Two-channel cavity QED: Stokes plus anti-Stokes emission with a classical pump field

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We consider a two-channel Λ-type three-level atom interacting simultaneously with two quantized cavity fields and a classically described cw laser field, all of different frequencies. The effective interactions are of the Raman type, in which the two cavity fields are regarded as Stokes and anti-Stokes modes, and the classical field acts as the pump mode, so that both the pump-Stokes and the pump–anti-Stokes transitions are included. We present the analytic solution of the system evolution operator. We also study the time dependence of the atomic inversion and the statistical properties of the cavity fields. We show that strongly correlated photons can be generated in our model.

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I. INTRODUCTION

Over the last decade, there has been much work on light-matter interactions of a single atom in an optical cavity [1]. Models of a two- or three-level atom interacting with one or two cavity modes in different configurations were studied extensively. Recently, Raman scattering of an atom with degenerate [2–4] or nondegenerate [5] levels in a cavity has attracted considerable interest. In a recent paper by Puri, Wang, and Eberly [6], a nondegenerate model of an atom interacting simultaneously in two channels with three quantized cavity fields was introduced. The interaction is essentially Raman type, in which both the pump-Stokes channel and the pump–anti-Stokes channel are included. All three mode frequencies, pump, Stokes, and anti-Stokes, are different. Since all the fields are dynamical variables, their model provides nontrivial features of multiwave mixing, including pump depletion, in the context of cavity QED. The system's dressed eigenstates are three-mode tangled states.

In this paper, we consider the same nondegenerate system but with the pump field replaced by a classically described laser field of constant amplitude, while the Stokes and the anti-Stokes cavity modes are kept quantized. The role of the pump field is significantly different in the original fully quantized model [6]; here it is not necessarily a cavity mode and does not evolve dynamically. In other words, our system is a classically driven atom in a two-mode cavity [7]. We will focus on the transient quantum dynamics of the system on a sufficiently short time scale that cavity damping is negligible.

This paper is organized as follows. In Sec. II we introduce the model Hamiltonian. By identifying the constant of motion, we find the effective interaction Hamiltonian. In Sec. III the system evolution operator is derived. The main purpose of this paper is to determine the dynamics of the atom and the statistical properties of the cavity fields. We consider a simple initial condition, in which the atom and the cavity fields all start in their ground states. The time dependence of the atomic inversion and various features of the photon states are discussed in Sec. IV. Finally, Sec. V provides a summary of our results.

II. THE MODEL HAMILTONIAN AND CONSTANT OF MOTION

The atomic energy levels and the radiative interactions of our model are shown in Fig. 1. We assume that the detuning Δ is sufficiently large that we can adiabatically eliminate the top level [6]. This approximation leads to a two-level atom that contains levels 1 and 3 only. The Hamiltonian describing this system, in the usual rotating-wave approximation, is given by

$$\begin{aligned} H = {} & E_{31}S_z + \omega_S a_S^\dagger a_S + \omega_A a_A^\dagger a_A \\ & + (\xi e^{-i\omega_P t} a_S^\dagger + \eta e^{i\omega_P t} a_A) S_+ \\ & + (\xi e^{i\omega_P t} a_S + \eta e^{-i\omega_P t} a_A^\dagger) S_- \end{aligned} \tag{2.1}$$

where we have taken $\hbar = 1$ for convenience. The symbols $a_i, a_i^\dagger$ $(i = S, A)$ represent the field operators for the Stokes and the anti-Stokes modes. The S's are the coher-

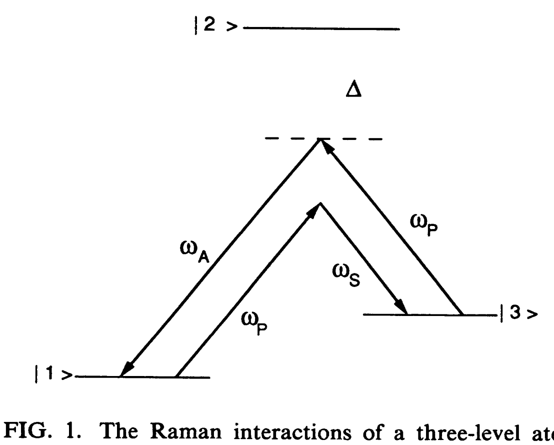

FIG. 1. The Raman interactions of a three-level atom interacting with the driven and the cavity fields simultaneously. We assume Δ to be large enough to eliminate the level $|2\rangle$.

ence operators for the atom, satisfying angular momentum (not Pauli matrix) commutation relations among themselves. The parameters ξ and η, which are proportional to the amplitude of the pump field, characterize the coupling strengths for the two transitions shown in Fig. 1. Both are assumed real. The energy difference of the atomic levels 3 and 1 is denoted E_{31}. The laser field frequency ω_P and the cavity mode frequencies ω_S and ω_A are supposed to satisfy the resonance conditions for Raman transitions:

$$\omega_P-\omega_S=\omega_A-\omega_P=E_{31} . \tag{2.2}$$

In order to eliminate the explicit time dependence of the Hamiltonian, we transform to a rotating frame of reference via the unitary operator,

$$T(t)=\exp[-i\omega_P t(a_S^\dagger a_S+a_A^\dagger a_A)] . \tag{2.3}$$

The new Hamiltonian is defined by

$$H'=T^\dagger(t)HT(t)-iT^\dagger(t)\frac{dT(t)}{dt} \tag{2.4}$$

and it is easy to show that H' takes the time-independent form:

$$H'=E_{31}C+(\xi a_S^\dagger+\eta a_A)S_+ +(\xi a_S+\eta a_A^\dagger)S_- , \tag{2.5}$$

where the operator C is defined by

$$C=(a_A^\dagger a_A-a_S^\dagger a_S+S_z) . \tag{2.6}$$

We see that this operator C commutes with H' and therefore is a constant of motion. This implies an important fact—that the photon number difference between the Stokes mode and the anti-Stokes mode can only change by one unit. In fact, if the system begins with the vacuum cavity field and the atomic ground state, the Fock states $|n_S,n_A\rangle$ that are allowed to be generated are confined to $|n,n\rangle$ and $|n+1,n\rangle$ where $n=0,1,2,\ldots$. Therefore the Stokes and the anti-Stokes photons are strongly correlated. We will discuss the dynamics of the photon generation in a later section.

The effective interaction Hamiltonian now reads

$$H_{\text{eff}}=(\xi a_S^\dagger+\eta a_A)S_+ +(\xi a_S+\eta a_A^\dagger)S_- . \tag{2.7}$$

III. TIME EVOLUTION OPERATOR

The dynamics of the system is described by the time evolution operator $U(t)$, which is written as

$$\begin{aligned}U(t)&=T(t)\exp(-iE_{31}Ct)\exp(-iH_{\text{eff}}t)T^\dagger(0)\\&=\exp[-it(E_{31}S_z+\omega_S a_S^\dagger a_S+\omega_A a_A^\dagger a_A)]\\&\quad\times\exp(-iH_{\text{eff}}t) .\end{aligned} \tag{3.1}$$

Equation (2.7) allows us to express $\exp(-iH_{\text{eff}}t)$ as

$$\begin{aligned}\exp(-iH_{\text{eff}}t)=&\sigma_{11}\cos[(A^\dagger A)^{1/2}t]\\&+\sigma_{33}\cos[(AA^\dagger)^{1/2}t]\\&-iA\frac{\sin[(A^\dagger A)^{1/2}t]}{(A^\dagger A)^{1/2}}\sigma_{31}\\&-iA^\dagger\frac{\sin[(AA^\dagger)^{1/2}t]}{(AA^\dagger)^{1/2}}\sigma_{13}\end{aligned} \tag{3.2}$$

where we have defined the operators $A=(\xi a_S^\dagger+\eta a_A)$ and $\sigma_{ij}=|i\rangle\langle j|$. It is not obvious how to obtain the explicit dynamics from $U(t)$ because of the presence of operators such as $\cos[(A^\dagger A)^{1/2}t]$ appearing in (3.2). The way to solve this problem is to work out the eigenvalues and eigenvectors of the operator $(A^\dagger A)^{1/2}$.

The operator $A^\dagger A$ is given by

$$A^\dagger A=(\eta^2 a_A^\dagger a_A+\xi^2 a_S^\dagger a_S+\xi\eta a_S^\dagger a_A^\dagger+\xi\eta a_A a_S+\xi^2) . \tag{3.3}$$

It describes two oscillators coupled together through a two-photon interaction. The eigenvalue problem can be solved analytically by decoupling the oscillators through a linear transformation. It is a well-known procedure. To avoid complication here, we will focus on the case where $\xi=\eta$. It means the Stokes channel and the anti-Stokes channel have the same coupling strength. We suggest that it is a good approximation even if in the actual situation $\xi\neq\eta$, provided that the cavity-atom interaction time is not too long [8]. Now let us write the field operators as

$$a_S=\frac{1}{\sqrt{2}}(p_S-ix_S),\quad a_A=\frac{1}{\sqrt{2}}(p_A-ix_A) \tag{3.4}$$

using the momentum and position representation. Then Eq. (3.3) takes a very simple form

$$A^\dagger A=\xi^2(p_+^2+x_-^2) \tag{3.5}$$

where we have used the transformations

$$p_\pm=\frac{p_S\pm p_A}{\sqrt{2}} , \tag{3.6}$$

$$x_\pm=\frac{x_S\pm x_A}{\sqrt{2}} . \tag{3.7}$$

The $p_\pm$ and $x_\pm$ satisfy the following commutation relations:

$$[p_+,p_-]=[x_+,x_-]=[p_+,x_-]=[p_-,x_+]=0 , \tag{3.8}$$

$$[x_+,p_+]=[x_-,p_-]=i . \tag{3.9}$$

Hence $(\pm)$ are two independent degrees of freedom.

Equation (3.5) contains only the kinetic energy associated with the $(+)$ coordinate and the potential energy associated with the $(-)$ coordinate. The eigenvalues are therefore continuously distributed on the positive number line and the eigenvectors can be denoted by $|p_+,x_-\rangle$:

$$A^\dagger A|p_+,x_-\rangle=\xi^2(p_+^2+x_-^2)|p_+,x_-\rangle \tag{3.10}$$

where $|p_+,x_-\rangle$ are the simultaneous eigenstates of p_+ and x_-. The projection of $|p_+,x_-\rangle$ onto the Stokes–anti-Stokes Fock states $|n,m\rangle$ is quite simple,

$$\langle n,m|p_+,x_-\rangle = N_{n,m}\int_{-\infty}^{\infty} dx_+ H_m\left[\frac{x_+ - x_-}{\sqrt{2}}\right] \times H_n\left[\frac{x_+ + x_-}{\sqrt{2}}\right] \times e^{-(x_+^2+x_-^2)}e^{ip_+x_+} \quad (3.11)$$

where $N_{n,m}=(\pi 2^{n+m+1}n!m!)^{-1/2}$ and $H_n(x)$ is the nth Hermite polynomial.

IV. DYNAMICS

For later purposes, we would like to define the dimensionless time

$$\tau=\sqrt{2}\xi t \quad (4.1)$$

and the operators

$$f=\frac{a_S^\dagger - a_A}{\sqrt{2}} , \quad (4.2)$$

$$g=\frac{a_S^\dagger + a_A}{\sqrt{2}} . \quad (4.3)$$

Notice that g is just the operator A besides a proportionality constant.

Suppose the system begins with the atom in its ground state and the vacuum cavity field,

$$|\Psi(0)\rangle = |0_S,0_A;1\rangle . \quad (4.4)$$

The state vector at time τ may be written in the form

$$|\Psi(\tau)\rangle = \sum_{n=0}^{\infty}[p_n(\tau)|n,n;1\rangle + q_n(\tau)|n+1,n;3\rangle] \quad (4.5)$$

where the abbreviation $|n,m;j\rangle$ denotes the state vector of n photons in the Stokes mode, m photons in the anti-Stokes mode, and the atomic level j is occupied. The time-dependent coefficients $p_n(\tau)$ and $q_n(\tau)$ are defined by

$$p_n(\tau)=\langle n,n|\cos[(g^\dagger g)^{1/2}\tau]|0,0\rangle \quad (4.6)$$

$$q_n(\tau)=-i\langle n+1,n|g\frac{\sin[(g^\dagger g)^{1/2}\tau]}{(g^\dagger g)^{1/2}}|0,0\rangle . \quad (4.7)$$

Using the results given in (3.10) and (3.11), they have the following integral representations:

$$p_n(\tau)=\int_{-\infty}^{\infty} dx_- \int_{-\infty}^{\infty} dp_+ \cos[\sqrt{(p_+^2+x_-^2)/2}\,\tau] \times\langle n,n|p_+,x_-\rangle\langle p_+,x_-|0,0\rangle \quad (4.8)$$

and

$$q_n(\tau)=-i(n+1)^{1/2} \times\int_{-\infty}^{\infty} dx_- \int_{-\infty}^{\infty} dp_+ \frac{1}{(p_+^2+x_-^2)^{1/2}} \times\sin[\sqrt{p_+^2+x_-^2)/2}\,\tau] \times\langle n,n+1|p_+,x_-\rangle\langle p_+,x_-|0,0\rangle . \quad (4.9)$$

Further calculation [9] shows that $p_n(\tau)$ and $q_n(\tau)$ may be written in more compact forms,

$$p_n(\tau)=\frac{(-1)^n n!\tau^{2n}}{2^n 2n!}\,{}_1F_1(n+1;n+\tfrac{1}{2};-\tau^2/8) , \quad (4.10)$$

$$q_n(\tau)=i\frac{(-1)^{n+1}n!\tau^{2n+1}}{2^n(2n+1)!}\left[\frac{n+1}{2}\right]^{1/2} \times\left[{}_1F_1(n+1;n+\tfrac{3}{2};-\tau^2/8)-\frac{\tau^2}{4(n+1)(2n+3)} \times{}_1F_1(n+2;n+\tfrac{5}{2};-\tau^2/8)\right] \quad (4.11)$$

where ${}_1F_1$ is the confluent hypergeometric function.

A. Atomic inversion

The atomic inversion $w(\tau)$ is determined by

$$w(\tau)=\langle\Psi(\tau)|S_z|\Psi(\tau)\rangle , \quad (4.12)$$

and it is easy to verify that

$$w(\tau)=-\langle 0,0|\cos[2(g^\dagger g)^{1/2}\tau]|0,0\rangle . \quad (4.13)$$

Since the eigenvalues of $g^\dagger g$ are continuous, the inversion has a simple integral representation,

$$w(\tau)=-2\int_0^{\infty} ds\, s e^{-s^2}\cos(\sqrt{2}s\tau) . \quad (4.14)$$

The numerical result is plotted in Fig. 2(a). Unexpectedly, the inversion has no oscillation at all and it quickly approaches the constant value zero. This is in contrast to the similar system with a one-channel transition [3], which would contain vacuum Rabi oscillations. In Fig. 2(b), we also plot the inversion when the initial field is not vacuum but the $|10,10\rangle$ state instead. Similarly, we see that the inversion becomes steady and the oscillations are just transient. These results demonstrate the different behavior due to the two-channel nature of our model. The physical mechanism for the two-channel atomic dynamics needs to be studied further; the phase difference between the Stokes and the anti-Stokes fields as well as the coherence of the atomic variables should be crucial.

B. Photon number and fluctuations

The average photon number $\langle N(\tau)\rangle$ is defined by

$$\langle N(\tau)\rangle = \langle\Psi(\tau)|a_A^\dagger a_A + a_S^\dagger a_S|\Psi(\tau)\rangle = \langle\Psi(\tau)|g^\dagger g + f^\dagger f|\Psi(\tau)\rangle - \tfrac{1}{2} . \quad (4.15)$$

The first term is trivially a constant because $[g^\dagger,g]=0$. The second term may be evaluated through the commutation relations $[g,f]=0$ and $[g,f^\dagger]=-1$. It is not difficult to show that

$$\exp(+iH_{\text{eff}}t)f\exp(-iH_{\text{eff}}t)=f+S_-\tau+g(g^\dagger S_- - S_+g)\frac{1}{2g^\dagger g}\left[\frac{\sin 2(g^\dagger g)^{1/2}\tau}{2(g^\dagger g)^{1/2}}-\tau\right]-gS_z\frac{\sin^2(g^\dagger g)^{1/2}\tau}{4g^\dagger g}\ . \qquad (4.16)$$

Therefore we can show that

$$\langle N(\tau)\rangle=\langle 0,0|\frac{\sin^4[(g^\dagger g)^{1/2}\tau]}{4g^\dagger g}+f^\dagger g\frac{\sin^2[(g^\dagger g)^{1/2}\tau]}{2g^\dagger g}+\frac{\sin^2[(g^\dagger g)^{1/2}\tau]}{2g^\dagger g}g^\dagger f+\frac{1}{4}\left[\frac{\sin^2[2(g^\dagger g)^{1/2}\tau]}{2g^\dagger g}-\tau\right]^2|0,0\rangle\ . \qquad (4.17)$$

Using the results given in (3.10) and (3.11), we find that $\langle N(\tau)\rangle$ finally reads

$$\langle N(\tau)\rangle=\int_0^\infty ds\, e^{-s^2}\left[\frac{1}{s}\sin^4\left[\frac{s\tau}{\sqrt{2}}\right]+\frac{2}{s}\sin^2\left[\frac{s\tau}{\sqrt{2}}\right]+\frac{s}{2}\left[\frac{\sin^2(\sqrt{2}s\tau)}{s^2}-\tau\right]^2\right]\ . \qquad (4.18)$$

When τ is sufficiently large [8], the integral has a simple limit:

$$\langle N(\tau)\rangle\approx\frac{\tau^2}{4}\ . \qquad (4.19)$$

As mentioned before, the photon numbers in the individual modes are approximately the same (at most they differ by 1), so we have

$$\langle a_A^\dagger a_A\rangle\approx\langle a_S^\dagger a_S\rangle\approx\frac{\tau^2}{8}\ . \qquad (4.20)$$

This result is interesting because the photon number in each cavity mode increases as τ^2, which is a feature of a fully coherent interaction without relaxation or phase interference. Alternatively, we can regard the increase of photon number as a consequence of the two-channel nature of the interaction because it does not happen in the same system with one transition channel.

Further calculation of the photon number variance is rather complicated. In Fig. 3 the numerical results are shown. We plot the ratio of the normally ordered photon variance to the mean photon number,

$$r_i=\frac{\langle :(\Delta N_i)^2:\rangle}{\langle N_i\rangle}=\frac{\langle a_i^\dagger a_i^\dagger a_i a_i\rangle-\langle a_i^\dagger a_i\rangle^2}{\langle a_i^\dagger a_i\rangle}\quad (i=S,A)\ . \qquad (4.21)$$

Interestingly, the fields first exhibit sub-Poisson statistics ($r_i<0$) in the time interval from zero to $\tau\approx 9$ and become super-Poissonian ($r_i>0$) eventually. The size of the fluctuation is important. We see in Fig. 3 that $\langle :(\Delta N_i)^2:\rangle$ is roughly equal to the mean during an intermediate time interval (up to $\tau\approx 40$). In other words, the fluctuation of the photon number increases quite slowly with time, and the photon number variance has the same order of magnitude as the mean photon number, so the width of the photon distribution is relatively narrow, like a coherent field. This is significantly different from thermal field statistics, although both are super-Poissonian.

In Fig. 4, we present the photon distributions at different scaled times τ. Both the probabilities $|p_n|^2$ and $|q_n|^2$ in (4.5) are plotted. As expected from the results in

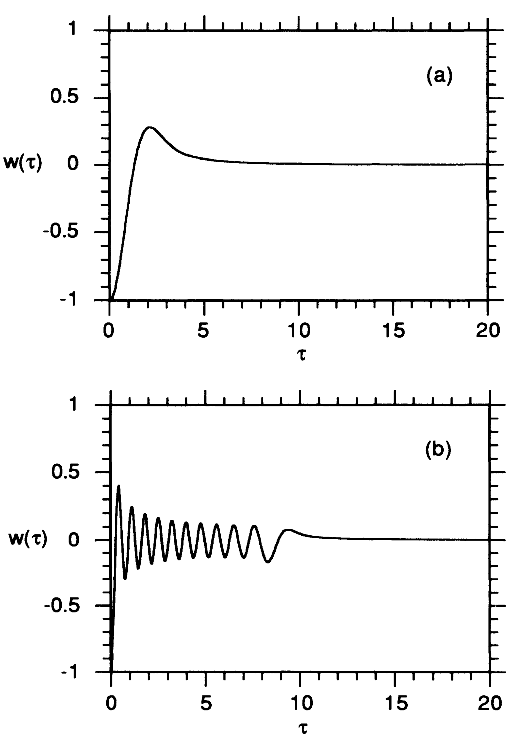

FIG. 2. Time-dependent atomic inversion. The atom is initially in its ground state. The initial field states are (a) $|0,0\rangle$ and (b) $|10,10\rangle$. The time scale is dimensionless, incorporating the pump intensity as defined in (4.1).

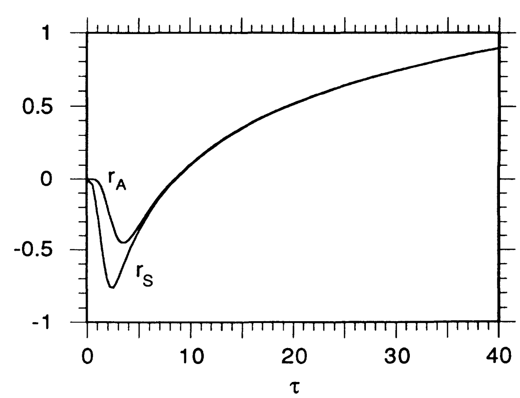

FIG. 3. Time evolution of r_S and r_A, which are the ratios of the normally ordered variance to the mean photon number for the Stokes fields and the anti-Stokes field, respectively.

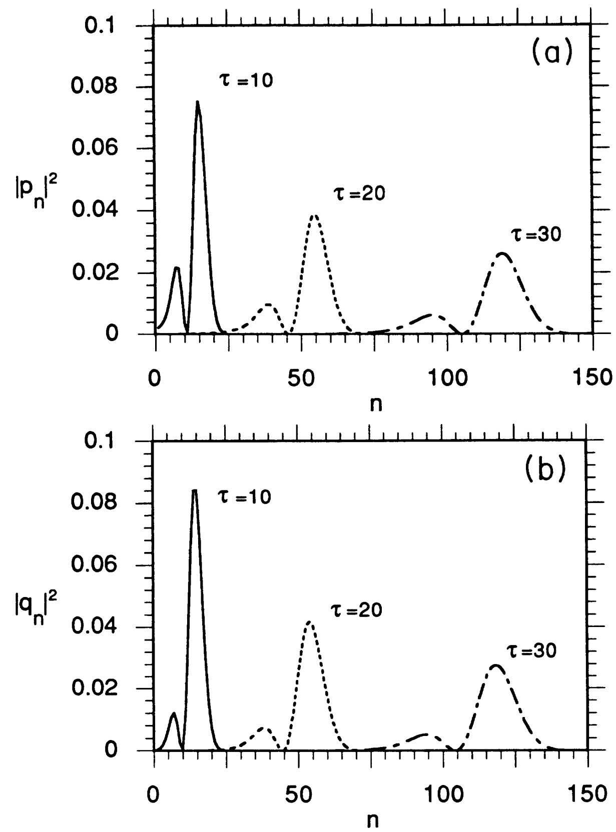

FIG. 4. (a) Probability distributions $|p_n|^2$ at three different values of τ. $|p_n|^2$ is the probability of finding the photon state to be in Stokes and anti-Stokes Fock state $|n,n\rangle$. (b) Probability distribution $|q_n|^2$ at the same values of τ as (a). $|q_n|^2$ is the probability of finding the photon state to be in Stokes and anti-Stokes Fock state $|n+1,n\rangle$.

the preceding discussion, the distributions have localized shapes and their center positions are well described by (4.20).

C. Cross correlation function

One of the most important features in our model is the production of strongly correlated photons. As already seen in (4.5), there are two kinds of paired photon states produced, namely, $|n,n\rangle$ and $|n+1,n\rangle$. In fact, if we make a measurement of the atomic state, collapse of the wave function would force the photon states to have either the $|n,n\rangle$ type or the $|n+1,n\rangle$ type. This implies a one-to-one correspondence between the Stokes and the anti-Stokes photons.

The pair states $|n,n\rangle$ and $|n+1,n\rangle$ generally show a nonclassical correlation between the two modes because of the following inequality:

$$\langle a_S^\dagger a_A^\dagger a_A a_S\rangle^2 - \langle a_S^\dagger a_S^\dagger a_S a_S\rangle\langle a_A^\dagger a_A^\dagger a_A a_A\rangle > 0 \tag{4.22}$$

which holds for all these pair states (except $|0,0\rangle$ and $|1,0\rangle$), and which indicates the violation of the corresponding Cauchy-Schwartz inequality that must be satisfied by any classical field.

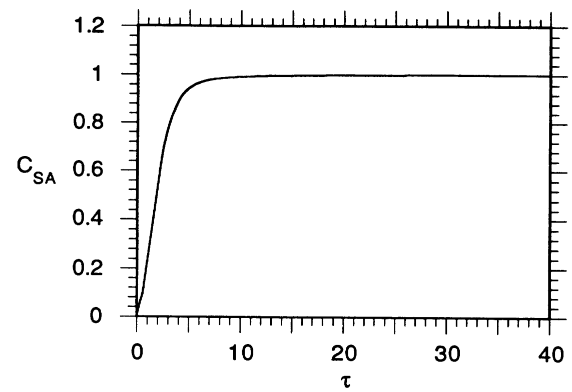

FIG. 5. Evolution of the cross correlation function of the Stokes and the anti-Stokes fields.

Finally, we demonstrate the strong cross correlation between the Stokes and the anti-Stokes fields by considering the quantity C_{SA} which is defined by

$$C_{SA} = \frac{\langle n_S n_A\rangle - \langle n_S\rangle\langle n_A\rangle}{[\langle(\Delta n_S)^2\rangle]^{1/2}[\langle(\Delta n_A)^2\rangle]^{1/2}} \tag{4.23}$$

where $n_i = \langle a_i^\dagger a_i\rangle$ $(i = S, A)$. This correlation function is bounded between -1 and $+1$. If the fields are independent of each other, C_{SA} has a value of zero. Using the results (4.10) and (4.11), the time dependence of C_{SA} is plotted in Fig. 5. Evidently, C_{SA} shows a positive correlation and it quickly approaches its maximum value $+1$, showing the strong correlation between the two cavity modes.

V. SUMMARY

In summary, we have found the solution of a model atom interacting simultaneously with two nondegenerate quantized cavity fields and a classically described (laser) field. The cavity frequencies are assumed to satisfy the resonance conditions for Raman transition processes, so that the cavity fields play the Stokes and anti-Stokes roles while the classical laser field acts as the pump.

We obtained explicit analytic expressions for the system evolution operator. We evaluated the atomic inversion when the atom is specially prepared in the ground state in the vacuum cavity field. The nature of the time dependence was found. We also studied the features of the photon states that are developed. The Stokes and the anti-Stokes fields exhibit nonclassical correlation. We found that the mean cavity photon number in each mode increases with τ^2. Our numerical results show that the photon distributions first exhibit sub-Poisson statistics and then become super-Poissonian, but the distributions are well localized in the Fock space.

ACKNOWLEDGMENTS

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[1] For reviews, see, for example, E. A. Hinds, in *Advances in Atomic, Molecular and Optical Physics,* edited by D. Bates and B. Bederson (Academic, New York, 1990), Vol. 29, p. 237; S. Haroche and D. Kleppner, Phys. Today **42** (1), 24 (1989); H. I. Yoo and J. H. Eberly, Phys. Rep. **118**, 239 (1985).

[2] P. L. Knight, Phys. Scr. **T12**, 51 (1986); S. J. D. Phoenix and P. L. Knight, J. Opt. Soc. Am. B **7**, 116 (1990).

[3] L. Schoendorff and H. Risken, Phys. Rev. A **41**, 5147 (1990).

[4] G. S Agarwal and R. R. Puri, Phys. Rev. A **43**, 3949 (1991).

[5] C. C. Gerry and J. H. Eberly, Phys. Rev. A **42**, 6805 (1990).

[6] R. R. Puri, Liwei Wang, and J.H. Eberly (private communication).

[7] There has been some previous work dealing with the steady-state response of a two-level atom in a multimode cavity subject to semiclassical excitation. See, for example, M. Sargent III, D. A. Holm, and M. S. Zubairy, Phys. Rev. A **31**, 3112 (1985); S. Stenholm, D. A. Holm, and M. Sargent III, *ibid.* **31**, 3124 (1985). Our paper is devoted to the transient regime, with Raman interactions.

[8] When $\xi \neq \eta$, the eigenvalues of $A^\dagger A$ can be shown to have discrete spacing $|\eta^2 - \xi^2|$. Our approximation $\xi = \eta$ is good if the interaction time is short compared with $|\eta^2 - \xi^2|^{-1/2}$.

[9] A direct use of (3.10) and (3.11) would yield rather complicated integrals. An easier way to derive (4.10) and (4.11) is to expand operators $\cos[(g^\dagger g)^{1/2}\tau]$ and $\sin[(g^\dagger g)^{1/2}\tau]$ in power series of τ and then operate on the vacuum state vector using the commutation relation $[g, g^\dagger] = 0$.