s-order nondiagonal quasiprobabilities

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(Received 26 November 1990; revised manuscript received 30 August 1991)

This paper presents a correspondence between the annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators and two independent complex variables α and β, which makes possible the definition of positive nondiagonal quasiprobabilities for any Cahill-Glauber s order. A generalized version of the Cahill and Glauber s-ordered displacement operator displacing the annihilation and creation operators by α and β is defined. Corresponding to this *nondiagonal* displacement, a nondiagonal ordering operator is introduced so that a map of s-ordered operators into c numbers $\hat{a}\rightarrow\alpha$, $\hat{a}^\dagger\rightarrow\beta$ can be defined. The Drummond-Gardiner projector and the Cahill-Glauber diagonal ordering operator are obtained as particular cases. In order to use the nondiagonal correspondence in the determination of quantum expectation values of observables, several families of quasiprobabilities are defined, generalizing Drummond-Gardiner (normal order and nondiagonal) positive-P-function and Cahill-Glauber (s-order and diagonal) quasiprobabilities. It is demonstrated that these nondiagonal quasiprobabilities exist as well-behaved functions where at least one is non-negative, for any state and any order. The time evolution of these quasiprobabilities is discussed both in the Schrödinger and in the Heisenberg pictures. In the Heisenberg picture, a method for obtaining c-number stochastic differential equations (SDE's) directly from operator equations using the s-order nondiagonal correspondence is described. The main difference between this method and the Langevin approach is that in the latter a diagonal correspondence is used, leading eventually to wrong results. The use of these SDE's to solve quantum optical problems is discussed, and an application to the nonlinearly damped degenerate parametric oscillator is made. In order to obtain the SDE, various (inequivalent) truncation schemes are necessary.

PACS number(s): 42.50.Dv, 03.65.Ca

I. INTRODUCTION

Similarities and differences between classical and quantum theories are particularly clear in the framework of the so-called quasiprobabilities (QP's) [1]. In order to use the QP's to perform quantum averages, an operator-ordering choice is necessary because the quantum observables are noncommuting linear operators [2]. The search for a formalism in which the state of a quantum system is expressed in terms of functions having the properties of a probability density started with the introduction of three functions: the Wigner function [3] for symmetrical combinations of $\hat{a}$ and $\hat{a}^\dagger$, which was further studied by Moyal [4]; the P function of Glauber [5,6] and Sudarshan [7] for normal order, based on the absorptive nature of photodetectors; and the Q function, for antinormal order [8]. These formalisms were unified by Cahill and Glauber [9,10], who defined an s-parameterized order for the annihilation and creation operators, such that normal, symmetric, and antinormal orders were obtained for $s=1$, 0, and -1, respectively. Other orders, such as the standard and antistandard ones, could be defined expressing the Cahill-Glauber formalism in terms of momentum and position, instead of annihilation and creation operators [9,10]. A more general formalism was developed by Agarwal and Wolf [11–13], who used filter functions in terms of which all the previous ordering schemes could be expressed from a unified point of view.

However, the initial objective of stochastic description of quantum systems could not be implemented successfully using these QP's because of problems such as negative values or nonexistence for some orders [6,10,14,15]. In antibunching [16,17] and optical bistability [18–21], the state of the field cannot be described as a Glauber-Sudarshan positive density. An important development was the introduction of the positive P function by Drummond and Gardiner [22]. This function was demonstrated to exist as a positive well-behaved function for any state and has been often used in quantum optics (see, e.g., Refs. [1,22–31]). Unfortunately, it is defined in a space having twice as many dimensions as the spaces associated with the Cahill-Glauber QP's [9,10], and this introduces a new difficulty: There is no simple interpretation of the two extra dimensions. Furthermore, the dimensions of the attractors are doubled, so that additional instabilities are expected for nonlinear dynamical systems [23]. Spikes and nonconjugacy were found in studies of quantization of chaotic systems using the positive P function [24–30], but the reason for the failure of the numerical simulations was not clear. Wolinsky and Carmichael found that subtle problems are expected if the positive P function is nonzero only in a bounded manifold of phase space [31]. Another problem, of practical nature, is that the transformation from the density-operator master equation to the c-number QP partial differential equation is a lengthy calculation [32], so that a more economical method is desirable. In classical physics the Langevin equation is derived first, and from the diffusion and drift terms, the Fokker-Planck equation is obtained. However, this is not possible in general (in quantum physics) for

other orders for the normal one, because the positive-P-function equation of motion is a Fokker-Planck one, whenever the Glauber-Sudarshan P function satisfies a pseudo or true Fokker-Planck equation [22]. In the investigation of the problems found in the simulations with the positive P function, it was clear that in some cases the Wigner function was a better choice [28], but it could be used in simulations only for a limited range of parameter values [29] where the diffusion was non-negative. The use of symmetric order in the other situations needed an extension of the ideas of Drummond and Gardiner [22].

Instead of giving the extension of the nondiagonal formalism to the symmetric order only, in this paper we present a formalism which is valid for all the Cahill-Glauber s orders [9,10]. In Sec. II the nondiagonal s-ordered displacement operator is introduced. The action of this operator is shown to be a displacement of the annihilation and creation operators by complex quantities α and β, respectively, which are completely independent from each other. Acting with this displacement upon the Cahill-Glauber ordering operator [9] at the origin, an analytic extension of the ordering operator is obtained. This can be used to define a nondiagonal correspondence between operators and c numbers of a four-dimensional phase space, instead of the diagonal correspondence obtained by Cahill and Glauber [9] in a two-dimensional phase space. We define a distribution function, which we will call the s-order nondiagonal quasiprobability, in this extended phase space. This function can be used together with the operator to c-number correspondence in order to evaluate quantum expectation values of s-ordered quantities. In Sec. III quasiprobabilities are used to describe the time evolution of quantum systems. We show how to transform from the density operator equation to the s-order QP equation. A method for finding stochastic differential equations directly from operator equations similar to the phenomenological Langevin method is described. Three examples of use of this method are presented: a harmonic oscillator, a parametric oscillator, and a nonlinearly damped parametric oscillator. We show in these examples that this is a much easier method of deriving the c-number equations than the previous method. Some limitations of the quasiprobability approach are discussed. Finally, a summary of the main results is presented in Sec. IV.

II. NONDIAGONAL FORMALISM

The formalism described in this paper is appropriate for systems with one degree of freedom. For these systems we define the annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators satisfying the commutation relation

$$[\hat{a},\hat{a}^\dagger]=1 \ . \tag{2.1}$$

The symmetrical and antisymmetrical linear combinations of these operators are proportional to the position and momentum operators of mechanical systems or to the in-phase and in-quadrature components of the electric field.

A. s-ordered power-series expansions

Glauber's displacement operator [6] has been generalized by Cahill and Glauber to a form in which the order is defined by a complex parameter s [9]. Here the nondiagonal form for the s-ordered displacement operator is defined by

$$\hat{D}(\xi,\eta,s)=\exp(\xi\hat{a}^\dagger-\eta\hat{a}+\tfrac{1}{2}s\xi\eta) \tag{2.2a}$$

$$=\exp(\xi\hat{a}^\dagger)\exp(-\eta\hat{a})\exp[\tfrac{1}{2}(s-1)\xi\eta] \tag{2.2b}$$

$$=\exp(-\eta\hat{a})\exp(\xi\hat{a}^\dagger)\exp[\tfrac{1}{2}(s+1)\xi\eta] \ , \tag{2.2c}$$

where the s-ordered nondiagonal displacement operator has been written in different orders: symmetric [Eq. (2.2a)], normal [Eq. (2.2b)], and antinormal [Eq. (2.2c)]. This operator acts displacing the annihilation and creation operators

$$\begin{aligned}\hat{D}^{-1}(\alpha,\beta,s)\hat{a}\hat{D}(\alpha,\beta,s)&=\hat{a}+\alpha \ ,\\ \hat{D}^{-1}(\alpha,\beta,s)\hat{a}^\dagger\hat{D}(\alpha,\beta,s)&=\hat{a}^\dagger+\beta \ ,\end{aligned} \tag{2.3}$$

where the displacements are independent of the parameter s. The displacement operator can be expanded in a Taylor series

$$\hat{D}(\xi,\eta,s)=\sum_{n,m}\frac{\{(\hat{a}^\dagger)^n\hat{a}^m\}_s}{n!m!}\xi^n(-\eta)^m \ , \tag{2.4}$$

where the s-ordered products are

$$\{(\hat{a}^\dagger)^n\hat{a}^m\}_s=\frac{\partial^{n+m}}{\partial\xi^n\partial(-\eta)^m}\hat{D}(\xi,\eta,s)\bigg|_{\xi=\eta=0} \ . \tag{2.5}$$

For $s=1$ we obtain the normal order, $s=0$ gives the symmetrical order, and $s=-1$ gives the antinormal order. For example,

$$\{\hat{a}^\dagger\hat{a}\}_s=\hat{a}^\dagger\hat{a}+\tfrac{1}{2}(1-s)$$

$$=\tfrac{1}{2}(1+s)\hat{a}^\dagger\hat{a}+\tfrac{1}{2}(1-s)\hat{a}\hat{a}^\dagger \quad (s \text{ order}) \ , \tag{2.6a}$$

$$\{\hat{a}^\dagger\hat{a}\}_1=\hat{a}^\dagger\hat{a} \quad (\text{normal order}) \ , \tag{2.6b}$$

$$\{\hat{a}^\dagger\hat{a}\}_0=\tfrac{1}{2}(\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger) \quad (\text{summetric order}) \ , \tag{2.6c}$$

$$\{\hat{a}^\dagger\hat{a}\}_{-1}=\hat{a}\hat{a}^\dagger \quad (\text{antinormal order}) \ . \tag{2.6d}$$

As another example, the Hamiltonian for the harmonic oscillator can be written in any s order as $\hat{H}_{\rm HO}=\hbar\omega(\{\hat{a}^\dagger\hat{a}\}_s+\frac{1}{2}s)$. Under certain conditions [9], operator functions $F(\hat{a},\hat{a}^\dagger)$ can be expanded in terms of the s-ordered displacement operators; i.e., they can be written in s-ordered form.

B. Ordering operator

Let us define the ordering operator $\hat{T}(\alpha,\beta,s)$,

$$\hat{T}(\alpha,\beta,s)=\frac{1}{\pi}\int d^2\xi\, e^{\alpha\xi^*-\beta\xi}\hat{D}(\xi,\xi^*,s) \ , \tag{2.7}$$

where $d^2\xi=d\,\mathrm{Re}(\xi)d\,\mathrm{Im}(\xi)$. Using Eq. (2.3), we can rewrite the ordering operator as

$$\hat{T}(\alpha,\beta,s)=\hat{D}(\alpha,\beta,0)\hat{T}(0,0,s)\hat{D}^{-1}(\alpha,\beta,0) \tag{2.8a}$$

$$=\hat{D}(\alpha,\beta,0)\frac{2}{1-s}\left[\frac{s+1}{s-1}\right]^{\hat{a}^\dagger\hat{a}}\hat{D}^{-1}(\alpha,\beta,0)\ , \tag{2.8b}$$

where we have written explicitly the *diagonal* ordering operator [9]. Using the nondiagonal displacement operator identifies [Eq. (2.3) into Eq. (2.8b)], the operator $\hat{T}(\alpha,\beta,s)$ can alternatively be written in the form

$$\hat{T}(\alpha,\beta,s)=\frac{2}{1-s}\left[\frac{s+1}{s-1}\right]^{(\hat{a}^\dagger-\beta)(\hat{a}-\alpha)} \tag{2.9a}$$

$$=\frac{2}{1-s}\exp\left[(\hat{a}^\dagger-\beta)(\hat{a}-\alpha)\ln\left[\frac{s+1}{s-1}\right]\right] \tag{2.9b}$$

$$=\frac{2}{1-s}\sum_n\left[\frac{s+1}{s-1}\right]^n|\underline{\alpha,\beta,n}\rangle\langle\overline{\alpha,\beta,n}|\ , \tag{2.9c}$$

where $|n\rangle$ is a number state and

$$|\underline{\alpha,\beta,n}\rangle=\hat{D}(\alpha,\beta,0)|n\rangle\ , \tag{2.10}$$

$$\langle\overline{\alpha,\beta,n}|=\langle n|\hat{D}^{-1}(\alpha,\beta,0)\ .$$

The set $\{|\underline{\alpha,\beta,n}\rangle\}$ forms a basis of the space of states of the system, and the set $\{|\overline{\alpha,\beta,n}\rangle\}$ is the dual basis. Note that $|\underline{\alpha,\beta,n}\rangle$ and $\langle\overline{\alpha,\beta,n}|$ are not Hermitean conjugates of each other. These states are a generalization of the diagonal displaced number states [33]. They are complete,

$$\sum_n|\underline{\alpha,\beta,n}\rangle\langle\overline{\alpha,\beta,n}|=\sum_n|\overline{\alpha,\beta,n}\rangle\langle\underline{\alpha,\beta,n}|=1\ , \tag{2.11}$$

and satisfy the orthonormality condition

$$\langle\overline{\alpha,\beta,i}|\underline{\alpha,\beta,j}\rangle=\delta_{ij}\quad(i,j=1,2,\ldots)\ . \tag{2.12}$$

Any state $|\psi\rangle$ can be expanded in terms of these states:

$$|\psi\rangle=\sum_n c_n|\underline{\alpha,\beta,n}\rangle\ , \tag{2.13}$$

where

$$c_n=\langle\overline{\alpha,\beta,n}|\psi\rangle\ . \tag{2.14}$$

The state $|\underline{\alpha,\beta,n}\rangle$ is an eigenstate of the nondiagonal inversely displaced number operator

$$\hat{n}(\alpha,\beta)=\hat{D}(\alpha,\beta,0)\hat{a}^\dagger\hat{a}\hat{D}^{-1}(\alpha,\beta,0)\ , \tag{2.15}$$

corresponding to the eigenvalue n ($n=0,1,2,\ldots$). Note that for each couple (α,β) we have a particular operator, so that Eq. (2.15) defines actually a two-complex-parameter family of operators. Using this operator, we can rewrite the nondiagonal ordering operator as

$$\hat{T}(\alpha,\beta,s)=\frac{2}{1-s}\left[\frac{s+1}{s-1}\right]^{\hat{n}(\alpha,\beta)}\ . \tag{2.16}$$

With this form for the ordering operator, extensions of the Cahill-Glauber formalism become more transparent. Particularly, expansion (2.9c) shows that the eigenvalue λ_n of the operator $\hat{T}(\alpha,\beta,s)$ corresponding to the eigenstate $|\underline{\alpha,\beta,n}\rangle$ is

$$\lambda_n=\frac{2}{1-s}\left[\frac{s+1}{s-1}\right]^n\ . \tag{2.17}$$

From Eq. (2.9c) the Drummond-Gardiner projector [22] is obtained for $s=-1$,

$$\hat{T}(\alpha,\beta,-1)=\frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle}\ , \tag{2.18}$$

which is similar to the Cahill-Glauber relationship $\hat{T}(\alpha,\alpha^*,-1)=|\alpha\rangle\langle\alpha|$ [9]. Having the same eigenvalues as the Cahill-Glauber operator, the trace of the nondiagonal ordering operator is unity as in the diagonal case [9].

C. *s*-order operator to *c*-number nondiagonal correspondence

The generalization of the Drummond-Gardiner formalism [22] can be done, for *s*-ordered operators, replacing the complex quantity α^*, which is usually associated with the annihilation operator, by a complex quantity β, which is independent of α:

$$\hat{a}\rightarrow\alpha\ ,\qquad \hat{a}^\dagger\rightarrow\beta\ . \tag{2.19}$$

Using this correspondence, a space with twice as many dimensions as in the Cahill-Glauber formalism can be defined, with an associated probability density which is well behaved for any state and any order. Integrals in the bidimensional α space are replaced by integrals in the four-dimensional $\alpha\beta$ space. Now we show an important result: Using the nondiagonal ordering operator, any *s*-ordered quantity, given by Eq. (2.5), can be transformed into a *c*-number quantity:

$$\mathrm{Tr}[\{(\hat{a}^\dagger)^n\hat{a}^m\}_s\hat{T}(\alpha,\beta,-s)]=\beta^n\alpha^m\ . \tag{2.20}$$

The proof follows:

$$\mathrm{Tr}[\{(\hat{a}^\dagger)^n\hat{a}^m\}_s\hat{T}(\alpha,\beta,-s)]$$

$$=\mathrm{Tr}[\{(\hat{a}^\dagger)^n\hat{a}^m\}_s\hat{D}(\alpha,\beta,0)\hat{T}(0,0,-s)\hat{D}^{-1}(\alpha,\beta,0)]$$

$$=\mathrm{Tr}[\{(\hat{a}^\dagger+\beta)^n(\hat{a}+\alpha)^m\}_s\hat{T}(0,0,-s)]=\beta^n\alpha^m\ ,$$

where in the last step we have used the result of Cahill-Glauber [9], for the particular case $\alpha=0$.

D. Nondiagonal quasiprobabilities

We will define the nondiagonal quasiprobability $\rho(\alpha,\beta,s)$ implicitly using the quantum characteristic function

$$\chi(\xi,\eta,s)=\mathrm{Tr}[\hat{\rho}\hat{D}(\xi,\eta,s)] \tag{2.21a}$$

$$=\frac{1}{\pi^2}\int d^2\alpha\, d^2\beta\,\rho(\alpha,\beta,s)e^{\xi\beta-\eta\alpha}\ . \tag{2.21b}$$

If there is such a function $\rho(\alpha,\beta,s)$, then averages can be computed with the help of Eq. (2.5):

$$\langle\{(\hat{a}^\dagger)^n\hat{a}^m\}_s\rangle=\frac{\partial^{n+m}}{\partial\xi^n\partial(-\eta)^m}\chi(\xi,\eta,s)\Big|_{\xi=\eta=o}$$
$$=\frac{1}{\pi^2}\int d^2\alpha\, d^2\beta\,\rho(\alpha,\beta,s)\beta^n\alpha^m$$
$$=\langle\beta^n\alpha^m\rangle\ . \tag{2.22}$$

The proof of the existence of $\rho(\alpha,\beta,s)$ is not so simple as in the Cahill-Glauber case, because Eq. (2.21b) cannot be inverted. First, note that the relationship between the characteristic functions for different orders s and t, for $\eta=\xi^*$, is [10]

$$\chi(\xi,\xi^*,s)=\exp[\tfrac{1}{2}(s-t)|\xi|^2]\chi(\xi,\xi^*,t)\ . \tag{2.23}$$

Taking the Fourier transform of the exponential function, and reexpressing the characteristic function in terms of the t-order diagonal quasiprobability $\rho'(\gamma,t)$ [10], we obtain

$$\chi(\xi,\xi^*,s)=\frac{1}{\pi^2}\int d^2\delta\, d^2\gamma\,\rho'(\gamma,t)\frac{2}{s-t}$$
$$\times\exp\left[-\frac{2}{s-t}|\delta|^2-\xi\delta^*-\xi^*\delta\right.$$
$$\left.+\xi\gamma^*-\xi^*\gamma\right]\ . \tag{2.24}$$

Now, making a transformation in the variables of integration,

$$\gamma=\frac{\alpha+\beta^*}{2}\ ,\quad \delta=\frac{\alpha-\beta^*}{2}\ , \tag{2.25}$$

the characteristic function can be written as

$$\chi(\xi,\xi^*,s)=\frac{1}{\pi^2}\int d^2\alpha\, d^2\beta\frac{1}{4}\frac{2}{s-t}$$
$$\times\exp\left[-\frac{2}{s-t}\left|\frac{\alpha-\beta^*}{2}\right|^2\right]$$
$$\times\rho'(\tfrac{1}{2}(\alpha+\beta^*),t)e^{\xi\beta-\xi^*\alpha}\ , \tag{2.26}$$

so that the function

$$\rho_t(\alpha,\beta,s)=\frac{1}{4}\frac{2}{s-t}\exp\left[-\frac{2}{s-t}\left|\frac{\alpha-\beta^*}{2}\right|^2\right]$$
$$\times\rho'(\tfrac{1}{2}(\alpha+\beta^*),t) \tag{2.27}$$

is a nondiagonal quasiprobability giving the correct characteristic function. This proves the existence of the nondiagonal quasiprobability $\rho_t(\alpha,\beta,s)$ for $\mathrm{Re}(t)<\mathrm{Re}(s)$. It is bounded everywhere in the α and β complex planes if the same condition is valid for the diagonal quasiprobability $\rho'(\frac{1}{2}(\alpha+\beta^*),t)$. Equation (2.27) defines a family of nondiagonal quasiprobabilities for each order s, with the parameter t designating a particular member of this family. The parameter t comes from the diagonal quasiprobability, but we remember that the ordering is defined by s, not by t. In the following a subscript t is used whenever we are referring to the form given by Eq. (2.27) and we will refer to that equation as our definition of the canonical forms of the nondiagonal quasiprobabilities. If the diagonal QP on the right-hand side of Eq. (2.27) is non-negative and s and t are real numbers with $s>t$, then $\rho_t(\alpha,\beta,s)$ is a non-negative function. Therefore $\rho_t(\alpha,\beta,s)$ is non-negative for $t\le -1$. The Drummond-Gardiner case is obtained for $s=1$ (normal order), $t=-1$ (in terms of the Q function), with a slightly different normalization. Only the value of one complex number $\frac{1}{2}(\alpha+\beta^*)$ is needed to obtain the value of the diagonal QP, whereas only the knowledge of $\frac{1}{2}(\alpha-\beta^*)$ is needed to obtain the value of the Gaussian function in Eq. (2.27). According to the definition, the nondiagonal QP cannot be an analytical function, because it is a real function whenever s and t are real. For $t=s$ the Gaussian function becomes a two-dimensional Dirac δ function

$$\rho_s(\alpha,\beta,s)=\pi\delta^2(\alpha-\beta^*)\rho'(\alpha,s)\ . \tag{2.28}$$

In this form we can rewrite the quantum averages [Eq. (2.22)] as

$$\langle\{(\hat{a}^\dagger)^n\hat{a}^m\}_s\rangle=\frac{1}{\pi}\int d^2\alpha\,\rho'(\alpha,s)\alpha^{*n}\alpha^m$$
$$=\langle\alpha^{*n}\alpha^m\rangle\ , \tag{2.29}$$

and we see that Cahill-Glauber quasiprobabilities [10] are a particular case of the nondiagonal ones, so that the nondiagonal quasiprobabilities can be chosen either having the same form as the diagonal ones (if one needs them) or having other new forms for situations where problems with the diagonal ones are found. For every order $s\in[-1,1]$, $\rho_t(\alpha,\beta,s)$ can be chosen to be non-negative because the Q function (the diagonal QP for $t=-1$) exists for any state and is non-negative. This demonstrates the existence of the s-order *positive* quasiprobabilities for any state for a choice of $t=-1$ (of course, they are non-negative and always exist as well for any $t<-1$). For other values of t, the nondiagonal QP exists only if the diagonal one for order t does.

There is another relationship between the diagonal QP for order p and the nondiagonal QP for order s. From Cahill and Glauber [10], we know that

$$\rho'(\gamma,t)=\frac{2}{p-t}\frac{1}{\pi}\int d^2\alpha'\exp\left[-\frac{2}{p-t}|\alpha'-\gamma|^2\right]$$
$$\times\rho'(\alpha',p)\ , \tag{2.30}$$

for $\mathrm{Re}(p)>\mathrm{Re}(t)$. Replacing γ by $\frac{1}{2}(\alpha+\beta^*)$ and multiplying by

$$\frac{1}{4}\frac{2}{s-t}\exp\left[-\frac{2}{s-t}\left|\frac{\alpha-\beta^*}{2}\right|^2\right]\ ,$$

we obtain

$$\rho_t(\alpha,\beta,s)=\frac{1}{4}\frac{2}{s-t}\frac{2}{p-t}\frac{1}{\pi}\times\int d^2\alpha' \exp\left[-\frac{2}{p-t}\left|\alpha'-\frac{\alpha+\beta^*}{2}\right|^2-\frac{2}{s-t}\left|\frac{\alpha-\beta^*}{2}\right|^2\right]\times\rho'(\alpha',p)\ , \tag{2.31}$$

and the nondiagonal QP for order s is obtained from the corresponding diagonal QP for order p (when this one exists) by a smoothing convolution with a Gaussian function. For the particular case of $p=s$, we obtain

$$\rho_t(\alpha,\beta,s)=\frac{1}{4}\left[\frac{2}{s-t}\right]^2\frac{1}{\pi}\times\int d^2\alpha' \exp\left[-\frac{1}{2}\frac{2}{s-t}(|\alpha'-\alpha|^2+|\alpha'-\beta|^2)\right]\times\rho'(\alpha',s)\ , \tag{2.32}$$

and the smoothing convolution is between a Gaussian function and diagonal QP for the *same* order s. This equation is another demonstration of the existence of the nondiagonal QP for a given order s, if the diagonal one for the same order s exists. For the particular case of $t=-1$, when the nondiagonal QP is positive semidefinite, we have the generalization for any order s of a theorem demonstrated by Drummond and Gardiner for normal order [22].

The density matrix can be expressed in terms of the ordering operator as

$$\hat{\rho}=\frac{1}{\pi^2}\int d^2\alpha\, d^2\beta\,\rho(\alpha,\beta,s)\hat{T}(\alpha,\beta,-s)\ , \tag{2.33}$$

and from this equation we reobtain the characteristic function

$$\begin{aligned}\chi(\xi,\xi^*,s)&=\mathrm{Tr}[\hat{\rho}\hat{D}(\xi,\xi^*,s)]\\&=\frac{1}{\pi^2}\int d^2\alpha\, d^2\beta\,\rho(\alpha,\beta,s)\\&\quad\times\mathrm{Tr}[\hat{T}(\alpha,\beta,-s)\hat{D}(\xi,\xi^*,s)]\\&=\frac{1}{\pi^2}\int d^2\alpha\, d^2\beta\,\rho(\alpha,\beta,s)e^{-\alpha\xi^*+\beta\xi}\ ,\end{aligned} \tag{2.34}$$

where we have used the correspondence between operators and c numbers [Eq. (2.20)].

III. TIME EVOLUTION

In this section we are going to describe two methods of studying the time development of quantum systems. The first one is the master-equation method [34–36], resulting in Fokker-Planck equations which can be transformed into stochastic differential equations [37]. The other approach is the quantum Langevin equations of the input-output theory [38,39], and we are going to describe a method that can be used to transform from the operator to the c-number equations.

A. Operator identities

In the first method, the master equation for the density operator,

$$\dot{\hat{\rho}}=-\frac{i}{\hbar}[\hat{H},\hat{\rho}]+\text{relaxation terms}\ , \tag{3.1}$$

is transformed into a partial differential equation for the quasiprobability,

$$\rho(\alpha,\beta,s)=\mathcal{L}\rho(\alpha,\beta,s)\ , \tag{3.2}$$

where $\mathcal{L}$ is a differential operator in α and β. This equation is obtained using the operator identities

$$\hat{a}\hat{\rho}\rightarrow\left[\alpha-\frac{s-1}{2}\frac{\partial}{\partial\beta}\right]\rho(\alpha,\beta,s)\ , \tag{3.3a}$$

$$\hat{a}^\dagger\hat{\rho}\rightarrow\left[\beta-\frac{s+1}{2}\frac{\partial}{\partial\alpha}\right]\rho(\alpha,\beta,s)\ , \tag{3.3b}$$

$$\hat{\rho}\hat{a}\rightarrow\left[\alpha-\frac{s+1}{2}\frac{\partial}{\partial\beta}\right]\rho(\alpha,\beta,s)\ , \tag{3.3c}$$

$$\hat{\rho}\hat{a}^\dagger\rightarrow\left[\beta-\frac{s-1}{2}\frac{\partial}{\partial\alpha}\right]\rho(\alpha,\beta,s)\ , \tag{3.3d}$$

which can be obtained using the characteristic function [Eq. (2.21b)]. The proofs are similar to the ones for the diagonal case, which can be found, e.g., in Ref. [40]. Using these identities, Eq. (3.1) is tranformed into Eq. (3.2), which will have in general third- and even higher-order derivatives (see, e.g., Refs. [4,9–12]). However, for a few cases the equation for the positive P function [22] is a Fokker-Planck equation in the phase-space variables (see, e.g., [22,30–32]). For a large class of problems, this is not true, but one can try to approximate the problem by truncating Eq. (3.2), transforming it into a Fokker-Planck equation. In Refs. [25–29,41], infinite-order derivatives were present, but they were neglected, based on a system-size argument, down to the third-order ones, and only the remaining second-order problem was analyzed; even with this approximation, only the small-noise limit was successfully understood, and a better comprehension of the full-noise second-order derivative situation is still waiting for a solution. The only possible analysis for these cases was limited in general to a normal order analysis ($s=1$), because the formalism of Drummond and Gardiner [22] was the only one available. But in the cases discussed in Refs. [23–30], this formalism was not successful in the chaotic region, because of problems such as loss of conjugacy and spikes. Only in one situation could the authors of Ref. [29] use the Wigner function, obtaining in this case better results with this function than with the positive-P-function simulations. But the simulation was not always possible using the Wigner function: The diffusion was negative in some cases, and

in other cases individual trajectories ran into negative-diffusion regions, invalidating the simulation. With the present formalism, the choice of symmetric order (between a number of possible choices) can be done without the problem of negative diffusion. The reason is that whenever the equation of motion for $\rho(\alpha,\beta,s)$ can be transformed into or approximated by a Fokker-Planck type equation, we can use the method of Drummond and Gardiner [22] to show that the diffusion is *positive semidefinite;* i.e., the nondiagonal QP equation of motion is a Fokker-Planck one, whenever the diagonal QP satisfies a pseudo or true Fokker-Planck equation [22]. For these cases we will say hereafter that we have a *diffusive* problem or a *diffusive* system.

B. Stochastic differential equations

In this section let us assume that we are dealing with a diffusive problem. However, instead of using the drift and diffusion terms of the Fokker Planck equation to write the stochastic differential equations, we introduce here a method which is closely related to the Langevin approach. The difference is that we rely on the existence of the *s*-order *nondiagonal* correspondence between operators and *c* numbers, given rigorously by Eq. (2.20). In the usual Langevin approach, the diffusive approximation for the problem is made. However, a diagonal correspondence is assumed, not the nondiagonal correspondence of this paper. The method will be described using it in examples.

1. Damped harmonic oscillator

We use the model discussed by Gardiner and Collett [39] for this system, with the quantum Langevin equations and quantum Wiener process defined by them. Using their results, we simplify the calculation of the diffusion matrix; consequently, it is easier to follow the steps which make up our method. The Hamiltonian for the system is

$$\begin{aligned}&\hat{H}=\hat{H}_0+\hat{H}_b+\hat{H}_i\ ,\\&\hat{H}_0=\hbar\omega\hat{a}^\dagger\hat{a}\ ,\\&\hat{H}_b=\hbar\int d\Omega\,\Omega\hat{b}^\dagger(\Omega)\hat{b}(\Omega)\ ,\\&\hat{H}_i=i\hbar\int d\Omega\,\kappa(\Omega)[\hat{b}^\dagger(\Omega)\hat{a}-\hat{a}^\dagger\hat{b}(\Omega)]\ .\end{aligned}\tag{3.4}$$

Here $\hat{H}_0$ is the free oscillator Hamiltonian, ω is the frequency, $\hat{H}_b$ is the free bath Hamiltonian, Ω is the frequency, and $\hat{b}(\Omega)$ and $\hat{b}^\dagger(\Omega)$ are the annihilation and creation operators, satisfying the commutation relation $[\hat{b}(\Omega),\hat{b}^\dagger(\Omega')]=\delta(\Omega-\Omega')$; $\hat{H}_i$ is the interaction between the heat bath and harmonic oscillator. The range of the integrations is $(-\infty,\infty)$, which is an idealization, as discussed by Gardiner and Collett [39]. In the Markov approximation, the quantum Langevin equations of motion for the harmonic oscillator are

$$\begin{aligned}&d\hat{a}=-(i\omega+\tfrac{1}{2}\gamma)\hat{a}\,dt+\sqrt{\gamma}d\hat{B}\ ,\\&d\hat{a}^\dagger=(i\omega-\tfrac{1}{2}\gamma)\hat{a}^\dagger dt+\sqrt{\gamma}d\hat{B}^\dagger\ ,\end{aligned}\tag{3.5}$$

where the quantum Wiener process is represented by $\hat{B}$ and $\hat{B}^\dagger$, with Gaussian statistics and zero averages, and fluctuations given by

$$\begin{aligned}&[d\hat{B}(t)]^2=[d\hat{B}^\dagger(t)]^2=0\ ,\\&d\hat{B}(t)d\hat{B}^\dagger(t)=(\bar{n}+1)dt\ ,\\&d\hat{B}^\dagger(t)d\hat{B}(t)=\bar{n}\,dt\ .\end{aligned}\tag{3.6}$$

Here $\bar{n}$ is the excitation of the heat bath. Now we use Eqs. (3.5) and (3.6) to obtain

$$d(\hat{a}^2)=-2(\tfrac{1}{2}\gamma+i\omega)\hat{a}^2dt+d\mathcal{F}\ ,\tag{3.7a}$$

$$d(\hat{a}^{\dagger 2})=-2(\tfrac{1}{2}\gamma-i\omega)\hat{a}^{\dagger 2}dt+d\mathcal{F}\ ,\tag{3.7b}$$

$$d(\hat{a}^\dagger\hat{a})=(-\gamma\hat{a}^\dagger\hat{a}+\gamma\bar{n})dt+d\mathcal{F}\ ,\tag{3.7c}$$

where $d\mathcal{F}$ means fluctuations for the second moments, and using the identify [9]

$$\{(\hat{a}^\dagger)^n\hat{a}^m\}_t=\sum_{k=0}^{\min(m,n)}\left[\frac{s-t}{2}\right]^k k!\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}\times\{(\hat{a}^\dagger)^{n-k}\hat{a}^{m-k}\}_s\ ,\tag{3.8}$$

with $t=1$, we transform Eq. (3.7c) into

$$d\{\hat{a}^\dagger\hat{a}\}_s=(-\gamma\{\hat{a}^\dagger\hat{a}\}_s+\gamma[\bar{n}+\tfrac{1}{2}(1-s)])dt+d\mathcal{F}\ ,\tag{3.9}$$

for the *s*-order second moment. The assumption that the process is diffusive (which is not an approximation in the present example) allows one to write down the equations of motion for the *c* numbers associated with the operators according to Eq. (2.7) as

$$\begin{aligned}&d\alpha=-(\tfrac{1}{2}\gamma+i\omega)\alpha\,dt+\Gamma_{\alpha\alpha}dW_{\alpha\alpha}+\Gamma_{\alpha\beta}dW_{\alpha\beta}\ ,\\&d\beta=-(\tfrac{1}{2}\gamma-i\omega)\beta\,dt+\Gamma_{\beta\beta}dW_{\beta\beta}+\Gamma_{\beta\alpha}dW_{\alpha\beta}\ ,\end{aligned}\tag{3.10}$$

where W_{ij} $(i,j=\alpha,\beta)$ are unit Wiener processes [37] satisfying

$$dW_{ij}dW_{kl}=\delta_{ik}\delta_{jl}dt\ ,\tag{3.11}$$

and Γ_{ij} $(i,j=\alpha,\beta)$ are quantities to be determined. From Eqs. (3.7) and (3.9), making the operator to *c*-number correspondence, we can write

$$\begin{aligned}&d(\alpha^2)=-2(\tfrac{1}{2}\gamma+i\omega)\alpha^2dt+d\mathcal{F}\ ,\\&d(\beta^2)=-2(\tfrac{1}{2}\gamma-i\omega)\beta^2dt+d\mathcal{F}\ ,\\&d(\alpha\beta)=\{-\gamma\alpha\beta+\gamma[\bar{n}+\tfrac{1}{2}(1-s)]\}dt+d\mathcal{F}\ ,\end{aligned}\tag{3.12}$$

whereas from Eq. (3.10) we obtain

$$\begin{aligned}&d(\alpha^2)=[-2(\tfrac{1}{2}\gamma+i\omega)\alpha^2+\Gamma_{\alpha\alpha}^2+\Gamma_{\alpha\beta}^2]dt+d\mathcal{F}\ ,\\&d(\beta^2)=[-2(\tfrac{1}{2}\gamma-i\omega)\beta^2+\Gamma_{\beta\beta}^2+\Gamma_{\beta\alpha}^2]dt+d\mathcal{F}\ ,\\&d(\alpha\beta)=(-\gamma\alpha\beta+\Gamma_{\alpha\beta}\Gamma_{\beta\alpha})dt+d\mathcal{F}\ .\end{aligned}\tag{3.13}$$

Comparison of Eq. (3.12) with Eq. (3.13) shows that

$$\begin{aligned}&\Gamma_{\alpha\alpha}^2+\Gamma_{\alpha\beta}^2=\Gamma_{\beta\beta}^2+\Gamma_{\beta\alpha}^2=0\ ,\\&\Gamma_{\alpha\beta}\Gamma_{\beta\alpha}=\gamma[\bar{n}+\tfrac{1}{2}(1-s)]\ .\end{aligned}\tag{3.14}$$

Therefore the drift vector is determined directly from the operator to c number correspondence, and the diffusion matrix is determined, using our method, by Eq. (3.13). The solution of Eq. (3.14) is not unique, but every solution will give one and the same diffusion matrix, so that which solution one is going to use does not matter. For a bath with zero temperature ($\bar{n}=0$) and normal order ($s=1$), the process is completely deterministic (zero diffusion).

We can thus obtain stochastic differential equations directly from the input-output equations, assuming that the process is a diffusive one. The use of Eq. (3.6) by us was a matter of convenience and is not essential for the present method. One could instead use the traditional method of Langevin forces (see, e.g., Ref. [35]), calculating the correlations between these random forces and using them to derive the c-number equations from the Heisenberg-Langevin equations.

If damping by the heat bath is not included, we make $\gamma=0$ in Eqs. (3.5)–(3.7) and (3.9)–(3.14). This gives an example of the application of this method to transform directly from a Heisenberg equation to a c-number equation.

2. *Degenerate parametric oscillator*

In this subsection we modify the example of the previous subsection by adding a degenerate resonant parametric interaction with a classical pump field. In this case it is convenient to make a transformation to a rotating frame, eliminating the natural oscillation of the harmonic oscillator, so that the parametric Hamiltonian to be added in Eq. (3.4) is

$$\hat{H}_p=\tfrac{1}{2}i\hbar G(\hat{a}^{\dagger 2}-\hat{a}^2)\ , \tag{3.15}$$

where G includes the coupling constant and the classical field amplitude. Applying the method described above [Eqs. (3.5)–(3.14)], the c-number equations of motion for order s will be

$$\begin{aligned} d\alpha&=(-\tfrac{1}{2}\gamma\alpha+G\beta)dt+\Gamma_{\alpha\alpha}dW_{\alpha\alpha}+\Gamma_{\alpha\beta}dW_{\alpha\beta}\ ,\\ d\beta&=(-\tfrac{1}{2}\gamma\beta+G\alpha)dt+\Gamma_{\beta\beta}dW_{\beta\beta}+\Gamma_{\beta\alpha}dW_{\alpha\beta}\ , \end{aligned} \tag{3.16}$$

with

$$\begin{aligned} &\Gamma_{\alpha\alpha}^2+\Gamma_{\alpha\beta}^2=\Gamma_{\beta\beta}^2+\Gamma_{\beta\alpha}^2=Gs\ ,\\ &\Gamma_{\alpha\beta}\Gamma_{\beta\alpha}=\gamma[\bar{n}+\tfrac{1}{2}(1-s)]\ . \end{aligned} \tag{3.17}$$

Comparing with Eqs. (3.5) and (3.14), additional terms due to the parametric coupling appear now in the drift vector and diffusion matrix in Eqs. (3.16) and (3.17). For the particular case of a heat bath at zero temperature ($\bar{n}=0$) and using normal order ($s=1$), comparing Eqs. (3.16) and (3.17) with Eq. (43) of Ref. [40], it is seen that the source of phase-dependent quantum noise giving squeezing in the weak-field limit is the parametriclike noise. Here, again, if we neglect the interaction with the heat bath, we have another example of the application of the method to obtain c-number stochastic differential equations from the Heisenberg equations.

C. Use of alternative orders

Smith and Gardiner [30] have demonstrated that the difficulties in the use of the positive P function [23–30] could not be attributed to the failure of the numerical simulation. They argued that the probable cause of the remaining problems, when better numerical methods are chosen in the simulation, is the form of the QP for large absolute values of the arguments. A related problem was studied by Wolinsky and Carmichael [31] who argued that the problems with the positive P appeared because the motion of the system must be restricted to a finite manifold contained in the real subspace of the $\alpha\beta$ phase space, and this condition was violated in the simulations, because of the finite step size available in computers. Using our method, we have obtained a set of equations for the problem studied in Ref. [30], in the diffusive approximation, for s order. For a particular order $s=\frac{2}{3}$, the problems with spikes found by Smith and Gardiner have disappeared [42].

It has been argued that the diffusive approximation gives differences of several orders of magnitude in the time spent by a parametric oscillator in each of the two stable steady states above threshold [43–45]. Because the equation of motion for the s-order quasiprobabilities involve in general higher-order derivatives, the truncation to second-order derivatives can give wrong results. But there must be cases where the differences are negligible. In Ref. [42] several cases are discussed, some of them being just cases where the positive-P-function simulation had problems, and good agreement between the normal and alternative orders is achieved in some situations. In the diffusive approximation, the s-order stochastic differential equations are

$$\begin{aligned} d\alpha&=\{[-1+g^2(1-s)]\alpha+\beta(\lambda-g^2\alpha^2)\}dt\\ &\quad+\Gamma_{\alpha\alpha}dW_{\alpha\alpha}+\Gamma_{\alpha\beta}dW_{\alpha\beta}\ ,\\ d\beta&=\{[-1+g^2(1-s)]\beta+\alpha(\lambda-g^2\beta^2)\}dt\\ &\quad+\Gamma_{\beta\beta}dW_{\beta\beta}+\Gamma_{\beta\alpha}dW_{\alpha\beta}\ , \end{aligned} \tag{3.18}$$

where the diffusion is given by

$$\begin{aligned} &\Gamma_{\alpha\alpha}^2+\Gamma_{\alpha\beta}^2=s(\lambda-g^2\alpha^2)\ ,\\ &\Gamma_{\beta\beta}^2+\Gamma_{\beta\alpha}^2=s(\lambda-g^2\beta^2)\ ,\\ &\Gamma_{\alpha\beta}\Gamma_{\beta\alpha}=(1-s)[1-g^2(1-s)+2g^2\alpha\beta]\ . \end{aligned} \tag{3.19}$$

The quantities used here are defined in Ref. [43]: λ is the fundamental pump strength, and g measures the nonlinear coupling with the heat bath. As observed in Ref. [45], the equations that they have been using for the Wigner function in their previous paper [43] had some terms missing, so that their Wigner-function simulations gave smaller tunneling times than the positive P function [45], in disagreement with their previous results giving much larger tunneling times [43]. From Eqs. (3.18) and (3.19) (for $s=0$) and Refs. [32,46], one sees that there is still one term missing in the equations used in Ref. [45], so that an additional calculation and simulation are needed in order to compare the positive-P- and Wigner-

function simulation results. For each s except $s=1$, the only approximation which we are making is the truncation of higher-order derivative terms. For symmetric and normal order, Ref. [42] presents results for simulation of the stochastic differential equations and the differences are not so dramatic as stated in Ref. [43].

IV. CONCLUSIONS

The nondiagonal displacement operator and nondiagonal ordering operator have been introduced. These operators allow us to map ordered operators into two c-number quantities, one for each power of the annihilation operator and the other (which is independent from the first one) for each power of the creation operator. We call this a nondiagonal correspondence, generalizing the Cahill-Glauber [9,10] map. For normal-ordered operators, the Drummond-Gardiner projector [22] is recovered. As in the Drummond-Gardiner case [22] and contrary to the diagonal case [9,10], the operator to the c-number map cannot be inverted.

Here families of quasiprobabilities for the s-order quantum-classical correspondence are introduced for systems with one degree of freedom. For any order, including the normal one, we have demonstrated that they can be defined in terms of other diagonal quasiprobabilities and can always be chosen as non-negative functions for any state. We show that the canonical form of the nondiagonal s-order QP has as a particular case the products of a δ function by the diagonal QP for the same order, whenever this one exists, so that the diagonal quasiprobabilities are a particular case of the more general nondiagonal ones. We have demonstrated that the nondiagonal QP can also be expressed as the convolution of a diagonal quasiprobability with a Gaussian function. The convolution has also the effect of averaging the irregularities, so that the nondiagonal QP can always be chosen as a positive function even when the diagonal QP oscillates between positive and negative values.

For diffusive problems the diffusion matrix of the Fokker-Planck equation for the s-ordered nondiagonal QP is found to be positive semidefinite, allowing the use of stochastic differential equations for the c numbers corresponding to the quantum operators. A rigorous method of derivation of stochastic differential equations directly from the Heisenberg or input-output equations, without the necessity of deriving first a Fokker-Planck equation, is described. Using this method, one can write down the Fokker-Planck equation from the stochastic differential equations, which is a much easier method for obtaining the Fokker-Planck equation than the usual one. Our method is therefore easier to be used than the traditional methods, so that better approximations can be found to quantum problems. We have already found advantages using our method with respect to methods traditionally used in the literature (compare the results of Ref. [46] with Refs. [43–45]). In order to obtain the stochastic differential equations, various (inequivalent) truncation schemes are necessary.

Our results imply that, associated with any one-dimensional system, there is a four-dimensional space with a well-defined and non-negative probability density everywhere, for any state, for any s order. More complicated systems can be mapped into several one-dimensional systems, through the Schwinger-Jordan map [47,48], so that this kind of correspondence can be used to study a large variety of quantum systems.

ACKNOWLEDGMENTS

We thank S. M. Barnett, B. J. Dalton, F. A. de Oliveira, F. A. P. Piôlho, and A. Schenzle for useful discussions, and Cid B. de Araújo for reading the manuscript. This work was partially supported by CNPq and Helisom. We would like to thank P. D. Drummond for sending us a copy of Ref. [45] prior to publication. Part of this work was completed when the author was with the Departamento de Física, Universidade Federal do Rio Grande do Norte.

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