Saturation effect in a laser at steady state

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The saturation properties of a single-mode laser are fully taken into account. The analytic expressions of the mean, variance, and skewness of the steady-state laser intensity are calculated. Compared with third-order laser theory and with experimental measurements, good agreement is obtained on the variance of the intensity, but noticeable deviations occur in the mean and skewness when the laser is operated far above threshold.

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I. INTRODUCTION

The applications of lasers are determined by the statistical properties in laser radiation. Experimental measurements [1–5] and theoretical analyses [6–9] of the intensity fluctuations in a laser operated at steady state showed that a nonlinear Brownian motion can be employed to describe the electric field in a laser. To characterize the statistical fluctuations of the laser light, an additive white noise is included in the conventional laser theory. However, the conventional laser model used in the previous analyses [1–9] only contains cubic nonlinearities. The theory thus includes saturation effects to third order in the field. The third-order theory, which is often called the cubic model, is valid when the laser is operated near or slightly above threshold. When the laser is operated far above threshold in practically every application, the saturation properties of the laser should be fully taken into account.

In this paper, both the cubic model and the model with full saturation effects are used to investigate the statistical fluctuations of a single-mode laser operated at steady state. In Sec. II, exact analytic expressions of the mean, variance, and skewness of the laser intensity are presented. In Sec. III, these solutions are compared with previous theoretical and experimental works [1–9]. A discussion of the results from the two laser models concludes the paper.

II. THEORETICAL ANALYSIS

The dimensionless complex electric field of a single-mode laser with cubic nonlinearities follows the Langevin equation

$$\frac{dE}{dt}=a_0E-A|E|^2E+q(t)\ , \tag{1}$$

with

$$\langle q_i(t)q_j(t')\rangle=P\delta_{ij}\delta(t-t')\quad (i,j=1,2)\ , \tag{2}$$

where a_0 and A are the net gain and self-saturation coefficients, $q(t)$ is the quantum noise, and P is the noise strength.

The single-mode laser with a full account of the saturation effects follows the equation [10]

$$\frac{dE}{dt}=-KE+\frac{F_1E}{1+A|E|^2/F_1}+q(t)\ , \tag{3}$$

where K is the decay constant for the electric field, F_1 is the gain parameter with $F_1=a_0+K$, and $q(t)$ is the same as that in Eq. (2). A simple binomial expansion of the denominator leads to the cubic model in Eq. (1).

The corresponding Fokker-Planck equation for the probability function $Q(I,t)$ of the laser intensity $I=|E|^2$ is given by [11]

$$\frac{\partial Q(I,t)}{\partial t}=-\frac{\partial}{\partial I}[F(I)Q(I,t)]+\frac{\partial^2}{\partial I^2}[D(I)Q(I,t)]\ , \tag{4}$$

where

$$F_c(I)=2a_0I-2AI^2+2P\ ,\quad D_c(I)=2PI\ , \tag{5}$$

for the cubic laser model [1–9], and

$$F_s(I)=-2KI+\frac{2F_1I}{1+AI/F_1}+2P\ ,\quad D_s(I)=2PI\ , \tag{6}$$

for the full saturation laser model.

The stationary solution of Eq. (4) can be obtained exactly for the two laser models. The analytic expressions of the mean, variance, and skewness of the steady-state laser intensity can be calculated without any difficulties. For the cubic laser model, the steady-state intensity distribution function is given by [7,12]

$$Q_c(I)=\left[\frac{2A}{P\pi}\right]^{1/2}\left[\frac{\exp\left[-\frac{A}{2P}\left[I-\frac{a_0}{A}\right]^2\right]}{1+\mathrm{erf}[a_0/(2PA)^{1/2}]}\right]\ . \tag{7}$$

The mean laser intensity is of the form [7]

$$\langle I\rangle_c=\frac{a_0}{A}+\left[\frac{2P}{A\pi}\right]^{1/2}\left[\frac{\exp(-a_0^2/2PA)}{1+\mathrm{erf}[a_0/(2PA)^{1/2}]}\right] . \tag{8}$$

The variance of the laser intensity is

$$\lambda_{2c}(0)=\frac{\langle(\Delta I)^2\rangle_c}{\langle I\rangle_c^2}=\frac{P}{A\langle I\rangle_c^2}+\frac{a_0}{A\langle I\rangle_c}-1 , \tag{9}$$

and the normalized skewness of the laser intensity is

$$\lambda_{3c}(0)=\frac{\langle(\Delta I)^3\rangle_c}{\langle I\rangle_c^3}=\frac{a_0P}{A^2\langle I\rangle_c^3}+\left[\left[\frac{a_0}{A}\right]^2+\frac{2P}{A}\right]\frac{1}{\langle I\rangle_c^2}-3\lambda_{2c}(0)-1 . \tag{10}$$

For the full saturation laser model, the steady-state intensity distribution function can be written as [12]

$$Q_s(I)=\frac{K[\alpha(1+AI/F_1)]^\beta\exp[-\alpha(1+AI/F_1)]}{P\Gamma(\beta+1,\alpha)} . \tag{11}$$

The mean intensity is of the form

$$\langle I\rangle_s=\frac{F_1(F_1-K)+PA}{AK}+\frac{P}{K}\left[\frac{\alpha^{\beta+1}e^{-\alpha}}{\Gamma(\beta+1,\alpha)}\right] . \tag{12}$$

The variance of the intensity is

$$\lambda_{2s}(0)=\frac{\langle(\Delta I)^2\rangle_s}{\langle I\rangle_s^2}=\frac{F_1P}{AK\langle I\rangle_s^2}+\frac{F_1(F_1-K)+2PA}{AK\langle I\rangle_s}-1 , \tag{13}$$

and the normalized skewness of the intensity is

$$\begin{aligned}\lambda_{3s}(0)&=\langle(\Delta I)^3\rangle_s/\langle I\rangle_s^3\\&=\frac{P}{\langle I\rangle_s^3}\left[\frac{F_1}{AK}\right]^2\left[(F_1-K)+\frac{3PA}{F_1}\right]+\left[\frac{F_1}{AK}\right]^2\left[(F_1-K)^2+\frac{3PA}{F_1}(F_1-K)+2PA\left[1+\frac{3PA}{F_1^2}\right]\right]\frac{1}{\langle I\rangle_s^2}\\&\quad-3\lambda_{2s}(0)-1 ,\end{aligned} \tag{14}$$

where

$$\alpha=KF_1/PA , \quad \beta=F_1^2/PA , \tag{15}$$

and $\Gamma(\beta+1,\alpha)$ is the incomplete gamma function, which is given by

$$\Gamma(\omega,z)=\int_z^\infty dx\, e^{-x}x^{\omega-1} . \tag{16}$$

III. COMPARISON OF THEORIES AND EXPERIMENTS

To check the accuracy of the cubic and full saturation laser models, the analytic results from the two theories have been compared with the experimental measurements from Refs. [1,4,15]. These experiments were carried out in the near-threshold region of a laser.

Figure 1 shows the steady-state distribution function in Eqs. (7) and (11) as a function of the laser intensity for different pump parameters. Curves (a), (b), and (c) are the distribution functions when the laser is operated below threshold ($a_0=-5$), at threshold ($a_0=0$), and above threshold ($a_0=5$). In order to see the differences of $Q_s(I)$ between the two laser models, $Q_s(I)$ is plotted on a logarithmic scale. The cavity decay constant K in a typical laser was taken to be 105 for a laser operated 6% above threshold and 5 for a laser operated 20% above threshold when the net gain coefficient a_0 was held fixed [13,14]. In Fig. 1, K is taken to be 20. It is clear that there is almost no difference between the two laser models when the laser is operated below or at threshold for $I<4$. There are noticeable deviations when the laser is operated above threshold or when $I\geq 4$ for $a_0\leq 0$.

The mean laser intensity $\langle I\rangle$ as a function of pump parameter a_0 for the two laser models is plotted in Fig. 2. It is shown that there is almost no difference between the two laser theories even for different values of K when the laser is operated below threshold. However, when the laser is operated above threshold, the curve of $\langle I\rangle$ from the full saturation model always gives larger values than that from the cubic model. For large values of K, the curves from the two laser models are close to each other. In this case, the binomial expansion of the denominator in Eq. (3) up to $|E|^2$ is a good approximation when the laser is operated not far above threshold.

The normalized variance $\lambda_2(0)$ and skewness $\lambda_3(0)$ of

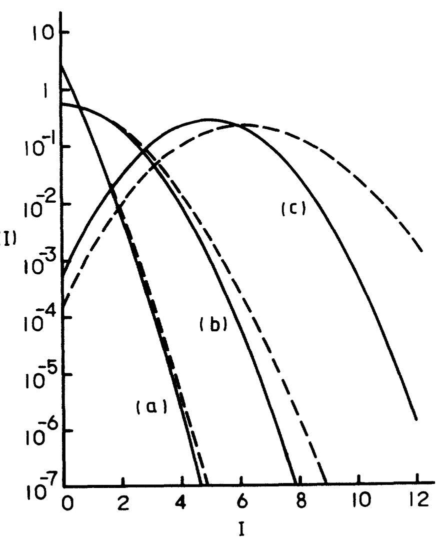

FIG. 1. Steady-state laser intensity distribution functions with $A=1$ and $P=2$. ——, Eq. (7); — — — Eq. (11) with $K=20$ and $F_1=a_0+K$. (a) $a_0=-5$; (b) $a_0=0$; (c) $a_0=5$.

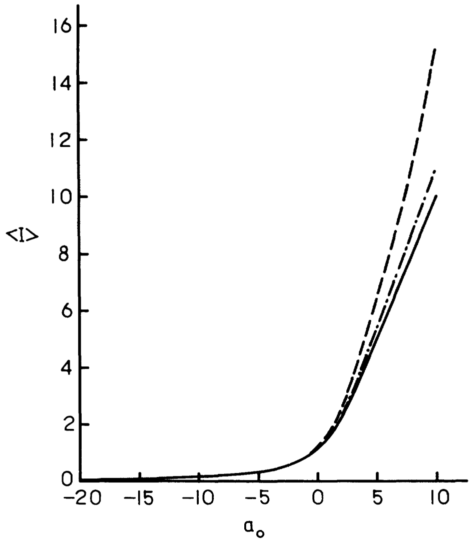

FIG. 2. Mean laser intensity as a function of the pump parameter a_0 with $A=1$ and $P=2$. ——, Eq. (8); — — — Eq. (12) with $K=20$; —·—·—· Eq. (12) with $K=100$.

the laser intensity are plotted in Figs. 3(a) and 3(b) together with the experimental data from Refs. [1] and [4]. Figure 3(a) shows that excellent agreement between the laser theories and the experimental measurements is obtained in $\lambda_2(0)$. There is almost no difference between the two laser models. The deviations from the two laser models are less than 2% even for very small values of K and are within the scope of the experimental errors [1–5]. It is also interesting to note that the curve from the full saturation model is higher than that from the cubic model when the laser is operated below and slightly above threshold but is lower when far above threshold. The turning point is at $a_0 \approx 2.5$ for $K=20$. Thus $\lambda_{2s}(0) > \lambda_{2c}(0)$ for $a_0 < 2.5$ and $\lambda_{2s}(0) < \lambda_{2c}(0)$ for $a_0 > 2.5$.

In Fig. 3(b), it is clear that noticeable deviations in $\lambda_3(0)$ occur between the two laser models even for the laser operated well below threshold. The full saturation model always gives larger values than the cubic model. However, the agreement between the theories and experiments is reasonably good.

In short, Fig. 3 is a plot of the laser fluctuations when the laser is operated in the threshold region. It is shown that the cubic laser model is an excellent approximation of the full saturation laser model and is valid when the laser is operated below and slightly above threshold.

When the laser is operated far above threshold, the asymptotic expressions of the variance and skewness of the laser intensity can be written as

$$\lambda_{2c}(0) \approx \frac{AP}{a_0^2}\ , \tag{17}$$

$$\lambda_{3c}(0) \approx \left[\frac{AP}{2\pi a_0^2}\right]^{1/2} \left[1-\frac{AP}{a_0^2}\right] \exp(-a_0^2/2AP) \tag{18}$$

for the cubic laser model and

$$\lambda_{2s}(0) \approx \frac{AP(F_1^2+AP)}{[F_1(F_1-K)+AP]^2}\ , \tag{19}$$

$$\lambda_{3s}(0) \approx \frac{2(AP)^2(F_1^2+AP)}{[F_1(F_1-K)+AP]^3} \tag{20}$$

for the full saturation laser model.

These equations are plotted in Figs. 4(a) and 4(b) when the laser is operated above threshold ($a_0 > 0$). For slightly above threshold ($a_0 < 10$), $\lambda_2(0)$, and $\lambda_3(0)$ are determined by Eqs. (9) and (13) and (10) and (14), respectively. Far above threshold ($a_0 > 10$), $\lambda_2(0)$, and $\lambda_3(0)$ can be approximated by Eqs. (17) and (19) and (18) and (20) without introducing large errors. In order to see the difference from the two laser theories, $\lambda_2(0)$ and $\lambda_3(0)$ are plotted on a logarithmic scale. Figure 4(a) is a plot of $\lambda_2(0)$ from the two laser models together with the mea-

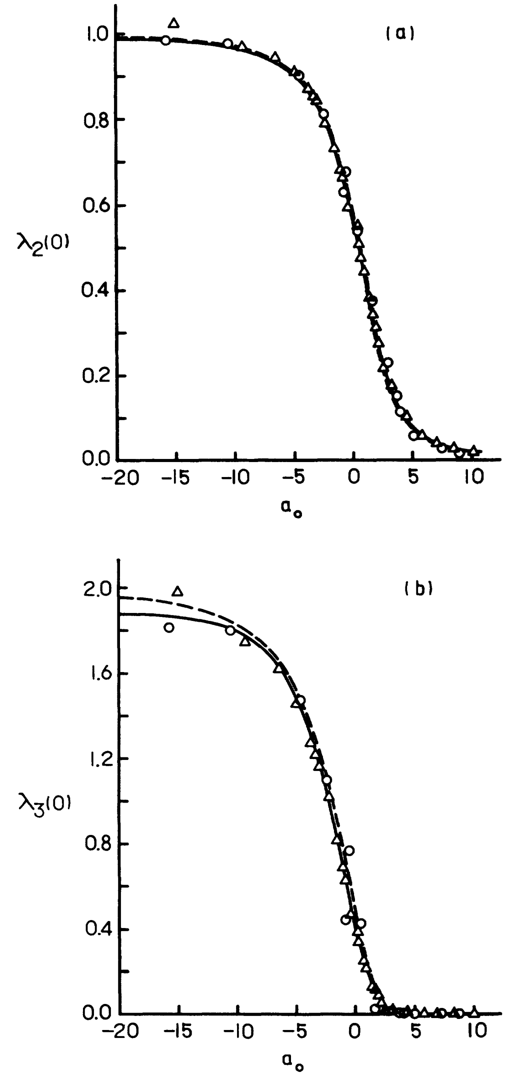

FIG. 3. Normalized variance and skewness of the laser intensity as a function of the pump parameter a_0 with $A=1$ and $P=2$. ○ and △, experimental measurements from Ref. [1] (○) and Ref. [4] (△); ——, Eq. (9); — — —, Eq. (13) with $K=20$ and $F_1=a_0+K$. (a) The variance $\lambda_2(0)$ as a function of a_0. (b) The skewness $\lambda_3(0)$ as a function of a_0.

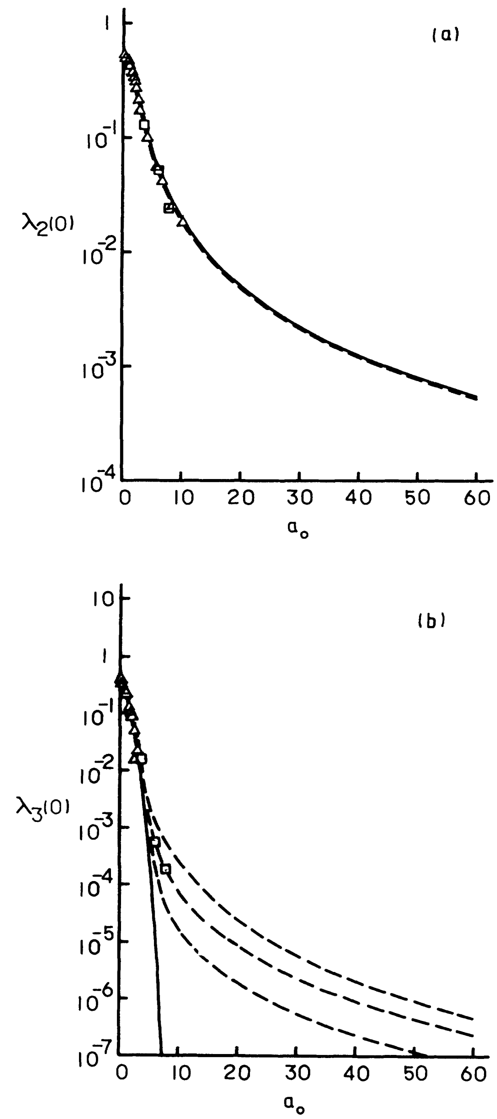

FIG. 4. Normalized variance and skewness of the laser intensity as a function of the pump parameter a_0 when the laser is operated above threshold ($a_0>0$) with $A=1$ and $P=2$. $\triangle$ and $\square$, experimental measurements from Ref. [4] ($\triangle$) and Ref. [15] ($\square$). (a) The variance of the laser intensity: ——, Eq. (9) for $a_0 \leq 10$ and Eq. (17) for $a_0 > 10$; — — —, Eq. (13) for $a_0 \leq 10$ and Eq. (19) for $a_0>10$ with $K=20$. (b) The skewness of the laser intensity: ——, Eq. (10) for $a_0 \leq 10$ and Eq. (18) for $a_0>10$; — — —, Eq. (14) for $a_0 \leq 10$ and Eq. (20) for $a_0 > 10$ with (from top to bottom) $K=20$, 100, and 500.

surements of Refs. [4] and [15]. It is clear that excellent agreement between the theories and the experiments is obtained. The curves from the two models are indistinguishable and within the thickness of the line. The relative error is no more than 1.5% when the laser is operated at threshold and decreases with increasing values of a_0.

Figure 4(b) is a plot of the skewness $\lambda_3(0)$ of the two laser models together with the experimental measurements from Refs. [4] and [15]. It is difficult to get correct values of the experimental data from Ref. [4] for a large value of a_0 since the measured data were less than 10^{-2} and cannot be determined from the figure when $a_0>4$. For large values of a_0, the measured data were taken from Ref. [15]. Though the maximum value of a_0 in the measurement of Ref. [15] is $a_0=8$, the tendency of the curve shows that the experimental data can be fitted by Eqs. (14) and (20) with $K=100$. It is clear that large deviations occur between the two models even though the laser is operated slightly above threshold for $a_0>5$. However, when the value of the cavity decay constant K is large, the difference between the two theories becomes small. In the near-threshold region, the results from the two theories are close to each other. This shows that the cubic model is valid only when the laser is operated below or slightly above threshold with a large value of K.

IV. DISCUSSION

The laser intensity distribution function given by Eq. (7) from the third-order theory is similar to a normal distribution with a peak at

$$I_{\max}=\begin{cases} a_0/A & (a_0 \geq 0) \\ 0 & (a_0<0)\ , \end{cases} \tag{21}$$

while that given by Eq (11) from the full saturation theory is similar to a Γ distribution with a maximum value at

$$I_{\max}=\begin{cases} \dfrac{F_1}{A}\left[\dfrac{F_1}{K}-1\right] & (F_1-K\geq 0) \\ 0 & (F_1-K<0)\ . \end{cases} \tag{22}$$

When the laser is operated above threshold, the shift of the peak of the distribution function introduces large errors in the mean and skewness of the laser intensity between the two laser models. However, the value of the variance is affected only a small amount, no more than 2%. If only the second-order intensity fluctuations are concerned, either of the laser models can be accepted in the theoretical analysis.

When the laser is operated below or slightly above threshold, the two laser theories give almost the same results. However, when the laser is operated far above threshold, the full saturation theory will give a correct picture of the laser behavior while the third-order theory will introduce relatively large errors in the analysis.

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