Λ-type lasers and masers: Lasing without inversion and a vector model

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We show that the problem of a Λ-type (and also a V-type) three-level single-mode laser (maser) is isomorphic to that of a two-level single-mode laser (maser) under the conditions of energy-level degeneracy and equal decay rates. This correspondence reveals that lasing without population inversion can occur in the Λ-type system with or without an initial atomic coherence, and also gives immediately the master equation of the laser (maser) field. We analyze in detail the operational and noise properties of a Λ-type laser using a Fokker-Planck approach. We also present a vector model to visualize the population trapping and the absorption and emission processes in such a laser (maser) system.

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I. INTRODUCTION

Recently various schemes have been proposed for achieving lasing action without population inversion [1–18]. These schemes can be classified into three classes. (i) If we inject active atoms into a cavity with an initial atomic coherence between the upper and lower transition levels, we can achieve lasing without population inversion in a two-level laser [3,4] and a two-level maser [6]. Such a laser and a maser with injected atomic coherence are the simplest systems for lasing without inversion. Lasing without inversion has also been found in other lasers with injected atomic coherence that involve more transition levels [15,19].

If we do not allow an initial atomic coherence between the upper and lower transition levels, lasing without inversion is still possible by using more atomic levels and/or using a driving field. (ii) For some schemes, one can find a pair of appropriate states between which there exists population inversion. These include a degenerate quantum-beat laser [2,12,15], some dressed-state lasers and masers [8–11,18] in which there exists population inversion between relevant dressed atom-field states [20,21], and others [14]. (iii) For other schemes [1,5,7,13,16,18], there is no population inversion between appropriate states [20,21]. In these schemes, a small atomic coherence generated by the atomic-relaxation processes contributes to the gain [17]; for example, in a resonantly pumped dressed-state laser, it is the dressed-state atomic coherence that leads to gain [18].

In classes (i) and (ii), the cavity fields build up due to a proper coherent dynamics of the atom-field interactions, i.e., due to a proper Rabi rotation of the Bloch vector [22] in an appropriate vector space. In general, if one finds lasing without inversion in a laser system, one also obtains lasing without inversion in a similar maser system [3,6,8], in which the effects of the atomic decay can be neglected. This is in contrast to class (iii), in which the atomic relaxation processes play a crucial role for providing gain in the absence of population inversion. Usually, no lasing without inversion is expected in a maser with a similar level structure. Because of the smallness of the atomic coherence, the linear gains in class (iii) are usually one order of magnitude [7,18], or even two orders of magnitude [7,13,16,17,23] if the lasing frequency is far off resonance with the appropriate (e.g., dressed) states, smaller than those in class (ii).

In this paper, we study a Λ-type three-level single-mode laser (maser), which belongs to classes (i) and (ii). This system is similar to, but simpler than, the degenerate quantum-beat laser [2]. This allows us to avoid the complexity caused by a *strong* microwave driving field, and to bring out the physics of the system in a simple and transparent manner. We show that the system is equivalent to a two-level single-mode laser (maser) under the conditions of energy-level degeneracy and equal decay rates. This correspondence reveals that lasing without population inversion can occur in the Λ-type system in several cases: (1) without any initial atomic coherence; (2) without initial atomic coherences between the upper and lower levels, but with an initial atomic coherence between the two lower levels; and (3) with an initial atomic coherence between the upper and lower levels. We analyze in detail the operational and noise properties of the Λ-type laser in cases (1) and (2) through a Fokker-Planck approach, and obtain the explicit expressions for the Glauber P function, the mean photon number, mode pulling, natural linewidth, and photon-number variance. We also present, in the resonant case, a vector model to explain the lasing without inversion in such a laser (maser) system.

Our vector model is different from those for two-level optical Bloch equations [22] and for a two-photon transition in a three-level system [24], in which the components of the Bloch vectors are combinations of density matrix elements. It is similar, however, to the vector model of an off-resonance three-level system [25], since probability amplitudes rather than density matrix elements are used in both vector models.

The paper is organized as follows. In Sec. II we show how to reduce Λ-type three-level single-mode laser (maser) systems to two-level laser (maser) systems. In Sec. III, using a Fokker-Planck approach, we study in detail the operational and noise properties of a Λ-type laser and derive an explicit expression for the Glauber P function. In Sec. IV, we present a vector model for resonant Λ-type laser and maser systems. Finally, we discuss in Sec. V how to modify the equations in Secs. II–IV to fit V-type three-level single-mode lasers and masers.

II. Λ-TYPE THREE-LEVEL LASERS AND MASERS: CORRESPONDENCE TO TWO-LEVEL LASERS AND MASERS

We consider Λ-type three-level active atoms interacting with a single mode of radiation field in a cavity (see Fig. 1). The atomic levels of the Λ-type three-level atom are denoted by $|a\rangle$, $|b\rangle$, and $|c\rangle$ with atomic energies $\hbar\omega_a$, $\hbar\omega_b$, and $\hbar\omega_c$, respectively. The upper level $|a\rangle$ is coupled with two lower levels $|b\rangle$ and $|c\rangle$ by direct electric dipole transitions. Both atomic transitions $a-b$ and $a-c$ are near resonant with the cavity field. The Hamiltonian for such an atom and the cavity field is (under the rotating-wave approximation)

$$H=H_0+V , \tag{2.1}$$

with

$$H_0=\hbar\Omega a^\dagger a+\sum_{\mu=a,b,c}\hbar\omega_\mu|\mu\rangle\langle\mu| , \tag{2.2a}$$

$$V=\hbar g_1|a\rangle\langle b|a+\hbar g_2|a\rangle\langle c|a+\text{H.c.} \tag{2.2b}$$

Here a and $a^\dagger$ are the field annihilation and creation operators, respectively; Ω is the passive cavity frequency; g_1 and g_2 are the atom-field coupling constants for the $a-b$ and $a-c$ transitions, respectively. Without loss of generality, we choose both g_1 and g_2 to be real in the following discussions.

We introduce two combination states for the atom,

$$|B\rangle=|b\rangle\cos\theta+|c\rangle\sin\theta , \tag{2.3a}$$

$$|C\rangle=-|b\rangle\sin\theta+|c\rangle\cos\theta , \tag{2.3b}$$

where

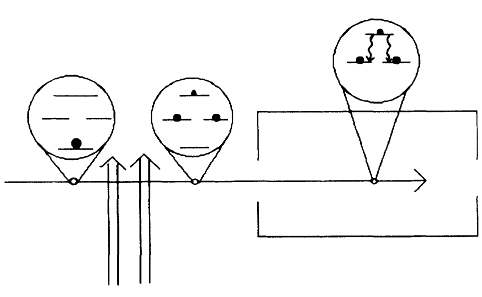

FIG. 1. Scheme of the Λ-type three-level lasers and masers. Before they enter the lasing cavity, the atoms in an atomic beam are pumped from the ground state to three excited states ($|a\rangle$, $|b\rangle$, and $|c\rangle$) which are involved in lasing action in the cavity. Initial atomic coherences ρ_{ab}, ρ_{ac}, and/or ρ_{bc} can be generated using this method.

$$\cos\theta=g_1/G,\quad \sin\theta=g_2/G ,\qquad G=(g_1^2+g_2^2)^{1/2} . \tag{2.4}$$

In the case of degeneracy of the two lower levels,

$$\omega_b=\omega_c , \tag{2.5}$$

we find that the Hamiltonians in Eqs. (2.2) become

$$H_0=\hbar\Omega a^\dagger a+\hbar\omega_a|a\rangle\langle a|+\hbar\omega_b(|B\rangle\langle B|+|C\rangle\langle C|) , \tag{2.6a}$$

$$V=\hbar G|a\rangle\langle B|a+\text{H.c.} \tag{2.6b}$$

in terms of the atomic combination states. Equation (2.6b) shows that the combination state $|C\rangle$ is decoupled from the cavity field, and we can omit the term $\hbar\omega_b|C\rangle\langle C|$ from Eq. (2.6a). Thus the problem of a Λ-type three-level single-mode maser is isomorphic to that of a two-level single-mode maser, with the atomic combination state $|B\rangle$ being the lower transition level. It follows from Eq. (2.3a) that the initial atomic population and coherence in the new basis are related to those in the old basis by the relation

$$\rho_{BB}=\rho_{bb}\cos^2\theta+\rho_{cc}\sin^2\theta+\tfrac{1}{2}(\rho_{bc}+\rho_{cb})\sin(2\theta) , \tag{2.7a}$$

$$\rho_{aB}=\rho_{ab}\cos\theta+\rho_{ac}\sin\theta . \tag{2.7b}$$

For a coherence-injection situation in which $\rho_{ab}\neq 0$ and $\rho_{ac}\neq 0$ (see Fig. 1 for the method of generating initial atomic coherences), by obtaining ρ_{BB} and ρ_{aB} from Eqs. (2.7), we can apply the results of Ref. [6] immediately and find lasing without population inversion in such a three-level maser. Such a Λ-type maser belongs to class (i) discussed in the Introduction.

In order to reduce the problem of a Λ-type three-level single-mode laser to that of a two-level single-mode laser, we need additional conditions regarding the relaxation processes in the Λ-type laser. In general, they should be in such a way that there is no population transfer decaying from the combination state $|C\rangle$ into the other one $|B\rangle$. In terms of the Scully-Lamb model [26,27] of lasers, in which transition levels decay to other lower lying levels, the density operator ρ_{af} for an atom and the field obeys the equation of motion

$$\dot\rho_{af}=-i\hbar^{-1}[H,\rho_{af}]-\tfrac{1}{2}(\hat\Gamma\rho_{af}+\rho_{af}\hat\Gamma) , \tag{2.8}$$

where

$$\hat\Gamma=\sum_{\mu=a,b,c}\Gamma_\mu|\mu\rangle\langle\mu| \tag{2.9}$$

is the atomic-decay operator with Γ_μ the decay rate of the atomic level $|\mu\rangle$ ($\mu=a,b,c$). Under the condition of equal decay rates

$$\Gamma_b=\Gamma_c , \tag{2.10}$$

the atomic-decay operator becomes

$$\hat\Gamma=\Gamma_a|a\rangle\langle a|+\Gamma_b(|B\rangle\langle B|+|C\rangle\langle C|) \tag{2.11}$$

in the new basis. In other words, under the condition (2.10), there is no relaxation coupling between the combination states $|B\rangle$ and $|C\rangle$. Consequently, under the conditions (2.5) and (2.10), the problem of a Λ-type three-level single-mode laser is isomorphic to that of a two-level single-mode laser. Again, for a coherence-injection situation in which $\rho_{ab}\neq 0$ and $\rho_{ac}\neq 0$, we can apply the result of Refs. [3,4] immediately, and obtain lasing without population inversion in such a three-level laser. Such a Λ-type laser also belongs to class (i) discussed in the Introduction.

In the rest of the section, we study in detail the situation of no initial atomic coherence between the upper and lower transition levels, $\rho_{ab}=\rho_{ac}=0$. This situation belongs to class (ii) discussed in the Introduction, since the problem of a Λ-type three-level single-mode laser (maser) reduces to that of an ordinary two-level single-mode laser (maser). Thus we know that the gain condition is $\rho_{aa}>\rho_{BB}$ [cf. Eq. (2.20)]. Two cases can be discussed within class (ii).

(1) No initial atomic coherence between the two lower levels, $\rho_{bc}=0$. In this ordinary case, lasing without inversion is possible when

$$\rho_{BB}=\tfrac{1}{2}(\rho_{bb}+\rho_{cc})<\rho_{aa}<\rho_{bb}+\rho_{cc}\ , \tag{2.12}$$

where we have used $g_1=g_2$, since the two lower levels are degenerate. Making use of $\rho_{aa}+\rho_{bb}+\rho_{cc}=1$, the condition for possible lasing without inversion becomes $\frac{1}{3}<\rho_{aa}<\frac{1}{2}$.

(2) With initial atomic coherence between the two lower levels, $\rho_{bc}\neq 0$. When $\rho_{bc}=-\sqrt{\rho_{bb}\rho_{cc}}$ and $\rho_{bb}/\rho_{cc}=\tan^2\theta$, we find that $\rho_{BB}=0$. Consequently, lasing without inversion is possible even when only a very small fraction of population is in the upper level $|a\rangle$, i.e., even when $\rho_{aa}\ll 1$. In this optimal situation, all the populations in the lower levels are in the other combination state $|C\rangle$ since $\rho_{CC}=\rho_{bb}+\rho_{cc}-\rho_{BB}=\rho_{bb}+\rho_{cc}$, and the population in the combination state $|C\rangle$ is coherently trapped [28].

To analyze the properties of Λ-type lasers and masers (including both $\rho_{bc}=0$ and $\rho_{bc}\neq 0$ cases) quantitatively, we need to obtain their master equations. Because of the correspondence, we get the desired master equations immediately from those of two-level lasers [4,26,27] and masers [29–31]. In the good-cavity limit, in which the cavity-loss rate γ is much smaller than the atomic decay rates Γ_μ ($\mu=a,b,c$) (in the laser case) or is much smaller than the inverse of the atom-field interaction time τ (in the maser case), the master equations in both laser and maser cases can be written in a general form,

$$\begin{aligned}\dot\rho_{nm}=&-\rho_{nm}(\rho_{aa}X_{n+1,m+1}+\rho_{BB}X^*_{nm})+\rho_{n-1,m-1}\rho_{aa}Y_{nm}\\&+\rho_{n+1,m+1}\rho_{BB}Y_{n+1,m+1}-i(\Omega-\nu)(n-m)\rho_{nm}\\&+\gamma\sqrt{(n+1)(m+1)}\rho_{n+1,m+1}-\tfrac{1}{2}\gamma(n+m)\rho_{nm}\ ,\end{aligned} \tag{2.13}$$

where ν is the oscillation frequency, and the explicit expressions for the coefficients X_{nm} and Y_{nm} depend on the values of the detuning Δ and, in the laser case, also on the relaxation rates. The Scully-Lamb model [26] of lasers assumes that atoms are injected into the laser cavity and the atomic levels involved in lasing action decay downward to other lower lying levels which are not involved in the lasing action. For simplicity, the atomic decay rates of the upper and lower levels involved in the lasing action are often taken to be equal [27]. For the Λ-type lasers studied in this paper, with equal downward decay rates $\Gamma_a=\Gamma_b=\Gamma_c\equiv\Gamma$, the coefficients in Eq. (2.13) are [4,26,27]

$$\begin{aligned}X^{(L)}_{nm}=&\tfrac{1}{2}\alpha[n+m+i(n-m)\delta\\&+(n-m)^2G^2/\Gamma^2]/\xi_{n-1,m-1}\ ,\end{aligned} \tag{2.14a}$$

$$Y^{(L)}_{nm}=\alpha\sqrt{nm}\,/\xi_{n-1,m-1}\ , \tag{2.14b}$$

$$\xi_{n-1,m-1}=1+\frac{\beta}{2\alpha}(n+m)+\frac{\beta^2}{16\alpha^2}(1+\delta)^2(n-m)^2\ . \tag{2.14c}$$

Here $\alpha=2r_aG^2/(\Gamma^2+\Delta^2)$, $\beta=8r_aG^4/(\Gamma^2+\Delta^2)^2$, and $\delta=\Delta/\Gamma$ are the linear gain, the saturation parameter, and the normalized detuning, respectively; r_a is the atomic injection rate; and $\Delta=\omega_{ab}-\nu=\omega_{ac}-\nu$ is the atom-field detuning [same for the $a-b$ and $a-c$ transitions due to the degeneracy condition (2.5)]. In the maser case, with zero detuning $\Delta=0$, the coefficients are [29–31]

$$X^{(M)}_{nm}=r_a[1-\cos(G\tau\sqrt{n})\cos(G\tau\sqrt{m})]\ , \tag{2.15a}$$

$$Y^{(M)}_{nm}=r_a\sin(G\tau\sqrt{n})\sin(G\tau\sqrt{m})\ . \tag{2.15b}$$

Note that there exist simple relations between the maser's and laser's coefficients,

$$\begin{aligned}&\int_0^\infty X^{(M)}_{nm}\Gamma e^{-\Gamma\tau}d\tau=X^{(L)}_{nm}\ ,\\&\int_0^\infty Y^{(M)}_{nm}\Gamma e^{-\Gamma\tau}d\tau=Y^{(L)}_{nm},\quad \Delta=0\ .\end{aligned} \tag{2.16}$$

For both lasers and masers, the diagonal coefficients X_{nn} and Y_{nn} satisfy the relation

$$X_{nn}=X^*_{nn}=Y_{nn}\equiv z_n\ . \tag{2.17}$$

Substitution of Eq. (2.17) into the master equation (2.13) gives rise to a detailed-balance relation in steady state for the diagonal elements ρ_{nn},

$$(n\gamma+\rho_{BB}z_n)\rho_{nn}=\rho_{aa}z_n\rho_{n-1,n-1}\ . \tag{2.18}$$

Equation (2.18) leads to the steady-state photon-number distribution in both laser and maser cases,

$$\rho_{nn}=\rho_{00}\prod_{l=1}^{n}\frac{\rho_{aa}z_l}{\gamma l+\rho_{BB}z_l}\ . \tag{2.19}$$

Let $\langle\hat n\rangle=n_0$ be the steady-state mean photon number, where $\hat n=a^\dagger a$ is the photon-number operator. Equation (2.18) also leads to the "semiclassical" equation for determining n_0 (or, in the maser case, positions at which the steady-state photon-number distribution peaks),

$$(\rho_{aa}-\rho_{BB})z_{n_0}=\gamma n_0\ . \tag{2.20}$$

Equation (2.20) confirms that the gain condition is

$\rho_{aa} > \rho_{BB}$. For the maser, numerical calculations are needed to determine the mean photon number n_0 and photon-number variance. For the laser, however, analytic expressions can be obtained for the mean photon number n_0 and the photon-number variance as well as for mode pulling and natural linewidth. In Sec. III, by using a Fokker-Planck approach, we calculate these quantities for a Λ-type laser which belongs to class (ii).

III. FOKKER-PLANCK APPROACH TO A Λ-TYPE LASER

We denote $|\mathcal{E}\rangle$ as the coherent state of the laser field, $a|\mathcal{E}\rangle = \mathcal{E}|\mathcal{E}\rangle$. We now convert the master equation (2.13) with coefficients (2.14) into a Fokker-Planck equation in the Glauber-Sudarshan P representation [32,33]. Using

$$\rho = \int P(\mathcal{E})|\mathcal{E}\rangle\langle\mathcal{E}| d^2\mathcal{E} , \tag{3.1}$$

we obtain the Fokker-Planck equation in the P representation,

$$\frac{\partial}{\partial t}P(\mathcal{E},t) = \left[-\frac{\partial}{\partial \mathcal{E}} d_{\mathcal{E}} + \frac{\partial^2}{\partial \mathcal{E}^* \partial \mathcal{E}} D_{\mathcal{E}^*\mathcal{E}} + \frac{\partial^2}{\partial \mathcal{E}^2} D_{\mathcal{E}\mathcal{E}} + \text{c.c.} \right] P(\mathcal{E},t) , \tag{3.2}$$

where the drift and diffusion coefficients can be derived under the assumption that the mean photon number in the field is much larger than unity, $n_0 \gg 1$. The derivation has been presented in Ref. [4] for more general initial atomic conditions. Specializing to the pumping situation of no initial atomic coherence between the upper and lower levels, the drift coefficient is

$$d_{\mathcal{E}} = \frac{\mathcal{E}}{2} \left[\frac{\alpha(\rho_{aa}-\rho_{BB})(1-i\delta)}{1+|\mathcal{E}|^2\beta/\alpha} - \gamma + 2i(\nu-\Omega) \right] , \tag{3.3}$$

and the diffusion coefficients are

$$D_{\mathcal{E}^*\mathcal{E}} = \frac{4\alpha\rho_{aa} + \beta(\rho_{aa}+\rho_{BB})(1+\delta^2)|\mathcal{E}|^2}{8(1+|\mathcal{E}|^2\beta/\alpha)} - \frac{\beta(\rho_{aa}-\rho_{BB})|\mathcal{E}|^2}{4(1+|\mathcal{E}|^2\beta/\alpha)^2} , \tag{3.4a}$$

$$D_{\mathcal{E}\mathcal{E}} = -\frac{\beta(\rho_{aa}+\rho_{BB})(1+\delta^2)\mathcal{E}^2}{8(1+|\mathcal{E}|^2\beta/\alpha)} - \frac{\beta(\rho_{aa}-\rho_{BB})(1-i\delta)\mathcal{E}^2}{4(1+|\mathcal{E}|^2\beta/\alpha)^2} . \tag{3.4b}$$

In order to study the intensity and phase properties of the laser field, we rewrite the Fokker-Planck equation (3.2) in terms of intensity and phase variables I and ϕ through the relation $\mathcal{E} = \sqrt{I}e^{i\phi}$,

$$\frac{\partial}{\partial t}P(I,\phi,t) = \left[-\frac{\partial}{\partial I}d_I - \frac{\partial}{\partial \phi}d_\phi + \frac{\partial^2}{\partial I^2}D_{II} + \frac{\partial^2}{\partial \phi^2}D_{\phi\phi} + 2\frac{\partial^2}{\partial I \partial \phi}D_{I\phi} \right] P(I,\phi,t) , \tag{3.5}$$

where

$$d_I(I) = \left[\frac{\alpha(\rho_{aa}-\rho_{BB})}{1+I\beta/\alpha} - \gamma \right] I , \tag{3.6a}$$

$$d_\phi(I) = \nu - \Omega - \frac{\alpha(\rho_{aa}-\rho_{BB})\delta}{2(1+I\beta/\alpha)} \tag{3.6b}$$

are the intensity- and phase-drift coefficients, respectively, and

$$D_{II}(I) = 2I[D_{\mathcal{E}^*\mathcal{E}} + \text{Re}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi})] = \frac{\alpha I(\rho_{aa}+\rho_{BB}I\beta/\alpha)}{(1+I\beta/\alpha)^2} , \tag{3.7a}$$

$$D_{\phi\phi}(I) = [D_{\mathcal{E}^*\mathcal{E}} - \text{Re}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi})]/2I = \frac{\alpha\rho_{aa} + \frac{1}{2}I\beta(1+\delta^2)(\rho_{aa}+\rho_{BB})}{4I(1+I\beta/\alpha)} , \tag{3.7b}$$

$$D_{I\phi}(I) = \text{Im}(D_{\mathcal{E}\mathcal{E}}e^{-i2\phi}) = \frac{\beta I(\rho_{aa}-\rho_{BB})\delta}{4(1+I\beta/\alpha)^2} \tag{3.7c}$$

are the intensity-, phase-, and cross-diffusion coefficients, respectively.

Above threshold $\alpha(\rho_{aa}-\rho_{BB}) \geq \gamma$, the laser field first builds up spontaneously in the cavity and then reaches its steady-state value n_0 ($\gg 1$), which is determined by the semiclassical equation

$$d_I(n_0) = 0 . \tag{3.8}$$

Such an n_0 is also the position at which the P function peaks in steady state. Substitution of Eq. (3.6a) into Eq. (3.8) gives the steady-state mean photon number

$$n_0 = \frac{\alpha}{\gamma} \frac{\alpha(\rho_{aa}-\rho_{BB})-\gamma}{\beta} , \tag{3.9}$$

which can be obtained alternatively by substituting Eqs. (2.17) and (2.14) into Eq. (2.20). Solution (3.9) is stable, since the "locking strength"

$$A_I \equiv \frac{\partial d_I(n_0)}{\partial I} = -\frac{\gamma}{\alpha}[\alpha - (\rho_{aa}-\rho_{BB})^{-1}\gamma] < 0 . \tag{3.10}$$

Another semiclassical equation $d_\phi(n_0)=0$ leads to the mode-pulling relation $\nu - \Omega = \frac{1}{2}\gamma\delta$, where use has been made of Eq. (3.9). The oscillation frequency is found to be

$$\nu = \frac{\Gamma\Omega + \frac{1}{2}\gamma\omega_{ab}}{\Gamma + \frac{1}{2}\gamma} . \tag{3.11}$$

In steady state the diffusion coefficients take their values at $I = n_0$. Making use of Eq. (3.9), we find from Eqs. (3.7) the steady-state intensity- and phase-diffusion coefficients,

$$D_{II}(n_0) = \frac{\gamma(\gamma+\alpha\rho_{BB})n_0}{\alpha(\rho_{aa}-\rho_{BB})} , \tag{3.12a}$$

$$D_{\phi\phi}(n_0)=\{\alpha(\rho_{aa}+\rho_{BB})+\gamma+(\rho_{aa}+\rho_{BB})[\alpha-(\rho_{aa}-\rho_{BB})^{-1}\gamma]\delta^2\}/8n_0 . \tag{3.12b}$$

Expression (3.12b) reduces to the result of Ref. [4] when $\delta=0$. The quantity $D_{\phi\phi}(n_0)$ is half the natural linewidth [34] of the Λ-type laser. Equation (3.12b) shows that the detuning δ increases phase fluctuations and the natural linewidth. When a Λ-type laser is far above threshold $\alpha(\rho_{aa}-\rho_{BB})\gg\gamma$, one finds from Eq. (3.12b) that $D_{\phi\phi}(n_0)=\alpha(\rho_{aa}+\rho_{BB})(1+\delta^2)/8n_0$, and thus the effect of the detuning δ becomes significant. Far above threshold, we may call the factors $(\rho_{aa}+\rho_{BB})$ and $(1+\delta^2)$ the population-induced and the detuning-induced linewidth enhancement factors, respectively.

Unlike the phase-diffusion coefficient $D_{\phi\phi}(n_0)$, the intensity-diffusion coefficient $D_{II}(n_0)$ itself is not a physical quantity and represents only a part of intensity fluctuations. We calculate the photon-number variance in the remaining of this section. Since the drift coefficients (3.6) and the diffusion coefficients (3.7) are independent of the phase variable ϕ, the steady-state solution of the Fokker-Planck equation (3.5) must be ϕ-independent too. Consequently, it satisfies the equation

$$\frac{\partial}{\partial I}\left[d_I-\frac{\partial}{\partial I}D_{II}\right]P(I)=0 . \tag{3.13}$$

The detailed-balance solution of Eq. (3.13) is

$$P(I)=\frac{C_0}{D_{II}(I)}\exp\left[\int_0^I\frac{d_I(x)}{D_{II}(x)}dx\right] , \tag{3.14}$$

where C_0 is a normalization constant. Using Eqs. (3.6a) and (3.7a), we obtain the following explicit expressions:

$$\int_0^I\frac{d_I(x)}{D_{II}(x)}dx=\frac{f_1+f_2\rho_{aa}\eta^{-1}}{\eta}I-\frac{f_2}{2\eta}I^2+\frac{f_0-(f_1+f_2\rho_{aa}\eta^{-1})\rho_{aa}\eta^{-1}}{\eta}\times\ln\left[1+\frac{\eta I}{\rho_{aa}}\right]\quad\text{when } \rho_{BB}\neq0 , \tag{3.15a}$$

$$\int_0^I\frac{d_I(x)}{D_{II}(x)}dx=\frac{1}{\rho_{aa}}(f_0I+\tfrac{1}{2}f_1I^2-\tfrac{1}{3}f_2I^3)\quad\text{when } \rho_{BB}=0 , \tag{3.15b}$$

where

$$f_0=\rho_{aa}-\rho_{BB}-\gamma/\alpha , \tag{3.16a}$$

$$f_1=(\rho_{aa}-\rho_{BB}-2\gamma/\alpha)\beta/\alpha , \tag{3.16b}$$

$$f_2=\gamma\beta^2/\alpha^3 , \tag{3.16c}$$

$$\eta=\rho_{BB}\beta/\alpha . \tag{3.16d}$$

As an approximation, we now expand the intensity-drift and intensity-diffusion coefficients d_I and D_{II} in Eq. (3.14) around $I=n_0$ up to first and zeroth order in $\delta I=I-n_0$, respectively, and obtain the linearized steady-state solution

$$P(I)=\frac{1}{\pi}\left[\frac{|A_I|}{2\pi D_{II}(n_0)}\right]^{1/2}\exp\left[-\frac{|A_I|(I-n_0)^2}{2D_{II}(n_0)}\right] . \tag{3.17}$$

As a function of I, $P(I)$ is a Gaussian distribution peaked at $I=n_0$ with a "variance" [by Eqs. (3.10) and (3.12a)]

$$\langle(\delta I)^2\rangle=\frac{D_{II}(n_0)}{|A_I|}=\frac{\alpha}{\beta}\left[1+\frac{\alpha\rho_{BB}}{\gamma}\right] . \tag{3.18}$$

Since the P function is a normal-ordering function, the photon-number variance is found after using Eqs. (3.18) and (3.9),

$$\langle(\Delta\hat{n})^2\rangle=\langle:(\Delta\hat{n})^2:\rangle+\langle\hat{n}\rangle=\langle(\delta I)^2\rangle+n_0=\frac{\alpha\rho_{aa}n_0}{\alpha(\rho_{aa}-\rho_{BB})-\gamma} . \tag{3.19}$$

Far above threshold $\alpha(\rho_{aa}-\rho_{BB})\gg\gamma$, the normalized photon-number variance reaches its lower limit $\langle(\Delta\hat{n})^2\rangle/\langle\hat{n}\rangle=\rho_{aa}/(\rho_{aa}-\rho_{BB})$, which is still larger than that of a Poisson distribution unless $\rho_{BB}=0$.

It is clear from Eqs. (3.9), (3.12b), and (3.19) that, for given parameters α, β, γ, and ρ_{aa}, a nonzero ρ_{BB} decreases the laser intensity n_0 and, at the same time, increases the natural linewidth $2D_{\phi\phi}(n_0)$ and the normalized photon-number variance $\langle(\Delta\hat{n})^2\rangle/\langle\hat{n}\rangle$ compared with a zero ρ_{BB}. Thus the coherence-injection case of $\rho_{bc}=-\sqrt{\rho_{bb}\rho_{cc}}$ is better than the usual case of $\rho_{bc}=0$, although one can find lasing without inversion in both cases.

IV. A VECTOR MODEL

The population trapping and the emission and absorption processes in the Λ-type lasers and masers can be visualized in terms of a vector model in the resonant case $\Delta=0$, in which there is no mode pulling $\nu=\Omega$ [by Eq. (3.11)]. Let us write the state vector of the coupled atom-field system as

$$|\Psi\rangle=\sum_{\mu=a,b,c}\sum_n\mu_n(t)|\mu\rangle|n\rangle e^{-i(\omega_\mu+n\Omega)t} , \tag{4.1}$$

where $\mu_n(t)$ are the probability amplitudes. Using the Schrödinger equation $i\hbar|\dot{\Psi}\rangle=H|\Psi\rangle$, we obtain the equations of motion for these probability amplitudes,

$$\frac{d}{dt}\begin{bmatrix}b_n\\ia_{n-1}\\c_n\end{bmatrix}=\begin{bmatrix}0&-g_1\sqrt{n}&0\\g_1\sqrt{n}&0&g_2\sqrt{n}\\0&-g_2\sqrt{n}&0\end{bmatrix}\begin{bmatrix}b_n\\ia_{n-1}\\c_n\end{bmatrix},\quad n=1,2,3,\ldots \tag{4.2}$$

where we have neglected the atomic relaxation terms for simplicity. Each set of equations in (4.2) can be put into the form of a vector model

$$\frac{d\mathbf{S}_n}{dt}=\mathbf{\Omega}_n\times\mathbf{S}_n, \quad n=1,2,3,\ldots \tag{4.3}$$

with an amplitude vector

$$\mathbf{S}_n=\begin{bmatrix} b_n \\ ia_{n-1} \\ c_n \end{bmatrix} \tag{4.4a}$$

and a driving field vector

$$\mathbf{\Omega}_n=\begin{bmatrix} -g_2\sqrt{n} \\ 0 \\ g_1\sqrt{n} \end{bmatrix}=G\sqrt{n}\begin{bmatrix} -\sin\theta \\ 0 \\ \cos\theta \end{bmatrix}. \tag{4.4b}$$

While the driving field vector $\mathbf{\Omega}_n$ is real, the amplitude vector $\mathbf{S}_n$ is complex. By separating $\mathbf{S}_n$ into real and imaginary parts $\mathbf{S}_n=\mathbf{v}_{1n}+i\mathbf{v}_{2n}$, the complex vector equation (4.3) can be decomposed into two real vector equations,

$$\frac{d\mathbf{v}_{ln}}{dt}=\mathbf{\Omega}_n\times\mathbf{v}_{ln}, \quad l=1,2 \tag{4.5}$$

where

$$\mathbf{v}_{1n}=\begin{bmatrix} \mathrm{Re}b_n \\ -\mathrm{Im}a_{n-1} \\ \mathrm{Re}c_n \end{bmatrix}, \quad \mathbf{v}_{2n}=\begin{bmatrix} \mathrm{Im}b_n \\ \mathrm{Re}a_{n-1} \\ \mathrm{Im}c_n \end{bmatrix}. \tag{4.6}$$

Equations (4.5) show that the two real amplitude vectors $\mathbf{v}_{1n}$ and $\mathbf{v}_{2n}$ rotate about the same driving field vector Ω_n as illustrated in Fig. 2.

Using the combination states $|B\rangle$ and $|C\rangle$ and the resonant condition $\Delta=0$, the state vector (4.1) becomes

$$|\Psi\rangle=\sum_n(a_{n-1}|a\rangle|n-1\rangle+B_n|B\rangle|n\rangle+C_n|C\rangle|n\rangle)e^{-i(\omega_b+n\Omega)t}. \tag{4.7}$$

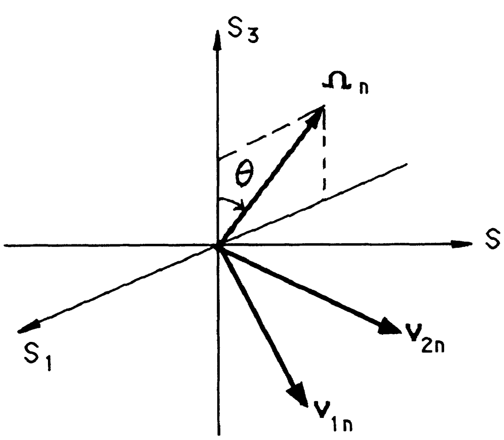

FIG. 2. The vector model of a resonant three-level system. Two real amplitude vectors $\mathbf{v}_{1n}$ and $\mathbf{v}_{2n}$ rotate about the same driving field vector $\mathbf{\Omega}_n$, $n=1,2,3,\ldots$.

The relations between the probability amplitudes in the two atomic basis are found after using Eqs. (2.3),

$$b_n=B_n\cos\theta-C_n\sin\theta, \tag{4.8a}$$

$$c_n=B_n\sin\theta+C_n\cos\theta. \tag{4.8b}$$

Thus the amplitude vector for the trapping state $|C\rangle|n\rangle$, for which we can write $B_n=0$ and $C_n=1$, is parallel to the driving field vector $\mathbf{\Omega}_n$,

$$\mathbf{S}_n^{(C)}=\mathbf{v}_{1n}^{(C)}=\frac{1}{G\sqrt{n}}\mathbf{\Omega}_n. \tag{4.9}$$

When an atom is initially in the trapping state $|C\rangle$, we find that $d\mathbf{S}_n^{(C)}/dt=0$, $n=1,2,3,\ldots$. Thus none of the amplitude vectors $\mathbf{v}_{ln}^{(C)}$ rotates. In other words, one finds population trapping but no absorption (since there is no atomic population transfer from the lower states to the upper state $|a\rangle$). On the other hand, the amplitude vector for the combination state $|B\rangle|n\rangle$, for which we can write $B_n=1$ and $C_n=0$, is perpendicular to the driving field vector $\mathbf{\Omega}_n$,

$$\mathbf{S}_n^{(B)}\cdot\mathbf{\Omega}_n=\mathbf{v}_{1n}^{(B)}\cdot\mathbf{\Omega}_n=0. \tag{4.10}$$

Thus if an atom is initially in the combination state $|B\rangle$, its amplitude vectors $\mathbf{v}_{1n}^{(B)}$ $(n=1,2,3,\ldots)$ will rotate about the driving field vectors $\mathbf{\Omega}_n$, and there will be absorption. While the absorption process can be canceled (e.g., when an atom is initially in the trapping state $|C\rangle$), the emission process always occurs when an atom is initially in the upper state $|a\rangle$, since its amplitude vectors are perpendicular to the driving field vectors $\mathbf{\Omega}_n$ and rotate.

Combining the above physical pictures for each of the three atomic states $|C\rangle$, $|B\rangle$, and $|a\rangle$ together, we can visualize lasing without inversion for various situations. For example, in the optimal situation discussed in Sec. II, we have the following picture: The amplitude vectors for atoms initially in the upper state $|a\rangle$ rotate about the driving field vectors $\mathbf{\Omega}_n$, but those for atoms initially in the lower state $|C\rangle$ do not rotate; i.e., there is emission but not absorption.

V. DISCUSSION

So far we have studied Λ-type three-level single-mode lasers and masers in which $\omega_a>\omega_b,\omega_c$. For a V-type three-level single-mode laser (maser) in which $\omega_a<\omega_b,\omega_c$, we now prove that it is also isomorphic to a two-level single-mode laser (maser) under the conditions (2.5) and (2.10) [condition (2.5)], with the atomic combination state $|B\rangle$ being the *upper* transition level. The proof goes from Eq. (2.1) to Eq. (2.11) with the exchange $a\leftrightarrow a^\dagger$ in the interaction Hamiltonian V in Eqs. (2.2b) and (2.6b). For V-type lasers and masers, "inversion without lasing" [2] will occur under conditions (2.12); Eqs. (2.13)–(3.19) are valid provided that we make the exchange $\rho_{aa}\leftrightarrow\rho_{BB}$ in them and redefine $\Delta=\omega_{ba}-\nu$; and Eqs. (4.1)–(4.10) are valid provided that we let $a_{n-1}\rightarrow a_{n+1}$, $|n-1\rangle\rightarrow|n+1\rangle$, and $\sqrt{n}\rightarrow\sqrt{n+1}$.

In summary, we have proved that, under the conditions (2.5) and (2.10) [condition (2.5)] the problem of a Λ-type three-level single-mode laser (maser) is isomorphic to that of a two-level single-mode laser (maser). This connection reveals that lasing without population inversion can occur in the Λ-type system in several cases: (1) without any initial atomic coherence, $\rho_{ab}=\rho_{ac}=\rho_{bc}=0$; (2) without initial atomic coherences between the upper and lower levels, $\rho_{ab}=\rho_{ac}=0$, but with an initial atomic coherence between the two lower levels, $\rho_{bc}\neq 0$; and (3) with initial atomic coherences between the upper and lower levels, $\rho_{ab}\neq 0$ and $\rho_{ac}\neq 0$. In cases (1) and (2), the problem reduces to that of an ordinary two-level laser (maser), and we have found laser (maser) master equations and photon statics. For the Λ-type laser, we have obtained in Sec. III the explicit expressions for the Glauber P function, the mean photon number, mode pulling, natural linewidth, and photon-number variance. In case (3), the problem is equivalent to that of a two-level laser (maser) with injected atomic coherence, which has been studied in Refs. [3,4] (Ref. [6]). In Sec. IV, we have presented a simple vector model to account for the population trapping and the lasing without inversion in such a laser (maser) system. Finally, we have shown that a V-type three-level single-mode laser (maser) is also isomorphic to a two-level single-mode laser (maser) under the conditions (2.5) and (2.10) [condition (2.5)].

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