Alternative derivation of relativistic two-body equations

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The derivation from a relativistic field theory of equations describing two interacting particles has, in the past, been based on the study of the matrix element of a product of field operators taken between a two-particle state and the vacuum. Instead, we propose here to study the matrix element of a single field operator connecting a two-particle state to a one-particle state; this introduces an asymmetry between the particle that is on and the one that is off the mass shell. We derive an equation for fermions that in the infinite-mass limit for the particle on the mass shell goes over into the correct (Dirac) equation for the light particle moving in the external field generated by the heavy one. For the equal-mass case, we show how symmetry between the two particles can be restored and a useful equation obtained at the level of approximation involving one-boson exchange and suppression of negative-energy intermediate states. The calculations are carried through for QED in the radiation gauge.

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I. INTRODUCTION

The purpose of the present paper is twofold. It is, first of all, to derive a two-body relativistic equation that "looks like a one-body equation," at least at one stage of the development. The impetus for such an effort came originally from the work of Wong and Becker [1], who were among the first to attempt to implement the suggestion that the positron and electron peaks observed in heavy-ion reactions [2,3] are evidence for a resonance of magnetic origin in the (e^-,e^+) system. The fact that the resonant system weighs 3–4 electron masses indicates that we are dealing with interactions of range within and perhaps well within the electron Compton wavelength. This circumstance places a cloud over these early calculations since they are based on equations of heuristic origin which are, at best, of semirelativistic validity. Nevertheless, since a resonance was reported in the calculations of Wong and Becker, it seemed desirable, at the time, to seek a description of a two-particle system, fully based on QED, that would resemble as closely as possible the formalism of these authors.

It turns out that the natural way to do this is to develop an equation in which one particle is on the mass shell and the other is not. Such an asymmetrical treatment has long been advocated by Gross [4], as a way of connecting the Bethe-Salpeter equation with single-time formalisms. Our approach is different in that the asymmetry is introduced from the very beginning by studying the matrix element of a single fermion field operator between a two-particle state and a one-particle state. Nevertheless, there is clearly an intimate connection between our approach and that of Gross, which we plan to explore in the future.

One of the purposes of the present paper is to show that in its asymmetrical form our equation solves in an irreducibly simply manner the problem of having a two-body equation that reduces to the exact one-body limit as the mass of one of the particles becomes infinite. As an example, for the electron-proton system, putting the *proton* on the mass shell, we derive an approximate equation describing one-photon exchange (in the Coulomb or radiation gauge) that, in the limit of large proton mass becomes the Dirac equation for the electron moving in both the Coulomb field and the magnetic dipole field of the proton. We invite the reader to compare this treatment with a recent study [5] of the same problem for a variety of relativistic two-body equations found in the literature. In the paper cited, a general method is developed for isolating factors that behave improperly in the one-body limit and for providing *ad hoc* replacements that will reduce properly and not do violence to the two-particle physics.

However, for the main application of interest to us, continuum (possibly resonant) states of the e^-e^+ system, a symmetrical treatment of the two particles is called for and the second purpose of this paper is to show how this can be done. Several steps are required. The first is to agree to study two *different* equations obtained by putting one or the other particle on the mass shell. The second point is recognize that if one drops the negative-energy components describing the particle off the mass shell, then the initially different amplitudes ("wave functions") occurring in the two equations can be identified, up to a phase, and consequently the two equations can be combined in a symmetrical fashion.

The resulting equation is of particular relevance in the light of a recent publication by Spence and Vary [6]. These authors report resonant solutions of an equation derived on the basis of the Tamm-Dancoff (TD) approach. Since the dependence of the retarded interaction kernel on the momentum transfer (and energy) is different for the TD equation than it is for our equation, it is a matter of some interest to try to understand to what degree the results reported depend on the particular two-body equation studied.

The partial-wave decomposition of our equation has been carried out, involving basically the same techniques as have been employed in the corresponding studies of one-boson-exchange potentials [7]; it is hoped to report in due course the results of solving the resulting integral equations. This preliminary report is offered independently because of the possible interest of the technique by which our two-body equation was obtained and because of the transparent connection with the one-particle limit. (In the meantime [8], we have carried out numerical calculations for a scalar version of QED, for which resonant scattering of particles and antiparticles has been reported [9], and have failed to confirm such behavior.)

In Sec. II, the derivation of the fully covariant but asymmetrical two-particle equation is described and its proper reduction to the one-body limit is pointed out. The contents of this section have a large overlap with a preliminary unpublished report [10]. In Sec. III we describe the steps necessary to obtain a symmetrized equation. Various equivalent forms of this equation are given, including a four-component version involving only Pauli matrices that provides the natural starting point for applications. The connection to standard semirelativistic approximations is also pointed out. Section IV is devoted to a summary and conclusions. Before turning to the details of the task that we have set, it is appropriate to add a brief discussion of and comparison with other recent work on relativistic two-body formalisms.

The long-term program of Crater and Van Alstine [11] is based on a quantization of a constrained relativistic two-particle classical mechanics that provides a possible *framework* for studying the two-body problem. This framework can be filled in by the choice of otherwise arbitrary potentials. In the recent work quoted, this formalism has been applied with some success to hadron spectroscopy. In addition, a partial connection has been established with single-time equations related to a field-theoretical starting point. Agreement with the spectrum of parapositronium to order α^4 has also been shown [12]. It is clear, however, that establishment of full equivalence with QED is beyond the capabilities of this formalism.

A second development to which we draw attention is the application of a variational approach to the two-fermion bound-state problem [13–15]. In effect, this approach is based on the standard Fock-space (Tamm-Dancoff) formalism, with the omission of self-interactions. The resulting equations are given serious numerical treatment that is carried into the strong-coupling domain. It is to be noted that the standard Tamm-Dancoff equation is the basis for the work of Ref. [6] and will be reexamined by us numerically using the method described in Ref. [8].

In a third independent development [16,17], the aim is to construct an approximate two-body equation that achieves the following goals: (i) to reduce to the correct one-body limit as either mass tends to infinity, (ii) to reduce to the eikonal approximation in the high-energy limit. The resulting formalism, which retains the full 16 components associated with two Dirac fields, has both positive- and negative-energy solutions. However, the full retardation effects associated with one-photon exchange are not included in lowest order, and therefore this formalism, ingenious as it is, is not suitable for our purposes.

II. ASYMMETRICAL COVARIANT TWO-BODY EQUATION IN THE ONE-PHOTON EXCHANGE APPROXIMATION

In the following, we derive an equation which, at the level of approximation considered, can apply to the system e-X, where e refers to the electron and X is any other charged, spin-$\frac{1}{2}$ fermion. This means that the annihilation interaction for charge-conjugate pairs is omitted in the present treatment. We start from the quantum field equation for the electron field, $\psi(x)$, interacting with the electromagnetic potential, $A_\mu(x)$ of a Maxwell field. In the notation of Bjorken and Drell [18] we have ($\hbar=c=1$)

$$(\mathbf{p}-e\,\mathbf{A}-m)\psi(x)=0\ , \tag{2.1}$$

where $\mathbf{p}=\gamma^0 p_0-\vec{\gamma}\cdot\vec{\mathrm{p}}$, and p_0 is the energy.

We study a selected matrix element of (2.1), and shall evaluate the matrix element of the product of operators, $\mathbf{A}\psi$, by summing over a small subset of intermediate states. Only the following states will be included in our treatment.

(i) $|\,\vec{\mathrm{P}},\Lambda\rangle$ is a two-particle state of the system e-X with total momentum $\vec{\mathrm{P}}$ (ultimately set to zero) and Λ is a complete set of internal quantum numbers.

(ii) $|\vec{\mathrm{k}},\sigma\rangle$ is a one-particle state for the partner of the electron, the X particle, with k the momentum and σ the spin variable. We now define a "wave function" for the electron

$$\Psi_{\vec{\mathrm{p}},\Lambda}(\vec{\mathrm{k}},\sigma|\vec{\mathrm{r}},t)=\langle -\vec{\mathrm{k}}\sigma|\psi(x)|\vec{\mathrm{P}},\Lambda\rangle=\exp\{-i[P_0-E(k)]t+i(\vec{\mathrm{P}}+\vec{\mathrm{k}})\cdot\vec{\mathrm{r}}\}\langle -\vec{\mathrm{k}},\sigma|\psi(0)|\vec{\mathrm{P}},\Lambda\rangle\ , \tag{2.2}$$

where $E(k)=(k^2+M^2)^{1/2}$ is the energy of the particle X, and, of course, we have used translation invariance. We also define

$$\Psi_{\vec{\mathrm{p}},\Lambda}(\vec{\mathrm{k}},\sigma)\equiv\langle -\vec{\mathrm{k}},\sigma|\psi(0)|\vec{\mathrm{P}},\Lambda\rangle\ . \tag{2.3}$$

By forming the matrix element (2.2), we derive from (2.1) the equation

$$\{\gamma^0[P_0-E(k)]-\vec{\gamma}\cdot(\vec{\mathrm{P}}+\vec{\mathrm{k}})-m\}\Psi_{\vec{\mathrm{p}},\Lambda}(\vec{\mathrm{k}},\sigma)$$
$$=\exp\{i[P_0-E(k)]-i(\vec{\mathrm{P}}+\vec{\mathrm{k}})\cdot\vec{\mathrm{r}}\}e\langle -\vec{\mathrm{k}},\sigma|[-\gamma^0 A_0(\vec{\mathrm{r}},t)-\vec{\gamma}\cdot\vec{\mathrm{A}}(\vec{\mathrm{r}},t)]\psi(\vec{\mathrm{r}},t)|\vec{\mathrm{P}},\Lambda\rangle\ . \tag{2.4}$$

The problem is to evaluate the operator products on the right-hand side of (2.4). To do so, we apply spectral decomposition to the product of the vector potentials with the field operator. Our major assumption is that the dominant contribution (even for the short distances of interest to us) comes from one-particle states of the particle X defined in (ii) above. We assume that contributions from intermediate states which contain additional photons or pairs ultimately contribute in an appropriately higher order in the fine-structure constant. Within the framework that we are erecting, this is, for the moment, only a plausibility argument, which may have as its most serious defect that in the case that X is the antiparticle of e, that the annihilation interaction is not included.

In consequence of the above assumption, we must calculate the electromagnetic potentials $[x=(\vec{r},t)]$

$$\langle -\vec{k},\sigma | A_0(x) | -k',\sigma' \rangle \ , \quad \langle -\vec{k},\sigma | A_i(x) | -\vec{k}',\sigma' \rangle \ , \tag{2.5}$$

which we shall evaluate in the radiation (Coulomb) gauge,

$$\vec{\nabla}\cdot\langle -\vec{k},\sigma | \vec{A}(x) | -\vec{k}'\sigma' \rangle = 0 \ . \tag{2.6}$$

In this gauge, Maxwell's equations take the form

$$\nabla^2 A_0 = -j_0 \ , \tag{2.7}$$

$$\partial^\mu \partial_\mu A_i = (\partial_0^2 - \nabla^2) A_i = (\delta_{ij} - \partial_i \partial_j / \nabla^2) j_j \ . \tag{2.8}$$

The form (2.8), which involves only the transverse part of the three-vector current, is reached by utilizing (2.7) and current conservation,

$$\partial_0 j_0 + \vec{\nabla}\cdot\vec{j} = 0 \ , \tag{2.9}$$

in Maxwell's equations.

In general, j_μ refers to the total electromagnetic current produced by the charged fields. For present purposes, it suffices to consider only the current generated by the field X, whose particles have charge q. Thus X serves here as the source of the field that acts on the electron. We write

$$j_\mu = \tfrac{1}{2} q [\bar{\psi}^X, \gamma_\mu \psi^X] \ , \tag{2.10}$$

where ψ^X refers to the field operator for the particle X. (Thus for the positron $q=-e$ and ψ^X is field charge conjugate to ψ.) The matrix element between one-particle states of X yields, in lowest order,

$$\langle \vec{k},\sigma | j_\mu(0) | \vec{k}',\sigma' \rangle = \frac{q}{(2\pi)^3} \frac{M}{\sqrt{EE'}} \bar{u}(\vec{k},\sigma)\gamma^\mu u(\vec{k}',\sigma') \ , \tag{2.11}$$

where it is understood that the spinors refer to particle X. This expression provides the source terms for the solution of Eqs. (2.7) and (2.8), which yield

$$\langle -\vec{k},\sigma | A_0(x) | -\vec{k}',\sigma' \rangle = q \exp\{ i[E(k)-E(k')]t + i(\vec{k}-\vec{k}')\cdot\vec{r} \} (\vec{k}-\vec{k}')^{-2} \bar{u}(-\vec{k},\sigma)\gamma^0 u(-\vec{k}',\sigma') \frac{1}{(2\pi)^3} \frac{M}{\sqrt{EE'}} \ , \tag{2.12}$$

$$\begin{aligned} \langle -\vec{k},\sigma | A_i(x) | -\vec{k}',-\sigma' \rangle &= q \exp\{ i[E(k)-E(k')]t + i(\vec{k}-\vec{k}')\cdot\vec{r} \} \{ (\vec{k}-\vec{k}')^2 - [E(k)-E(k')]^2 \}^{-1} \\ &\quad \times \left[\delta_{ij} - \frac{(\vec{k}-\vec{k}')_i (\vec{k}-\vec{k}')_j}{(\vec{k}-\vec{k}')^2} \right] \bar{u}(-\vec{k},\sigma)\gamma^j u(-\vec{k}',\sigma') \frac{1}{(2\pi)^3} \frac{M}{\sqrt{EE'}} \ . \end{aligned} \tag{2.13}$$

If we insert these results in Eq. (2.4), set $\vec{P}=\vec{0}$, dropping it from the notation, and also set $P_0=W$, $\vec{q}=\vec{k}-\vec{k}'$, and

$$Q^2 = q^2 - [E(k)-E(k')]^2 \ , \tag{2.14}$$

we obtain the main result of this section, the equation

$$\begin{aligned} &\{ \gamma^0 [W - E(k)] - \vec{\gamma}\cdot\vec{k} - m \} \Psi_\Lambda(\vec{k},\sigma) \\ &= \frac{eq}{(2\pi)^3} \int d\vec{k}' \frac{M}{\sqrt{EE'}} \sum_{\sigma'} \{ q^{-2} [\bar{u}(-\vec{k},\sigma)\gamma^0 u(-\vec{k}',\sigma')]_X \gamma^0 - Q^{-2} (\delta_{ij} - q_i q_j q^{-2}) [\bar{u}(-\vec{k},\sigma)\gamma^i u(-\vec{k}',\sigma)]_X \gamma^j \} \\ &\qquad \times \Psi_\Lambda(\vec{k}'\sigma') \ . \end{aligned} \tag{2.15}$$

This is our two-body equation that "looks like a one-body equation." It is the most convenient form that we know for producing the one-body limit $M\to\infty$. In this limit, the quantity in curly braces on the right-hand side of (2.15) becomes

$$(1/q^2)\delta_{\sigma,\sigma'}\gamma^0 + (i/2Mq^2)\vec{\sigma}\times\vec{q}\cdot\vec{\gamma} \ , \tag{2.16}$$

where spin matrices refer to the particle X and Dirac matrices to the electron. Here the second term represents the magnetic dipole field produced at the position of the electron by the Dirac magnetic moment of X. Naturally convection currents associated with the heavy particle have been set to zero. The contribution of the anomalous magnetic moment has to be added by hand in this approach.

On the other hand, if the field-producing particle is a positron, we have an equation that somewhat resembles that used in Ref. [1] except that our equation has actually been derived from QED and contains a correctly retarded magnetic interaction. On the other hand, it treats the electron and positron in asymmetrical fashion, and in this regard must be accounted unsatisfactory. In the next section, we show how symmetry in the treatment of the two particles may be restored for the e^-e^+ system.

III. TRANSFORMATION TO A SYMMETRICAL FORM

Further considerations are restricted to the e^-e^+ system; we shall henceforth use superscripts e and p on spinors and Dirac matrices to distinguish these two particles. An expression of the asymmetry in the treatment of the two particles in Eq. (2.15) is the fact that it is an eight-component equation, four (Dirac indices) for the electron and two (spin components) for the positron. A *first* step toward restoring parity is taken by introducing a 16-component amplitude,

$$\Phi^{(e)}_{\alpha\beta}(\vec{k}) \equiv \sum_\sigma u^{(p)}_\beta(-\vec{k},\sigma)\langle -\vec{k},\sigma|\psi^{(e)}_\alpha(0)|0,\Lambda\rangle \ . \tag{3.1}$$

With the standard definition (and with $\gamma^0 \to \gamma_0$)

$$\Lambda_+(\vec{k}) = \sum_\sigma u(\vec{k},\sigma)\bar{u}(\vec{k},\sigma) = \frac{\gamma_0 E(k) - \vec{\gamma}\cdot\vec{k} + m}{2m} \ , \tag{3.2}$$

we find that Eq. (2.15) may be rewritten in the form

$$\{\gamma^{(e)}_0[W-E(k)] - \vec{\gamma}^{(e)}\cdot\vec{k} - m\}\Phi^{(e)}(\vec{k})$$
$$= -\frac{e^2}{(2\pi)^3}\int d\vec{k}'\frac{m}{\sqrt{EE'}}\Lambda^{(p)}_+(-\vec{k})\{q^{-2}\gamma^{(e)}_0\gamma^{(p)}_0 - Q^{-2}[\vec{\gamma}^{(e)}\cdot\vec{\gamma}^{(p)} - q^{-2}(\vec{\gamma}^{(e)}\cdot\vec{q})(\vec{\gamma}^{(p)}\cdot\vec{q})]\}\Phi^{(e)}(\vec{k}') \ . \tag{3.3}$$

Let us write this equation symbolically as

$$D(\gamma^{(e)},\gamma^{(p)},\vec{k})\Phi^{(e)}(\vec{k}) = 0 \ . \tag{3.4}$$

In the light of the definition (3.1), it seems reasonable to next study the amplitude

$$\Phi^{(p)}_{\alpha\beta} = \sum_\sigma u^{(e)}_\beta(-\vec{k},\sigma)\langle -\vec{k},\sigma|\psi^{(p)}_\alpha(0)|0,\Lambda\rangle \ . \tag{3.5}$$

In fact, the amplitude (3.5) satisfies an equation obtained from (3.3) by the interchange of superscripts e and p. But because in (3.3) the first index of $\Phi^{(e)}$ is an electron index, whereas in (3.5) it refers to positrons, and the Dirac matrices that are superscripted with e or p are one and the same, it follows that the transformed equation is indistinguishable from (3.3). Therefore another tack is required to obtain a distinct equation with e and p interchanged.

For this purpose consider the equation

$$D(\gamma^{(p)},\gamma^{(e)},-\vec{k})\Phi^{(p)}(-\vec{k}) = 0 \ . \tag{3.6}$$

This is still Eq. (3.3) with $\vec{k} \to -\vec{k}$. But now let us evaluate the amplitude involved under the simplifying assumption that $|\vec{k},\sigma\rangle$ is a simple Fock-space state, i.e.,

$$|\vec{k},\sigma\rangle^{(e)} \equiv b^\dagger(\vec{k},\sigma)|\mathrm{vac}\rangle \ , \tag{3.7a}$$

$$|\vec{k},\sigma\rangle^{(p)} \equiv d^\dagger(\vec{k},\sigma)|\mathrm{vac}\rangle \ , \tag{3.7b}$$

where $|\mathrm{vac}\rangle$ is annihilated by the corresponding annihilation operators. [This is exactly the approximation used in the evaluation of the matrix element of the current in Eq. (2.11).] In detail

$$\begin{aligned}\Phi^{(p)}_{\alpha\beta}(-\vec{k}) &= \sum_\sigma u^{(e)}_\beta(\vec{k},\sigma)\langle\vec{k},\sigma|\psi^{(p)}(0)|e^-e^+\rangle \\ &\cong \sum_{\sigma,\sigma'} \mathrm{const}\times u^{(e)}_\beta(\vec{k},\sigma)u^{(p)}_\alpha(-\vec{k},\sigma')\langle\mathrm{vac}|b(\vec{k},\sigma)d(-\vec{k},\sigma')|e^-e^+\rangle \\ &= -\sum_{\sigma'}u^{(p)}_\alpha(-\vec{k},\sigma)^{(p)}\langle -k,\sigma|\psi^{(e)}_\beta(0)|e^-e^+\rangle = -\Phi^{(e)}_{\beta\alpha}(\vec{k}) \ .\end{aligned} \tag{3.8}$$

The combination of this result and Eq. (3.6) shows that in the approximation considered, $\Phi^{(e)}(\vec{k})$ also satisfies an equation that is distinct from (3.4), namely,

$$D(\gamma^{(p)},\gamma^{(e)},-\vec{k})\Phi^{(e)}(\vec{k}) = 0 \ . \tag{3.9}$$

We can then get a symmetrized equation by taking the average of (3.4) and (3.9), namely,

$$\{\gamma^{(e)}_0[W-E(k)] - \vec{\gamma}^{(e)}\cdot\vec{k} - m + \gamma^{(p)}_0[W-E(k)] + \vec{\gamma}^{(p)}\cdot\vec{k} - m\}\Phi^{(e)}(\vec{k})$$
$$= -\frac{e^2}{(2\pi)^3}\int d\vec{k}'\frac{m}{\sqrt{EE'}}[\Lambda^{(e)}_+(\vec{k}) + \Lambda^{(p)}_+(-\vec{k})]\left[\frac{1}{q^2}\gamma^{(e)}_0\gamma^{(p)}_0 - \frac{1}{Q^2}\left[\vec{\gamma}^{(e)}\cdot\vec{\gamma}^{(p)} - \frac{(\gamma^{(e)}\cdot\vec{q})(\vec{\gamma}^{(p)}\cdot\vec{q})}{q^2}\right]\right]\Phi^{(e)}(\vec{k}') \ . \tag{3.10}$$

[We remind the reader to refer to Eq. (3.1) to see how the Dirac matrices act on $\Phi^{(e)}$.] This equation will be further transformed in several steps. We first introduce the symmetrical expansion

$$\Phi^{(e)}_{\alpha\beta}(\vec{k}) = \sum_{\sigma,\sigma'} u^{(e)}_\alpha(\vec{k},\sigma)u^{(p)}_\beta(-\vec{k},\sigma)\left[\frac{m}{E}\right]^{1/2}\psi_{\sigma\sigma'} \ . \tag{3.11}$$

This will take us from a 16-component amplitude to a 4-component one, a reduction that is possible, naturally, only because we have excluded negative-energy states.

When we introduce (3.11) into (3.10), use the Dirac equation for a free particle and take scalar products with Dirac spinors $u^\dagger$, which are normalized according to $u^\dagger u=(E/m)$, we obtain the equation

$$\begin{aligned}[][W-2E(k)]\psi_{\sigma\sigma'}(\vec{k}) &= \frac{-e^2}{(2\pi)^3}\int d\vec{k}'\frac{m^2}{EE'}\sum_{\sigma'',\sigma'''}(q^{-2}[\bar{u}(\vec{k},\sigma)\gamma_0^{(e)}u(\vec{k}',\sigma'')][\bar{u}(-\vec{k}\sigma')\gamma_0^{(p)}u(-\vec{k}',\sigma''')] \\ &\quad -Q^{-2}\{[\bar{u}(\vec{k},\sigma)\vec{\gamma}^{(e)}u(\vec{k}',\sigma'')]\cdot[\bar{u}(-\vec{k}\sigma')\vec{\gamma}^{(p)}u(-\vec{k}',\sigma'')] \\ &\quad -q^{-2}[\bar{u}(\vec{k},\sigma)\vec{\gamma}^{(e)}\cdot\vec{q}u(\vec{k}',\sigma'')][\bar{u}(-\vec{k}',\sigma')\vec{\gamma}^{(p)}\cdot\vec{q}u(-\vec{k}',\sigma'')]\})\psi_{\sigma''\sigma'''}(\vec{k}')\ . \end{aligned} \tag{3.12}$$

One more transformation will suffice to bring the equation to a suitable starting point for practical work. We introduce, instead of ψ, the combination ($\alpha,\beta=1,2$ only).

$$\Psi_{\alpha\beta}(\vec{k})=\sum_{\sigma,\sigma'}\chi_\alpha^{(e)}(\sigma)\chi_\beta^{(p)}(\sigma')\psi_{\sigma\sigma'}(\vec{k})\ , \tag{3.13}$$

where all the χ's are two-component Pauli spinors. This substitution plus the partial evaluation of the matrix elements of the Dirac spinors, that reduces them in the standard fashion to the matrix elements of Pauli matrices between two-component spinors, leads to the version

$$[W-2E(k)]\Psi(\vec{k})=\frac{-e^2}{(2\pi)^3}\int d\vec{k}' I(\vec{k},\vec{k}'|\vec{\sigma}^{(e)},\vec{\sigma}^{(p)})\Psi(\vec{k}')\ , \tag{3.14}$$

where I is a 4×4 matrix function of the vectors $\vec{k}$ and $\vec{k}'$. With the help of the vectors

$$\vec{\kappa}=\frac{\vec{k}}{E+m}\ ,\quad \vec{\kappa}'=\frac{\vec{k}'}{E'+m}\ , \tag{3.15}$$

I is given by the expression

$$\begin{aligned}\frac{4EE'}{(E+m)(E'+m)}I &= q^{-2}[1+\vec{\kappa}\cdot\vec{\kappa}'+i\vec{\sigma}^{(e)}\cdot(\kappa\times\kappa')][1+\vec{\kappa}\cdot\vec{\kappa}'+i\vec{\sigma}^{(p)}\cdot(\vec{\kappa}\times\vec{\kappa})] \\ &\quad +Q^{-2}\vec{\sigma}^{(e)}\cdot\vec{\sigma}^{(p)}[(\sigma^{(e)}\cdot\vec{\kappa})(\vec{\sigma}^{(p)}\cdot\vec{\kappa})-(\sigma^{(e)}\cdot\vec{\kappa})(\vec{\sigma}^{(p)}\cdot\vec{\kappa}')-(\sigma^{(e)}\cdot\vec{\kappa}')(\vec{\sigma}^{(p)}\cdot\vec{\kappa})+(\vec{\sigma}^{(e)}\cdot\vec{\kappa}')(\vec{\sigma}^{(p)}\cdot\vec{\kappa}')] \\ &\quad +Q^{-2}(4\vec{\kappa}\cdot\vec{\kappa}'+4i\vec{S}\cdot\vec{\kappa}\times\vec{\kappa}')+(q^{-2}-Q^{-2})(1+\vec{\kappa}\cdot\vec{\kappa}'+i\vec{\sigma}^{(e)}\cdot\vec{\kappa}\times\vec{\kappa}') \\ &\quad \times(1+\vec{\kappa}\cdot\vec{\kappa}'+i\vec{\sigma}^{(p)}\cdot\vec{\kappa}\times\vec{\kappa}')\ ,\end{aligned} \tag{3.16}$$

with $\vec{S}=\frac{1}{2}(\vec{\sigma}^{(e)}+\vec{\sigma}^{(p)})$ as the total spin. By inspection, we see that I is invariant under rotations, inversion, and the interchange of the two spins. Thus $J^2=(\vec{L}+\vec{S})^2, J_z$, parity and S^2 are good quantum numbers. This makes a partial-wave decomposition straightforward, since, in addition, the required angular averages of all functions that occur in (3.16) are well known. In other words, from the technical point of view, we have a problem that parellels that already encountered in the study of the one-boson approximation for nuclear forces. Further details on this will be reported when numerical results become available.

We finish this account with remarks on several other properties of our equation. We have verified, of course, that it gives the spectrum of positronium correctly to $O(\alpha^4)$ Refs. [19,20]. Secondly, it contains the Breit equation [21] as a special limit. This is most easily obtained if we start from Eq. (3.12) and use, instead of (3.13), the following transformation:

$$X_{\alpha\beta}(\vec{k})=\sum_{\sigma,\sigma'}\tilde{u}_\alpha^{(e)}(\vec{k},\sigma)\tilde{u}_\beta^{(p)}(-\vec{k},\sigma')\psi_{\sigma\sigma'}(\vec{k})\ , \tag{3.17}$$

where $\tilde{u}$ is related to the previous Dirac spinors u by a rescaling

$$\tilde{u}=\left[\frac{m}{E}\right]^{1/2}u\ . \tag{3.18}$$

With the help of (3.17) and the definitions

$$H(k)=\vec{\alpha}\cdot\vec{k}+\beta m\ , \tag{3.19}$$

$$\Omega_+(\vec{k})=\sum_\sigma\tilde{u}(\vec{k},\sigma)\tilde{u}^\dagger(\vec{k},\sigma)\ , \tag{3.20}$$

we can transform (3.12) into the version

$$[W-H^{(e)}(\vec{k})-H^{(p)}(-\vec{k})]X(\vec{k})=-e^2\Omega_+^{(e)}(\vec{k})\Omega_+^{(p)}(-\vec{k})\frac{1}{(2\pi)^3}\int d\vec{k}'\left[\frac{1}{q^2}-\frac{1}{Q^2}\left[\vec{\alpha}^{(e)}\cdot\vec{\alpha}^{(p)}-\frac{(\vec{\alpha}^{(e)}\cdot\vec{q})(\vec{\alpha}^{(p)}\cdot\vec{q})}{q^2}\right]\right]X(\vec{k}')\ . \tag{3.21}$$

This becomes the equation historically identified as the Breit equation if one replaces the product $\Omega_{+}^{(e)}\Omega_{+}^{(p)}$ of positive-energy projection operators by unity and neglects recoil by the replacement $Q^2 \rightarrow q^2$. One can then transform to coordinate space and recognize the combination sought, where in place of the Coulomb term r^{-1}, one finds

$$\frac{1}{r}\left[1-\frac{1}{2}\left[\vec{\alpha}^{(e)}\cdot\vec{\alpha}^{(p)}+\frac{(\vec{\alpha}^{(e)}\cdot\vec{\mathrm{r}})(\vec{\alpha}^{(p)}\cdot\vec{\mathrm{r}})}{r^2}\right]\right] . \tag{3.22}$$

IV. SUMMARY AND CONCLUSIONS

We have described an alternative method for deriving a relativistic two-particle equation from a field theory, using a technique of taking matrix elements of the field equation and using a completeness relation to evaluate the matrix element of a product of field operators. Because we take the matrix element of a single fermion field operator between a two-particle and a one-particle state, we obtain an equation with superficial resemblance to a one-particle equation. This equation is asymmetric in appearance, since it puts one particle on the mass shell and keeps the other off. In the case of unequal masses, if the particle on the mass shell is the heavier one, in the limit of "infinite" mass, one regains almost trivially the Dirac equation for the lighter particle moving in the field of the heavier one. Concentrating thereafter on the case of equal masses, we showed how an equation symmetric in its treatment of the two particles can be derived. This equation, in contradistinction to that derived from the Tamm-Dancoff approximation, has an interaction which, though nonlocal, is energy independent. It yields the correct spectrum of positronium to order $O(\alpha^4)$ and also contains the Breit equation in a suitable limit. A partial-wave analysis leads, depending on the channel, to one-dimensional integral equations, or else to a coupled pair of such equations. This analysis plus results for e^-e^+ scattering based upon it will be reported later.

In terminating this report, we have left open a larger question of whether the methods initiated here can be made the starting point of a systematic approach to QED. We must then confront the whole problem of treating self-interactions. Though there are no historic grounds for optimism concerning one's ability to deal with this problem in a nonmanifestly covariant formalism, there is one positive feature that may be worth mentioning. The electromagnetic field enters in the form of a sequence of matrix elements, the simplest sets having already appeared in the derivation given. For these fields, we have the option of choosing different gauges for different sets, and thus we may choose a manifestly covariant gauge for the study of contributions requiring renormalization. It remains to be seen if this suggestion will be useful.

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[1] C. Y. Wong and R. L. Becker, Phys. Lett. B **182**, 251 (1986).
[2] W. Koenig *et al.*, Phys. Lett. B **218**, 12 (1989).
[3] P. Salabara *et al.*, Phys. Lett. B **245**, 153 (1990).
[4] F. Gross, Phys. Rev. **186**, 1448 (1969); Phys. Rev. C **26**, 2203 (1982).
[5] E. D. Cooper and B. K. Jennings, Nucl. Phys. A **500**, 553 (1989).
[6] J. R. Spence and J. P. Vary, Phys. Lett. B **254**, 1 (1991).
[7] K. Erkelenz, Phys. Rep. **5**, 191 (1974).
[8] N. R. Walet, A. Klein, and R. M. Dreizler, Phys. Lett. B **273**, 1 (1991).
[9] B. A. Arbuzov *et al.*, Phys. Lett. B **240**, 447 (1990).
[10] A. Klein, University of Pennsylvania Report No. UPR-0041NT, 1987 (unpublished).
[11] H. W. Crater and P. Van Alstine, Ann. Phys. (N.Y.) **148**, 57 (1983); Phys. Rev. D **37**, 1982 (1988), and other references found in this latter work.
[12] P. Van Alstine and H. Crater, Phys. Rev. D **34**, 1932 (1986).
[13] J. W. Darewych and M. Horbatsch, J. Phys. B **22**, 973 (1989); **23**, 337 (1990).
[14] W. Dykshoorn, R. Koniuk, and R. Muñoz-Tapia, Phys. Rev. A **41**, 60 (1990).
[15] W. Dykshoorn and R. Koniuk, Phys. Rev. A **41**, 64 (1990).
[16] V. B. Mandelzweig and S. J. Wallace, Phys. Lett. B **197**, 469 (1987).
[17] S. J. Wallace and V. B. Mandelzweig, Nucl. Phys. A **503**, 673 (1989).
[18] J. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).
[19] R. A. Ferrel, Phys. Rev. **84**, 858 (1951).
[20] R. Karplus and A. Klein, Phys. Rev. **86**, 257 (1952).
[21] G. Breit, Phys. Rev. **34**, 553 (1929); **36**, 383 (1930); **39**, 616 (1932).