial\alpha^*},\beta,\beta^*,\frac{\vec{\partial}}{\partial\beta},\frac{\vec{\partial}}{\partial\beta^*}\right]\hat{\Delta}_b\hat{\Delta}_f \ . \qquad (22)$$

With the explicit form of the Hamiltonian (18), $\vec{\mathrm{L}}$ is found to be

$$-i\vec{\mathrm{L}}=\Omega\left[\alpha^*-\frac{1}{2}\frac{\vec{\partial}}{\partial\alpha}\right]\left[\alpha+\frac{1}{2}\frac{\vec{\partial}}{\partial\alpha^*}\right]+\omega\left[\left[\beta^*+\frac{1}{2}\frac{\vec{\partial}}{\partial\beta}\right]\left[\beta+\frac{1}{2}\frac{\vec{\partial}}{\partial\beta^*}\right]-\frac{1}{2}\right]+g\left[(\alpha+\alpha^*)+\frac{1}{2}\left[\frac{\vec{\partial}}{\partial\alpha^*}-\frac{\vec{\partial}}{\partial\alpha}\right]\right]\times c\left[(\beta^*-\beta)+\frac{1}{2}\left[\frac{\vec{\partial}}{\partial\beta}-\frac{\vec{\partial}}{\partial\beta^*}\right]\right] \ . \qquad (23)$$

Thus, we have the generalized Fokker-Planck equation

$$\frac{\partial W}{\partial t}+\vec{\mathrm{L}}W+W\overleftarrow{\mathrm{L}}^*=0 \ , \qquad (24)$$

which may be regarded as a certain "superfield equation."

Now we note that the WDF depends on the fermionic variables only through the combinations βc and $c\beta^*$ [12]. Therefore, the expansion of the WDF in terms of these nilpotent variables has the following form:

$$W=W_0(\alpha,\alpha^*)+W_1^*(\alpha,\alpha^*)\beta c+W_1(\alpha,\alpha^*)c\beta^*+W_2(\alpha,\alpha^*)\beta\beta^* \ . \qquad (25)$$

Clearly, the expansion coefficients are the ordinary functions of the radiation field variables alone. W_0 and W_2 are real, while W_1 is complex. This should be compared with the usual superfield expansion [8] where some of the expansion coefficients are fermionic.

To see the physical meanings of these functions, we calculate the phase-space averages with respect to the atom variables. From Eqs. (10a) and (12), the Wigner equivalents and their G Fourier transforms are, respectively, found to be

$$1\leftrightarrow 1; \quad \tilde{1}=2\beta\beta^* \ , \qquad (26a)$$

$$\hat{\sigma}_+=c\hat{b}^\dagger\leftrightarrow c\beta^*; \quad (c\beta^*)^\sim=c\beta^* \ , \qquad (26b)$$

$$\hat{\sigma}_-=\hat{b}c\leftrightarrow\beta c; \quad (\beta c)^\sim=-\beta c \ , \qquad (26c)$$

$$\hat{\sigma}_z=2\hat{b}^\dagger\hat{b}-1\leftrightarrow 2\beta^*\beta; \quad (2\beta^*\beta)^\sim=-1 \ . \qquad (26d)$$

Then, using Eq. (11) and keeping point (ii) in mind, we obtain

$$\langle 1 \rangle_{\rm atom} = \int d^2\beta\, W(\alpha,\alpha^*,\beta,\beta^*) 2\beta\beta^* = 2W_0(\alpha,\alpha^*) \ , \tag{27a}$$

$$\langle \hat{\sigma}_+ \rangle_{\rm atom} = \int d^2\beta\, W(\alpha,\alpha^*,\beta,\beta^*) c\beta^* = W_1^*(\alpha,\alpha^*) \ , \tag{27b}$$

$$\langle \hat{\sigma}_- \rangle_{\rm atom} = -\int d^2\beta\, W(\alpha,\alpha^*,\beta,\beta^*) \beta c = W_1(\alpha,\alpha^*) \ , \tag{27c}$$

$$\langle \hat{\sigma}_z \rangle_{\rm atom} = -\int d^2\beta\, W(\alpha,\alpha^*,\beta,\beta^*) = -W_2(\alpha,\alpha^*) \ . \tag{27d}$$

Thus, the set of coefficient functions gives the phase-space representation of the *effective theory* for the radiation field.

Finally, we derive the set of the generalized Fokker-Planck equations for these functions. Substitution of expression (25) into Eq. (24) leads to an identity, which gives rise to

$$\frac{\partial W_0}{\partial t} + i\Omega \left[\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha} \right] W_0 + \frac{i}{2} g \left[\frac{\partial}{\partial \alpha^*} - \frac{\partial}{\partial \alpha} \right] (W_1 + W_1^*) = 0 \ , \tag{28a}$$

$$\frac{\partial W_1^*}{\partial t} + i \left[\Omega \left[\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha} \right] - \omega \right] W_1^* + ig \left[\frac{\partial}{\partial \alpha^*} - \frac{\partial}{\partial \alpha} \right] W_0 - ig(\alpha+\alpha^*) W_2 = 0 \ , \tag{28b}$$

$$\frac{\partial W_1}{\partial t} + i \left[\Omega \left[\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha} \right] + \omega \right] W_1 + ig \left[\frac{\partial}{\partial \alpha^*} - \frac{\partial}{\partial \alpha} \right] W_0 + ig(\alpha+\alpha^*) W_2 = 0 \ . \tag{28c}$$

$$\frac{\partial W_2}{\partial t} + i\Omega \left[\alpha^* \frac{\partial}{\partial \alpha^*} - \alpha \frac{\partial}{\partial \alpha} \right] W_2 + 2ig(\alpha+\alpha^*)(W_1 - W_1^*) = 0 \ . \tag{28d}$$

This set of equations determines the effective phase-space dynamics of the radiation field. Up to the linear combinations, it reproduces correctly that derived from the Wigner matrix formalism.

We have studied the phase-space quantum theory of a coupled boson-fermion system. We have constructed the WDF on the extended phase space and applied it to the kinetic theory of the Dicke model with a two-level atom.

Recently, the optical models with two-level atoms have been revisited in the context of supersymmetry [13]. In the present approach, the atom and radiation field variables are treated on a completely equal footing with each other. Therefore, we expect that this approach enables us to discuss such a symmetry geometrically on the phase space.

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