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# Phase-space representation of amplitude-squared squeezing

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Phase-space pictures have proven to be useful in analyzing standard squeezing. Here such pictures are developed for amplitude-squared squeezing, a form of higher-order squeezing. They are used to examine amplitude-squared squeezing of the squeezed vacuum state and to suggest a method of producing amplitude-squared squeezed states using a nonlinear interferometer. The phase-space representation of a variable which is a mixture of the usual amplitude-squared squeezing variables is also discussed.

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## I. INTRODUCTION

Phase-space pictures have proven to be a useful technique in the analysis of problems in quantum optics. There have been two main approaches to the use of such pictures. The first describes a single-mode field state by means of a complex amplitude and an error box which corresponds to the uncertainty of the amplitude. The shape of the error box depends on the state. For example, it is circular for a coherent state and it is elliptical for a squeezed state. Pictures of this type have been used in an analysis of interferometers by Caves [1] and in a study of the validity of the parametric approximation by Caves and Crouch [2]. A second approach, that of interference in phase space, has been explored by Schleich and Wheeler [3–5]. Here one examines the overlap between phase-space regions corresponding to different quantum states. The inner product of two states can be built up from a knowledge of these overlaps. Development of these techniques allowed Schleich and Wheeler to predict oscillations in the photon-number distribution of squeezed states. These techniques have also been extended to two modes and a four-dimensional phase space [6].

The phase space which we shall consider has the real part of the mode amplitude on the $x$ axis and the imaginary part on the $y$ axis. Quantum mechanically the complex amplitude of a mode corresponds to the mode-annihilation operator $a$ and the real and imaginary parts correspond to the operators

$$X_1=(a^\dagger+a)/2,\quad X_2=i(a^\dagger-a)/2\ , \tag{1.1}$$

respectively. In this phase space, "eigenstates" of $X_1$ and $X_2$ are represented by lines parallel to the $x_2$ ($y$ axis) and the $x_1$ ($x$ axis) axes, respectively. Note that in the preceding sentence mode operators are written with capital letters and their corresponding $c$-number quantities have been written with small letters. This convention will be adhered to for the rest of this paper. The use of quotation marks is to indicate that normalizable eigenstates do not exist though states whose uncertainty is arbitrarily small in either of these variables do.

As was mentioned previously a coherent state is represented as a circular region. The circle is centered on the point $(x_1,x_2)=(\langle X_1\rangle\ \langle X_2\rangle)$ and has a radius of $\frac{1}{2}$. The reason for this is as follows. The variable whose "eigenstates" correspond to lines making an angle of $\theta+\pi/2$ with the $x_1$ axis is

$$X(\theta)=(e^{i\theta}a^\dagger+e^{-i\theta}a)/2=X_1\cos\theta+X_2\sin\theta\ . \tag{1.2}$$

The magnitude of the fluctuations of a state $|\psi\rangle$ in the direction at an angle $\theta$ to the $x_1$ axis is given by

$$\Delta X(\theta)=\{\langle\psi|[X(\theta)]^2|\psi\rangle-\langle\psi|X(\theta)|\psi\rangle^2\}^{1/2}\ . \tag{1.3}$$

For a coherent state $\Delta X(\theta)=\frac{1}{2}$ for all values of $\theta$ and this leads to the circular error box of radius $\frac{1}{2}$. What this representation is meant to describe is that the amplitude of the state has a mean value of $\langle a\rangle=\langle X_1\rangle+i\langle X_2\rangle$ but fluctuates within the error box (see Fig. 1). As is well known the complex amplitude cannot be specified precisely because the operators corresponding to its real and imaginary parts do not commute [7]. In fact, the size of the error box is constrained by the requirement that $\Delta X_1\Delta X_2\geq\frac{1}{4}$.

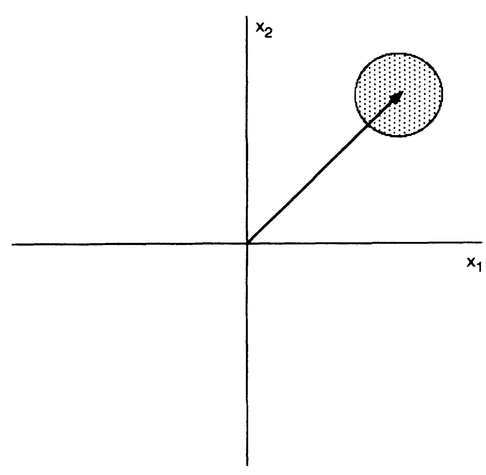


FIG. 1. Phase-space representation of a coherent state. The circular "error box" has a radius of $\frac{1}{2}$ and represents the uncertainty in the complex field amplitude of the state.

 

In this paper these techniques will be applied to amplitude-squared squeezing, a form of higher-order squeezing [8,9]. In this type of squeezing the fluctuations of the square of the mode amplitude are smaller in some directions in phase space than they can be for a classical state. Amplitude-squared squeezing is an example of su(1,1) squeezing; the variables which describe it form a representation of the su(1,1) Lie algebra [10]. We shall also consider phase-space representations for more general types of su(1,1) squeezing which are in fact, variants of amplitude-squared squeezing. An illustration of how these techniques can be used to suggest ways of producing amplitude-squared squeezed states, which makes use of an idea due to Kitagawa and Yamamoto [11], will also be presented.

## II. AMPLITUDE-SQUARED SQUEEZING

Amplitude-squared squeezing is described in terms of the real and imaginary parts of the square of the field amplitude. These quantities correspond to the operators

$$Y_1=(a^{\dagger 2}+a^2)/2=X_1^2-X_2^2 ,$$
$$Y_2=i(a^{\dagger 2}-a^2)/2=X_1X_2+X_2X_1 , \tag{2.1}$$

respectively. These operators obey the uncertainty relation [8]

$$\Delta Y_1 \Delta Y_2 \geq \langle N+\tfrac{1}{2}\rangle , \tag{2.2}$$

where $N=a^\dagger a$ is the photon number. A state is said to be amplitude-squared squeezed in the $Y_1$ direction if

$$(\Delta Y_1)^2 < \langle N+\tfrac{1}{2}\rangle , \tag{2.3}$$

and such a state is nonclassical.

Let us now proceed to the phase-space representation of these variables. An "eigenstate" of $Y_1$ with eigenvalue $y_1$ corresponds to the curve $y_1=x_1^2-x_2^2$. This is a hyperbola with asymptotes $x_2=\pm x_1$. This is pictured in Fig. 2. If $y_1>0$, then the hyperbola intersects the $x_1$ axis at $x_1=\pm\sqrt{y_1}$, and if $y_1<0$, then it intersects the $x_2$ axis at $x_2=\pm\sqrt{|y_1|}$. An "eigenstate" of $Y_2$ with eigenvalue $y_2$ corresponds to the hyperbola $y_2=2x_1x_2$. Hyperbolas of this form have the $x_1$ and $x_2$ axes as asymptotes and lie in the first and third quadrants if $y_2>0$ and the second and fourth quadrants if $y_2<0$. This is shown in Fig. 3.

The third operator which appears in the condition for amplitude-squared squeezing is the number operator $N$. In phase space an eigenstate of $N$ with eigenvalue $n$ is represented as a circular band [4]. This is shown in Fig. 4. The band has a radius $\sqrt{n+1/2}$ and a width of $1/2\sqrt{n}$. This follows from the fact that in terms of $X_1$ and $X_2$, $N$ is given by

$$N=X_1^2+X_2^2-\tfrac{1}{2} . \tag{2.4}$$

Therefore a number state $|n\rangle$ corresponds to the curve $x_1^2+x_2^2=n+\frac{1}{2}$, which is a circle of radius $\sqrt{n+1/2}$. However, unlike the eigenvalues of $Y_1$ or $Y_2$ which are continuous, the eigenvalues of $N$ are discrete. On the other hand, the union of all the phase-space representations of number states must fill phase space, because the number states are complete, i.e., $I=\sum_{n=0}^{\infty}|n\rangle\langle n|$, where $I$ is the identity operator. This means that the phase-space representation of each number state must have a finite area, which implies that each circle must have a width. The spacing between circles is $\delta r_n=\sqrt{(n+1)+1/2}-\sqrt{n+1/2}\cong 1/2\sqrt{n}$ for $n\gg 1$, and we associate to each circle this width. That is, the number state $|n\rangle$ is represented by the circular band with inner radius $\sqrt{n+1/2}-1/4\sqrt{n}$ and outer radius $\sqrt{n+1/2}+1/4\sqrt{n}$. Each of these bands has an area of $\pi$ and all of the bands taken together fill phase space.

Let us make use of these pictures to show that a squeezed vacuum state is also an amplitude-squared-squeezed state. A squeezed vacuum state is represented by an elliptical region centered on the origin, i.e., the amplitude of this state has a mean value of zero and has a

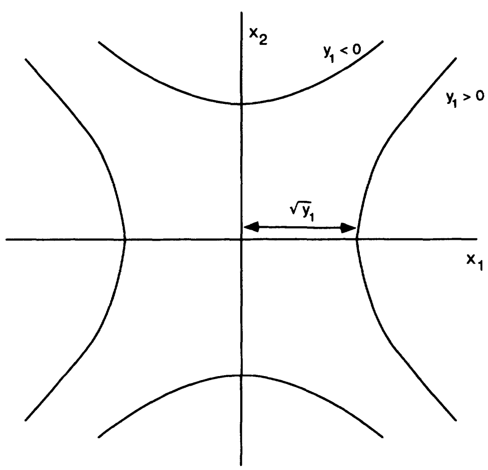


FIG. 2. Representations of eigenstates of $Y_1$. The hyperbola which intersects the $x_1$ axis corresponds to an eigenstate whose eigenvalue is positive, and the hyperbola intersecting the $x_2$ axis corresponds to one whose eigenvalue is negative.

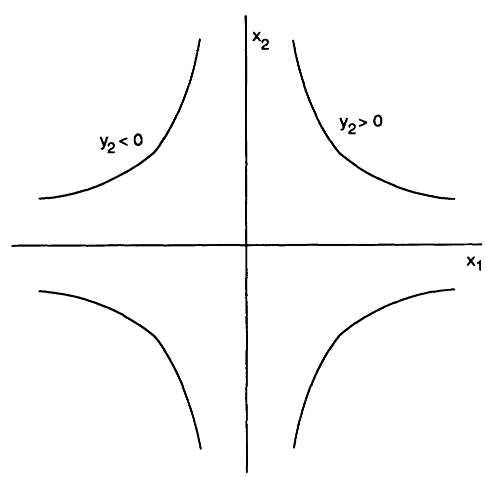


FIG. 3. Representations of eigenstates of $Y_2$. The hyperbola which lies in the first and third quadrants corresponds to an eigenstate whose eigenvalue is positive, and the hyperbola which lies in the second and fourth quadrants corresponds to one whose eigenvalue is negative.

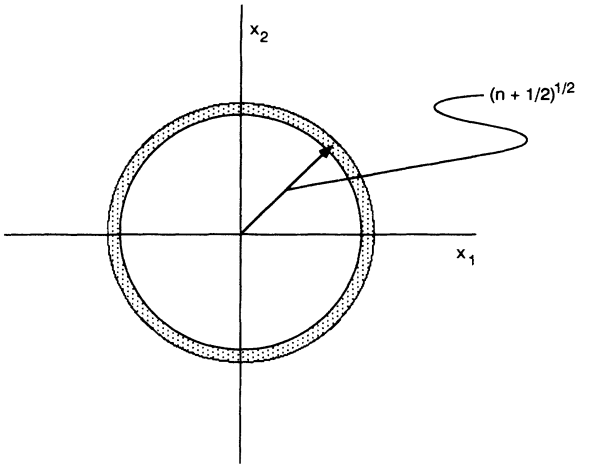


FIG. 4. Phase-space representation of the number state $|n\rangle$. It consists of a circular band with inner radius $\sqrt{n+1/2}-1/4\sqrt{n}$ and outer radius $\sqrt{n+1/2}+1/4\sqrt{n}$.

substantial probability of being found anywhere in the ellipse. Let us assume that the state is squeezed in the $x_2$ direction which implies that $\Delta x_1 > \Delta x_2$. Then the semimajor axis of the ellipse has a length $\Delta x_1$, the semiminor axis has a length $\Delta x_2$, and $\Delta x_1 \Delta x_2 = \frac{1}{4}$.

We now want to find $\Delta y_1$, and $\Delta y_2$. We do this by examining the overlap of the elliptical region with the hyperbolas representing the eigenstates of $Y_1$ and $Y_2$. The hyperbolas which correspond to $Y_1$ eigenstates and which intersect the ellipse range in value from $y_1 = -(\Delta x_2)^2$ to $y_1 = (\Delta x_1)^2$. This is illustrated in Fig. 5. Consequently we set

$$\Delta y_1 = (\Delta x_1)^2 + (\Delta x_2)^2 \ . \tag{2.5}$$

The situation for $\Delta y_2$ is more complicated. We again want to find the hyperbolas which represent $Y_2$ eigenstates and which overlap with the ellipse (see Fig. 6). The

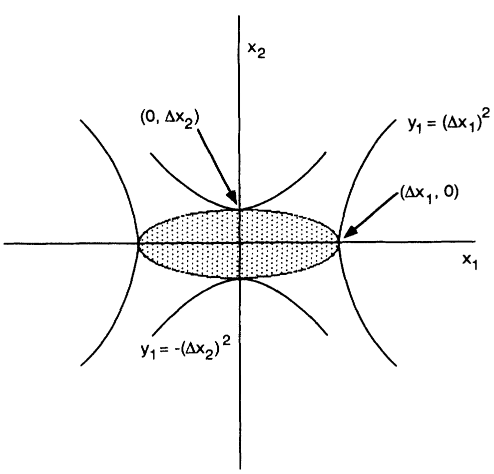


FIG. 5. Overlap of squeezed vacuum state with $Y_1$ eigenstates.

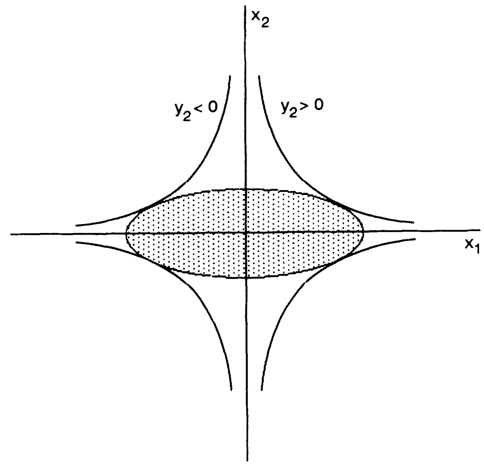


FIG. 6. Overlap of squeezed vacuum state with $Y_2$ eigenstates.

equation for the ellipse (actually the boundary of the elliptical region) is

$$(x_1/\Delta x_1)^2 + (x_2/\Delta x_2)^2 = 1 \ . \tag{2.6}$$

The equation for the $Y_2$ hyperbola corresponding to the eigenvalue $y_2$ is $y_2 = 2x_1x_2$, and solving this equation and Eq. (2.6) simultaneously to find the intersection points gives

$$(x_1/\Delta x_1)^2 + (y_2/2x_1\Delta x_2)^2 = 1 \ . \tag{2.7}$$

If we now note that $\Delta x_1 \Delta x_2 = \frac{1}{4}$ and also set $u = x_1/\Delta x_1$ the above equation becomes

$$u^4 - u^2 + 4y_2^2 = 0 \ . \tag{2.8}$$

Equation (2.8) has real solutions for $u$, corresponding to intersection points between the hyperbola and the ellipse, if

$$1 - 16y_2^2 \geq 0 \ , \tag{2.9}$$

so that the hyperbola intersects the ellipse if $-\frac{1}{4} \leq y_2 \leq \frac{1}{4}$. Therefore we set $\Delta y_2 = \frac{1}{2}$.

Finally, we must determine the mean value of $N$. The "radius" of the ellipse in the $x_1$ direction is $\Delta x_1$, and in the $x_2$ direction it is $\Delta x_2$. For the mean value we can take $(\Delta x_1^2 + \Delta x_2^2)^{1/2}$. As Eq. (2.4) demonstrates, the square of the radial coordinate, $r = (x_1^2 + x_2^2)^{1/2}$, is equal to $n + \frac{1}{2}$. Therefore, for the mean value of $N + \frac{1}{2}$ we take the square of the average radius which is $\Delta x_1^2 + \Delta x_2^2$. With $\Delta x_1 \Delta x_2 = \frac{1}{4}$ this quantity has a minimum value of $\frac{1}{2}$ which occurs when $\Delta x_1 = \frac{1}{2}$. We now note that this implies that $(\Delta y_2)^2 = \frac{1}{4}$ is less than the mean value of $N + \frac{1}{2}$ so that the state is amplitude-squared squeezed.

Let us now compare the values for different quantities which we have found from phase-space methods to the actual values of those quantities calculated by standard operator techniques. For a squeezed vacuum state with an uncertainty in $X_1$ of $\Delta X_1$ and an uncertainty in $X_2$ of $\Delta X_2$ we find

$$(\Delta Y_1)^2=\sqrt{2}[(\Delta X_1)^2+(\Delta X_2)^2], \quad \Delta Y_2=1/\sqrt{2}$$
$$\langle N+\tfrac{1}{2}\rangle=(\Delta X_1)^2+(\Delta X_2)^2 . \tag{2.10}$$

The value of $\langle N+\frac{1}{2}\rangle$ agrees with its phases-space derived value and the values of $\Delta Y_1$ and $\Delta Y_2$ are off by a factor of $\sqrt{2}$. The phase-space pictures, therefore, do not give exact values for the above quantities, but, considering that the methods are somewhat crude, the answers that they do give are not far off. What recommends their use is that by providing an easy way to visualize the fluctuation behavior of quantum states in the variables $Y_1$, $Y_2$, and $N$, the phase-space representations provide a greater understanding of what properties a state must have to have in order to have small fluctuations in these variables. In particular, we see that for large $\Delta x_1$ the elliptical region representing the squeezed vacuum will overlap with a large number of $N$ eigenstates but with only a small subset of the hyperbolas representing $Y_2$ eigenstates. The result is amplitude-squared squeezing.

## III. MIXED VARIABLES

Let us suppose that we are measuring the photon number at the output of a degenerate parametric amplifier. If the input mode is described by the annihilation operator $a_{\rm in}$ and its adjoint $a_{\rm in}^\dagger$, then the annihilation operator for the output mode, $a_{\rm out}$, is given by

$$a_{\rm out}=\cosh(r)a_{\rm in}+e^{i\theta}\sinh(r)a_{\rm in}^\dagger , \tag{3.1}$$

where the parameters $r$ and $\theta$ depend on the pump strength and phase and also on the interaction time. For simplicity let us consider the case $\theta=0$. The output number operator is then

$$N_{\rm out}=a_{\rm out}^\dagger a_{\rm out}$$
$$=\cosh(2r)N_{\rm in}+\sinh(2r)Y_{1,\rm in}+\sinh^2(r) , \tag{3.2}$$

where $N_{\rm in}$ and $Y_{1,\rm in}$ are the number and $Y_1$ operators for the input mode. Therefore, by measuring the photon number at the output, one is measuring the variable [the $c$-number contribution in Eq. (3.2) has been dropped]

$$Z(r)=\cosh(2r)N+\sinh(2r)Y_1 , \tag{3.3}$$

for the input field. The eigenstates of $Z(r)$ are squeezed number states and are given explicitly by

$$|\psi_n\rangle=e^{-r(a^{\dagger 2}-a^2)/2}|n\rangle . \tag{3.4}$$

The eigenvalue of $Z(r)$ corresponding to this state is $n-\sinh^2(r)$. The eigenstates $|\psi_n\rangle$ form a complete set.

What does such a variable correspond to in phase space? In order to find out we begin by expressing $Z(r)$ in terms of $X_1$ and $X_2$

$$Z(r)=e^{2r}X_1^2+e^{-2r}X_2^2-\tfrac{1}{2}\cosh(2r) . \tag{3.5}$$

Rephrasing this equation in terms of the phase-space quantities $x_1$ and $x_2$ and setting $Z$ equal to its eigenvalue $n-[\cosh(2r)-1]/2$ gives

$$1=(x_1/L_1)^2+(x_2/L_2)^2 , \tag{3.6}$$

where

$$L_1(r,n)=e^{-r}(n+\tfrac{1}{2})^{1/2}, \quad L_2(r,n)=e^{r}(n+\tfrac{1}{2})^{1/2} . \tag{3.7}$$

This is the equation for an ellipse which has its semimajor axis of length $L_2$ in the $x_2$ direction and its semiminor axis of length $L_1$ in the $x_1$ direction. As in the case of number states, the union of all the areas in phase space which represent the eigenstates of $Z(r)$ must fill phase space. This follows from the completeness relation for these states. Therefore the eigenstates of $Z(r)$ correspond to elliptical bands which are centered on the ellipses given in Eq. (3.6). This is pictured in Fig. 7. In particular, the elliptical band in phase space corresponding to the eigenstate of $Z(r)$ with eigenvalue $n-[\cosh(2r)-1]/2$ lies between the ellipses whose equations are

$$1=x_1^2/[e^{-2r}(n+\tfrac{1}{2}\pm\tfrac{1}{2})]+x_2^2/[e^{2r}/(n+\tfrac{1}{2}\pm\tfrac{1}{2})] , \tag{3.8}$$

where the upper signs indicate the outer boundary and the lower signs the inner boundary of the band. The band has an area equal to $\pi$.

In order to determine the condition for nonclassical behavior in $Z(r)$ we first express $[\Delta Z(r)]^2$ in normally ordered form

$$[\Delta Z(r)]^2=\langle :[Z(r)-\langle Z(r)\rangle]^2:\rangle+\langle Z(2r)\rangle$$
$$+[\cosh(4r)-1]/4 . \tag{3.9}$$

For a classical state the normally ordered expectation value in the above equation is always greater than or equal to zero. Therefore, a state is nonclassical if

$$[\Delta Z(r)]^2<\langle Z(2r)\rangle+[\cosh(4r)-1]/4 . \tag{3.10}$$

Let us use phase-space methods in order to see what properties a state must have to satisfy this condition. Consider first the eigenstate of $Z(r)$ with eigenvalue $n-[\cosh(2r)-1]/2, |\psi_n\rangle$. As has been pointed out, this state is represented by an elliptical band centered on the ellipse with semiminor axis in the $x_1$ direction of length $L_1(r,n)$ and semimajor axis in the $x_2$ direction of length

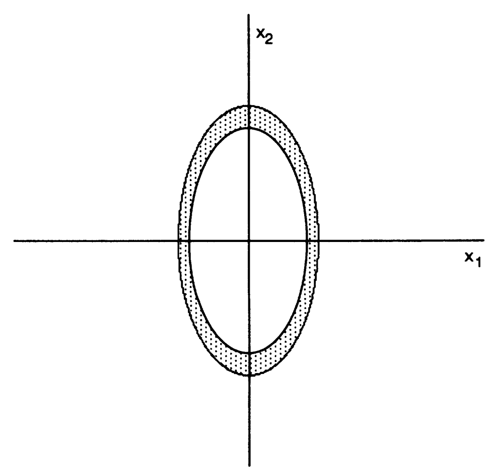


FIG. 7. Representation of an eigenstate of $Z(r)$.

$L_2(r,n)$. We need to calculate the expectation value of $Z(2r)$, and in doing so it is useful to have a representation of the eigenstates of this variable. In phase space the eigenstate of $Z(2r)$ with eigenvalue $m+[\cosh(4r)-1]/2$ is an elliptical band centered on the ellipse with semiminor axis in the $x_1$ direction of length $L_1(2r,m)$ and semimajor axis in the $x_2$ direction of length $L_2(2r,m)$. The ellipses will overlap if $L_2(2r,m)\geq L_2(r,n)$ and $L_1(2r,m)\leq L_1(r,n)$ (see Fig. 8), i.e., if

$$e^{-2r}(n+\tfrac{1}{2})<m+\tfrac{1}{2}<e^{2r}(n+\tfrac{1}{2}) . \quad (3.11)$$

In terms of values of $z(2r)$, the phase-space $c$ number corresponding to the operator $Z(2r)$, which is given by

$$z(2r)=m-[(\cosh(4r)-1]/2 , \quad (3.12)$$

this condition becomes

$$e^{-2r}(n+\tfrac{1}{2})-\tfrac{1}{2}\cosh(4r) < z(2r) < e^{2r}(n+\tfrac{1}{2})-\tfrac{1}{2}\cosh(4r) . \quad (3.13)$$

If we use the value of $z(2r)$ in the middle of this range for the expectation value of $Z(2r)$ we have

$$\langle Z(2r)\rangle=(n+\tfrac{1}{2})\cosh(2r)-\tfrac{1}{2}\cosh(4r) . \quad (3.14)$$

This is also the value of $\langle Z(r)\rangle$ which one obtains by operator methods.

For the state $|\psi_n\rangle$ we have $\Delta Z(r)=0$. Substituting this result and Eq. (3.14) into Eq. (3.10) tells us that the state $|\psi_n\rangle$ is nonclassical if

$$[\cosh(4r)+1]/4<(n+\tfrac{1}{2})\cosh(2r) . \quad (3.15)$$

Therefore not all eigenstates of $Z(r)$ satisfy Eq. (3.10); $n$ must be sufficiently large. Let us note that for $r>0$ all of the states $|\psi_n\rangle$ are nonclassical states. What Eq. (3.15) states is that for the nonclassical behavior to manifest itself in the variable $Z(r)$, $n$ must be sufficiently large.

This tells us what kind of state will satisfy the condition in Eq. (3.10). The region which corresponds to it in phase space must first overlap with a small number of the ellipses given by Eq. (3.6). This guarantees that $\Delta Z(r)$ will be small. It is also necessary that the ellipses with which it does overlap have values of $n$ which satisfy Eq. (3.15), i.e., the ellipses must be sufficiently large.

## IV. NONLINEAR INTERFEROMETER

A Kerr medium changes the phase-fluctuation properties of the light which passes through it. A coherent state at the input will be changed into what is called a generalized coherent state at the output. In phase space the circular region which describes the coherent state is elongated into a crescent. If the crescent is displaced so that it overlaps a relatively small number of the circles representing number states, then the resulting state has sub-Poissonian photon statistics. Kitigawa and Yamamoto showed that this elongation followed by a displacement can be accomplished by a Mach-Zehnder interferometer with a Kerr medium in one leg (see Fig. 9) [11].

Here we want to use the same device to produce an amplitude-squared squeezed state. In doing so the phase-space representation will serve as a guide. The basic idea is to again create the crescent by means of the Kerr medium, but this time to displace and rotate it so that it overlaps a small subset of the hyperbolas representing the eigenstates of $Y_1$. This will produce a state which is amplitude-squared squeezed in $Y_1$. It should be noted that it is possible to create a state with a small amount of amplitude-squared squeezing using a Kerr medium alone [12].

A single mode in a Kerr medium is described by the Hamiltonian

$$H=\omega a^\dagger a+\lambda(a^\dagger a)^2 , \quad (4.1)$$

and the resulting time evolution operator is $U(t)=e^{-iHt}$. The coupling constant $\lambda$ is proportional to the third-order nonlinear susceptibility of the material. The time evolution described by $U(t)$ is relatively simple because

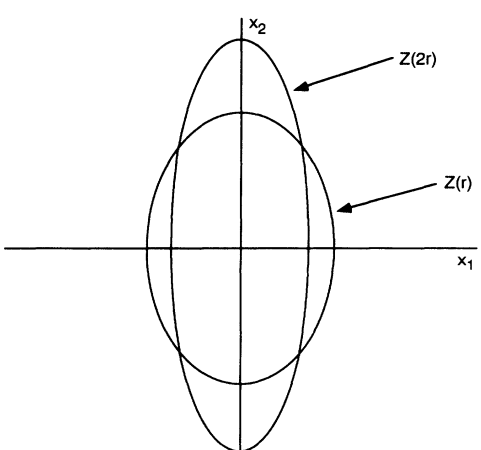


FIG. 8. Overlap of an eigenstate of $Z(r)$ and an eigenstate of $Z(2r)$. The eigenstates are elliptical bands but have been drawn as ellipses for simplicity.

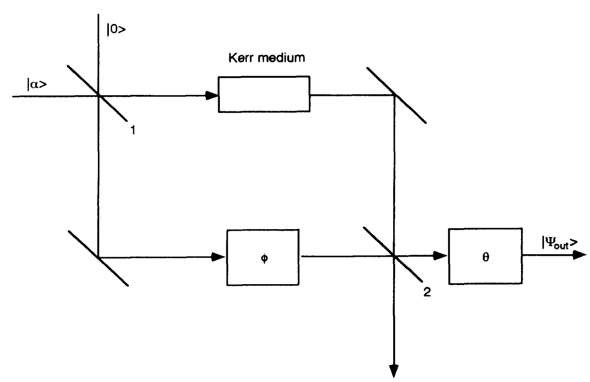


FIG. 9. Mach-Zehnder interferometer with a Kerr medium in one of the two legs. The boxes labeled by angles represent linear media which shift the phase of the light by the designated angle.

$H$ commutes with the photon-number operator. As a result, the action of $U(t)$ does not charge the photon statistics of a state but does modify its phase properties.

The interferometer which we wish to consider is shown in Fig. 9. A coherent state $|\alpha\rangle$ is introduced into one of the input ports. Part of it is transmitted and goes directly to the second beam splitter after a phase shift of $\varphi$. The rest passes through the Kerr medium before arriving at beam splitter no. 2. The light from one of the output ports is passed through a linear medium which shifts the phase of the light by $\theta$. After emerging from the second phase shifter the state of the light is [11]

$$|\Psi_{\rm out}\rangle = R(\theta)D(\xi)U(t)|\alpha_1\rangle \ . \tag{4.2}$$

Here $R(\theta)$ is the rotation operator $R(\theta)=\exp(i\theta a^\dagger a)$, $D(\xi)$ is the displacement operator $D(\xi)=\exp(\xi a^\dagger - \xi^* a)$, and $U(t)$ describes the evolution of the state in the Kerr medium ($t$ is the time it takes the light to pass through the medium). The state $|\alpha_1\rangle$ is a coherent state whose amplitude is $\alpha_1=(1-R_1)^{1/2}\alpha$ where $R_1$ is the reflectivity of the first beam splitter. The displacement parameter $\xi$ is given by $\xi=(1-R_2)^{1/2}e^{i\varphi}\sqrt{R_1}\alpha$ where $R_2$ is the reflectivity of the second beam splitter. In deriving Eq. (4.2) it has been assumed that $1-R_2 \ll 1$ and $|\alpha\sqrt{R_1}| \gg 1$.

Let us now consider the values we want to choose for $\theta$ and $\xi$. After emerging from the Kerr medium the light is in the state $U(t)|\alpha_1\rangle$. In phase space this state corresponds to a crescent whose center is located at

$$\begin{aligned}\langle \alpha_1|[U(t)]^{-1}aU(t)|\alpha_1\rangle \\ &= \alpha_1 e^{-i(\omega+\lambda)t} \\ &\quad \times \exp(-|\alpha_1|^2\{[1-\cos(4\lambda t)] \\ &\qquad + i\sin(2\lambda t)\}) \ .\end{aligned} \tag{4.3}$$

If we assume that we are in the regime where $\lambda t|\alpha_1| \ll 1$, which we shall do for the rest of this section, then $|\alpha_1|^2[1-\cos(4\lambda t)] \cong 1$. Therefore the center point of the crescent is a distance of $|\alpha_1|$ from the origin and at an angle of $\theta_\alpha - (\omega+\lambda)t - |\alpha_1|^2\sin(2\lambda t)$ from the $x_1$ axis, where $\alpha_1 = |\alpha_1|\exp(i\theta_\alpha)$. The crescent has a radius of curvature approximately equal to $|\alpha_1|$. The hyperbola $y_1 = x_{1^2} - x_2^2$ has a radius of curvature at the point $(-\sqrt{y_1},0)$ of $\sqrt{y_1}$. This suggests that we should displace and rotate the crescent so that its center is at the point $(-|\alpha_1|,0)$, and so that it lines up with the hyperbola $|\alpha_1|^2 = x_1^2 - x_2^2$ which passes through this point. In this way the crescent should overlap with a small subset of the hyperbolas $y_1 = x_1^2 - x_2^2$, and, thereby, the quantum state represented by the displace and rotated crescent should have a $(\Delta Y_1)^2$ small enough for it to be amplitude-squared squeezed. This is illustrated in Fig. 10.

These considerations suggest that we take for the displacement

$$\xi = -2\alpha_1 e^{-i(\omega+\lambda)t}\exp[-i|\alpha_1|^2\sin(2\lambda t)] \ , \tag{4.4}$$

and for the angle of rotation

$$\theta = (\omega+\lambda)t + |\alpha_1|^2\sin(2\lambda t) - \theta_\alpha \ . \tag{4.5}$$

This sequence of a displacement followed by a rotation will put the center of the crescent approximately where we want it. In order to improve the result we choose instead for the displacement

$$\begin{aligned}\xi = e^{-i(\omega+\lambda)t}&\exp[-i|\alpha_1|^2\sin(2\lambda t)] \\ &\times(-2\alpha_1 + \zeta e^{i\theta_\alpha}) \ ,\end{aligned} \tag{4.6}$$

where $\zeta$ represents a small correction, and leave our choice of $\theta$ unchanged. We shall now calculate $(\Delta Y_1)^2 - \langle N + \frac{1}{2}\rangle$ for $|\Psi_{\rm out}\rangle$ and minimize the result with respect to $\zeta$.

The calculation is straightforward and so only the

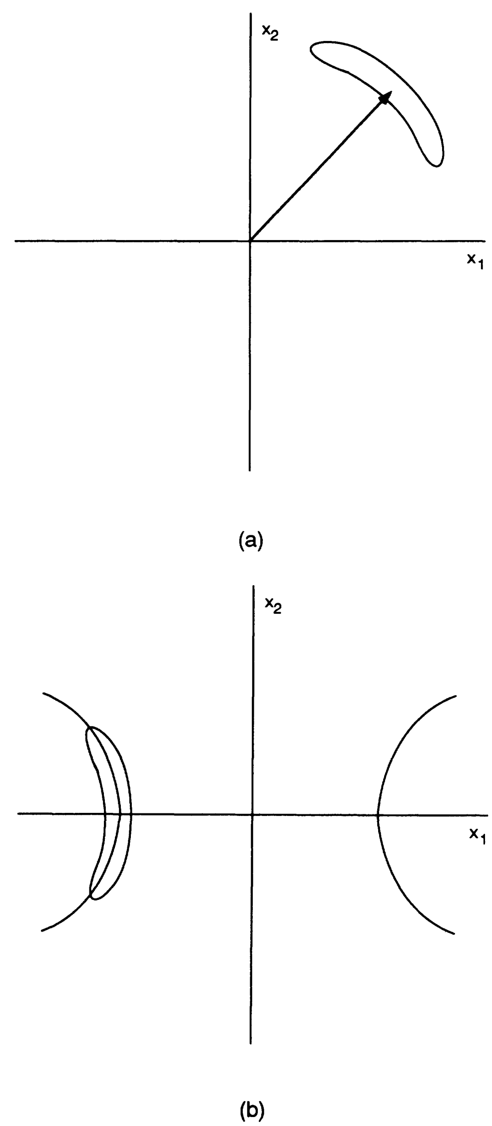


FIG. 10. (a) Phase-space representation of the state of the light after it has emerged from the Kerr medium. (b) After passing through the second beam splitter and the phase shifter the crescent has been displaced and rotated so that it lines up with a $Y_1$ eigenstate.

essential parts will be summarized here. In order to make the final result simpler we can make use of the results of Kitagawa and Yamamoto [11]. They found that they could obtain the highest level of sub-Poissonian photon statistics using a nonlinear interferometer if $\lambda t = O(1/|\alpha_1|^{4/3})$. We shall assume that this condition is satisfied here and only the highest-order terms will be kept. Setting $\zeta = x + iy$ we find that

$$(\Delta Y_1)^2 - \langle N + \tfrac{1}{2} \rangle = |\alpha_1|^2[-|\alpha_1|(8\lambda t y + 96\gamma^3 x) + 8\gamma(y^2 + \gamma x^2)] , \quad (4.7)$$

where $\gamma = 2(|\alpha_1|\lambda t)^2$. Minimizing this with respect to $x$ and $y$ we find that

$$(\Delta Y_1)^2 - \langle N + \tfrac{1}{2} \rangle = -|\alpha_1|^2 + O(|\alpha_1|^{4/3}) , \quad (4.8)$$

so that the state is amplitude-squared squeezed. Therefore phase-space pictures have led to an alternative method of producing amplitude-squared squeezed states.

## V. CONCLUSION

The phase-space representations of "eigenstates" of variables which are Hermitian linear combinations of single-mode creation and annihilation operators are straight lines. Here Hermitian operators quadratic in the creation and annihilation operators have been considered, and it was found that phase-space representations of their "eigenstates" are curves. In particular, we examined the amplitude-squared squeezing variables $Y_1$ and $Y_2$, whose eigenstates correspond to hyperbolas, and a variable $Z(r)$, which is a linear combination of $Y_1$ and the number operator, whose eigenstates are elliptical bands.

Once one has a representation of the eigenstates of a variable one can then try to construct states whose fluctuations in that variable are below the classically allowed level. Such a state should correspond to a region in phase space which overlaps with only a small set of the curves representing eigenstates. This procedure led us to a method for producing amplitude-squared squeezing by using a Mach-Zehnder interferometer with a nonlinear element.

With standard squeezing, phase-space representations have served as both a tool for understanding and a guide to thinking. What has been shown here is that they can play a similar role for higher-order squeezing.

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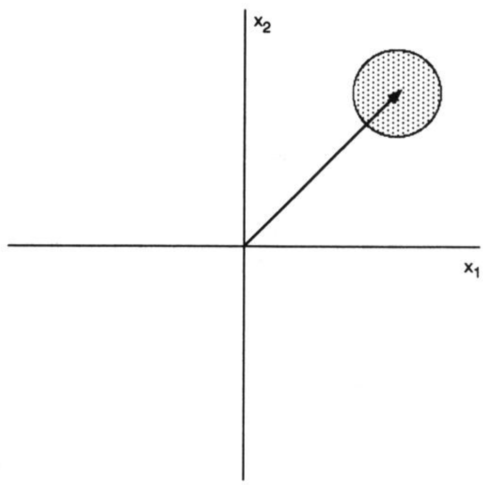


FIG. 1. Phase-space representation of a coherent state. The circular "error box" has a radius of $\frac{1}{2}$ and represents the uncertainty in the complex field amplitude of the state.

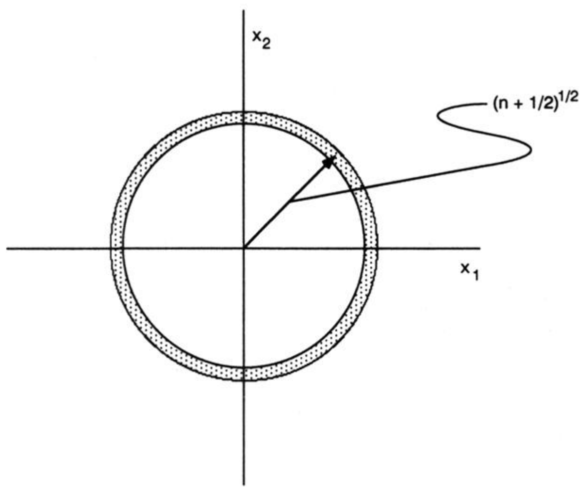


FIG. 4. Phase-space representation of the number state $|n\rangle$. It consists of a circular band with inner radius $\sqrt{n+1/2}-1/4\sqrt{n}$ and outer radius $\sqrt{n+1/2}+1/4\sqrt{n}$.

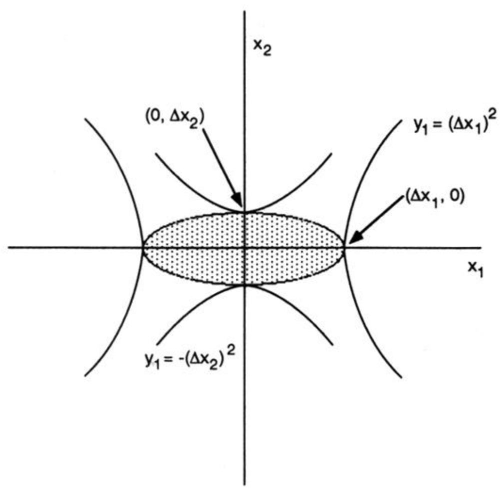


FIG. 5. Overlap of squeezed vacuum state with $Y_1$ eigenstates.

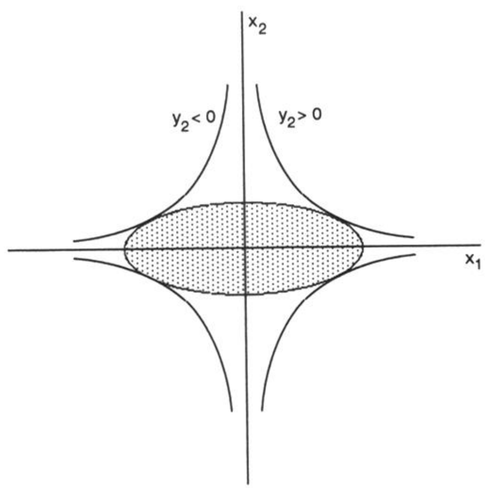


FIG. 6. Overlap of squeezed vacuum state with $Y_2$ eigenstates.

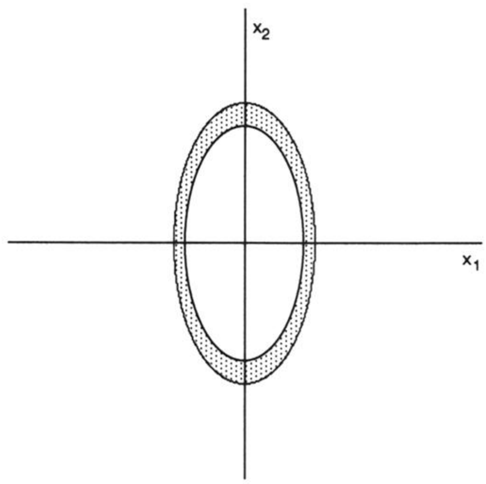


FIG. 7. Representation of an eigenstate of $Z(r)$.