Classical eigenspinors and the Dirac equation

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The four-velocity and orientation of an "elementary" particle is given classically by the Lorentz transformation from the rest frame of the particle to the observer's frame. This transformation, first discussed in detail by Gürsey [Nuovo Cimento **5**, 784 (1957)], is the classical *eigenspinor* of the particle; it is shown here to satisfy a trivial four-momentum relation that is the exact analog of the Dirac equation of relativistic quantum theory. Although the classical elementary particle can spin at an arbitrary rate, it is constrained by the Lorentz-force equation and by the linearity of its time evolution to have a g factor of 2. Bilinear covariants of its eigenspinor include the four-velocity and spin, and these as well as the transformations of the eigenspinor under **P**, **T**, and **C**, have the same form as in quantum theory. However, the bilinear covariants giving the spacelike Frenet vectors orthogonal to the spin have no counterparts among the usual bilinear covariants $\bar{\psi}\Gamma\psi$ of quantum theory, although they can be expressed in the form $\bar{\psi}\Gamma\psi^c$, where Γ is a sum of products of Dirac matrices and ψ^c is the charge-conjugated spinor wave function. The relation of the classical and quantum theories is strengthened by a discussion of the superposition of eigenspinors and the eigenspinor field of a classical distribution of particles.

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I. INTRODUCTION

The Dirac equation and its spinor solutions lie at the very foundations of relativistic quantum theory. Appropriate solutions govern the motion of elementary fermions in external fields and thereby describe not only the electromagnetic interactions of leptons but also the hadronic interactions of quarks. The success of quantum electrodynamics (QED) in describing leptons and their interactions is well documented. With few exceptions [1] it is based on perturbation expansions in the fine-structure constant $\alpha = ke^2/(\hbar c) \approx \frac{1}{137.036}$ [where k is 1 in gaussian units and $1/(4\pi\epsilon_0)$ in SI units]. QED has served as a model for quantum chromodynamics (QCD) to describe the interactions of quarks, but the much larger size of the strong-coupling constant at typical interaction energies has severely restricted the accuracy and variety of QCD applications. It appears that significant advances in the realm of QCD calculations hinge largely on finding nonperturbative approaches.

New insights into the meaning of relativistic quantum theory and its classical limit may be crucial in the development of new computational approaches. There is a long history of papers [2] that have sought classical analogs of the Dirac equation, but as pointed out previously [3], the common technique of letting $\hbar \to 0$ fails to give us the relativistic classical limit in QED or, for that matter, in any perturbative theory dependent on the convergence of expansions in a parameter proportional to $\hbar^{-1}$. A number of classical models interpret the fact that components of the "velocity operator" $\boldsymbol{\alpha}$ have eigenvalues $\pm c$ in terms of a point charge which moves at or greater than the speed of light c on a helical path.

Here, an approach based on the covariant Pauli algebra is used to demonstrate close relationships between classical and quantum theories of "elementary" fermions. The relationships established are more direct than those of the usual prescription for quantizing classical theories, in which classical variables of the Hamiltonian or Lagrangian are replaced by operators, and Poisson brackets, by commutators. The fermion is not required to be a point particle and no luminal or superluminal velocities are required.

It is well known that Dirac spinors can be constructed to represent the covering group SL(2,C) of restricted Lorentz transformations, which includes boosts, rotations, and their products. In his study of the Frenet tetrad in relativistic kinematics, Gürsey [2] associated the Dirac wave function with a representation of the Lorentz transformation of the particle. Hestenes [4], in his Dirac theory in the real Dirac algebra, made a similar identification and, like Huang [5], related Schrödinger's *Zitterbewegung* to the electron spin and the wave nature of matter. Barut's approach [6] to relating the classical and quantum theories of the electron was to find a classical Lagrangian whose quantization gives the Dirac equation. In a recent paper, Tisza [7] introduced the concept of a "wave simplex" in an algebraic model for a new non-Newtonian mechanics designed to integrate quantum and classical pictures.

In this paper, Gürsey's classical "proper matrix" or "proper four-spinor" is further developed and becomes the *eigenspinor* of an elementary particle in the covariant Pauli algebra; it is shown here to obey an equation identical in form to the quantum Dirac equation. Bilinear covariants of the eigenspinor are formed, interpreted, and associated with their quantum counterparts. These covariants include the four-velocity, associated with the quantum-mechanical current density, and the spin. The concept of an elementary particle forms an important link between the classical and quantal worlds: it is shown to lead to a g factor constrained by the Lorentz-force equation to be equal to 2. Classical negative-energy and zero-mass solutions and the transformations of the eigenspinor under spatial inversion (**P**), time reversal (**T**), and charge conjugation (**C**) are also shown to correspond closely to those in the quantum theory. The full quantum Dirac theory is even more closely approached by a spinor field formed from a superposition of classical eigenspinors which describe a free rest-frame rotation at the *Zitterbewegung* frequency [5].

The following section reviews essential elements of the Pauli algebra. The classical eigenspinor of an elementary particle is introduced in Sec. III, and is shown to obey a trivial "classical Dirac equation" in Sec. IV. Bilinear covariants and basic transformation laws of the classical eigenspinor are compared to their quantum counterparts in Sec. V. The constraint on the g factor of the particle is also derived in this section. In Sec. VI, an attempt is made to relate quantum solutions to the Dirac equation to a classical spinor field of rotating particles.

II. ELEMENTS OF THE PAULI ALGEBRA

The arguments are developed here in the Pauli algebra $\mathcal{P}$, which has been extended to a covariant formulation of special relativity [8–10]. In $\mathcal{P}$, which mathematicians know as the Clifford algebra of real three-dimensional Euclidean space, an associative but generally noncommutative multiplication of vectors admits products to any order. As shown elsewhere [8], $\mathcal{P}$ has the same power for problems in special relativity as the closely related real Dirac algebra [11] (the Clifford algebra of Minkowski space), but is significantly simpler and more intuitive. The brief introduction presented here contains all the background needed in the following sections.

An arbitrary element $p\in\mathcal{P}$ can be expanded in products of the three unit vectors $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$. The condition that the square of any vector $\mathbf{a}=\alpha^j\mathbf{e}_j$ (summed over $j=1,2,3$) is the square length $\mathbf{a}\cdot\mathbf{a}$ gives the basic multiplication rule

$$\mathbf{e}_j\mathbf{e}_k+\mathbf{e}_k\mathbf{e}_j=2\delta_{jk} \ . \tag{1}$$

This rule relates products which differ only in the ordering of the basic vectors and allows any product of basis vectors containing two or more factors of a given $\mathbf{e}_j$ to be reduced. There are, as a result, only eight linearly independent products, namely $1,\mathbf{e}_j\mathbf{e}_j\mathbf{e}_k$ with $j<k$, and $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, representing real scalars, polar vectors, planes, and volumes, respectively. These products constitute the basis set of the algebra.

The set is simplified by noting that the "canonical element" $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ of $\mathcal{P}$ commutes with all elements and has a square $(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^2=-1$; it plays the role of the imaginary i in the algebra: $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3=i$. Furthermore, $\mathbf{e}_1\mathbf{e}_2=\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_3=i\mathbf{e}_3$ and similarly $\mathbf{e}_2\mathbf{e}_3=i\mathbf{e}_1$ and $\mathbf{e}_3\mathbf{e}_1=i\mathbf{e}_2$. It may be seen that $i\mathbf{e}_j$ is a pseudovector representing the plane normal to $\mathbf{e}_j$, whereas i times a scalar is a pseudoscalar and represents a volume. Any $p\in\mathcal{P}$ can thus be written as the sum of real-scalar, pseudoscalar, polar-vector, and pseudovector parts

$$p=p_0+\mathbf{p}=(p_s+p_{ps})+(\mathbf{p}_v+\mathbf{p}_{pv}) \ , \tag{2}$$

where p_s and components of $\mathbf{p}_v$ are pure real, whereas p_{ps} and components of $\mathbf{p}_{pv}$ are pure imaginary. Thus the Clifford algebra $\mathcal{P}$ of real three-dimensional Euclidean space spans a complex four-dimensional space: any element is the sum of a scalar and a vector: $p=p_0+p^k\mathbf{e}_k$, the components of which may be complex (or zero).

The simplest faithful representation of $\mathcal{P}$ is a 2×2 matrix representation in which the unit vectors $\mathbf{e}_j$ may be represented by the Pauli spin matrices $\underline{\sigma}_j$, and indeed it is from this representation that the Pauli algebra takes its name. To every element $p=p_0+p^k\mathbf{e}_k\in\mathcal{P}$ there correspond two closely related elements: the *Hermitian* conjugate $p^\dagger\equiv p_0^*+p^{k*}\mathbf{e}_k$, obtained by taking complex conjugates of the components, and the *spatial reverse* $\bar{p}=p_0-p^k\mathbf{e}_k$. Both Hermitian conjugation and spatial reversal are said to be "antiautomorphisms" because when applied to a product pq of $\mathcal{P}$, the order is reversed, as is readily verified: $(pq)^\dagger=q^\dagger p^\dagger$ and $\overline{pq}=\bar{q}\,\bar{p}$. Of course the combination is an automorphism: $\overline{(pq)}^\dagger=\bar{p}^\dagger\bar{q}^\dagger$. The order in which the antiautomorphisms are applied is immaterial: the spatial reversal of a Hermitian conjugate is the same as the Hermitian conjugate of the spatial reversal. An element equal to its Hermitian conjugate is *real* and its matrix representations are *Hermitian*; an element equal to its spatial reversal is a (possibly complex) *scalar.*

The scalar part of a product pq is indicated by the dot product

$$p\cdot q=\tfrac{1}{2}(pq+\overline{pq})=p_0q_0+\mathbf{p}\cdot\mathbf{q} \ . \tag{3}$$

Since $1\in\mathcal{P}$, $p\cdot q=1\cdot(pq)=1\cdot(qp)$. Although the product $p^2=pp$ of any element with itself is not generally a scalar, $p\bar{p}$ is, since $\overline{(p\bar{p})}=p\bar{p}$. In other words, $p\bar{p}=p\cdot\bar{p}=\bar{p}p$. If $p\bar{p}=0$, the element p is *null.* Every non-null element p has an *inverse*

$$p^{-1}=\bar{p}/(p\bar{p}) \ . \tag{4}$$

One of the beautiful features of the Pauli algebra is the natural way it yields the structure of Minkowski spacetime. Minkowski four-vectors appear as real elements of $\mathcal{P}$; for example, the four-momentum $p=p_0+\mathbf{p}=p^\mu\mathbf{e}_\mu$ where $\mathbf{e}_0\equiv1$, and μ is summed over the values $\mu=0,1,2,3$. One may associate p with contravariant components and its spatial reversal with covariant ones: $\bar{p}=p^\mu\bar{\mathbf{e}}_\mu\equiv p_\mu\mathbf{e}_\mu$. The scalar $p\bar{p}$ is the Lorentz norm $p\bar{p}=p_0^2-\mathbf{p}^2=p^\mu p_\mu$. Similarly, $p\cdot\bar{q}=p_0q_0-\mathbf{p}\cdot\mathbf{q}$ is the Lorentz-invariant scalar product $p\cdot\bar{q}=p^\mu q_\mu$. If $p\cdot\bar{q}=0$, then p and q are four-orthogonal to each other.

A restricted Lorentz transformation of a four-vector like the four-velocity $u=p/m=\gamma+\mathbf{u}$ is given by [8,9]

$$u\to u'=LuL^\dagger \ , \tag{5}$$

where $L=\exp(-i\boldsymbol{\theta}/2+\mathbf{w}/2)$ is a unimodular element $(L\bar{L}=1)$ of $\mathcal{P}$ and a member of the group SL(2,C), whose six parameters are given explicitly by the components of $\mathbf{w}-i\boldsymbol{\theta}$. If $\boldsymbol{\theta}=0$, L is Hermitian and describes a pure boost with boost parameter $\mathbf{w}$. On the other hand, if $\mathbf{w}=0$, then L is unitary and describes a pure rotation by the angle θ in the plane $i\hat{\boldsymbol{\theta}}$ about the axis $\hat{\boldsymbol{\theta}}$. Note that L is the exponential of a vector, but such expressions make perfect sense in $\mathcal{P}$ since any product of a vector is defined by (1). Indeed, using the rule $\mathbf{a}^2=\mathbf{a}\cdot\mathbf{a}$, one can expand any analytic function of a vector $\mathbf{a}$ into a scalar part and a vector part parallel to $\mathbf{a}$. For example, $\exp(\mathbf{a})=\cosh\sqrt{\mathbf{a}\cdot\mathbf{a}}+(\mathbf{a}/\sqrt{\mathbf{a}\cdot\mathbf{a}})\sinh(\sqrt{\mathbf{a}\cdot\mathbf{a}})$.

Several four-vectors will be used in the following discussion, all in units with $c=1$. These include the space-time position $x=t+\mathbf{x}$, the electromagnetic four-potential $A=\phi+\mathbf{A}$, the particle current density $j=\rho+\mathbf{j}$, and the four-vector differential operator $\partial=\partial/\partial t-\nabla$. As with other four-vectors, one can expand ∂ in the basis elements $\mathbf{e}_\mu$: $\partial=\mathbf{e}_\mu\partial/\partial x_\mu=\mathbf{e}_\mu\partial^\mu$. If p is a constant four-vector, note that $\partial(p\cdot\bar{x})=p$. Products of four-vectors with spatially reversed four-vectors evidently transform distinctly from four-vectors themselves. For example, the electromagnetic field $\mathbf{F}=\partial\bar{A}-\partial\cdot\bar{A}=\mathbf{E}+i\mathbf{B}$ transforms as

$$\mathbf{F}=\partial\bar{A}-\partial\cdot\bar{A}\to L\partial L^\dagger\bar{L}^\dagger\bar{A}\bar{L}-\partial\cdot\bar{A}=L\mathbf{F}\bar{L}\ . \tag{6}$$

Such transformations characterize "six-vectors" (with three polar-vector components plus three pseudovector components), which correspond to antisymmetric second-rank Minkowski-space tensors. Another six-vector is the exponent of L itself.

III. THE CLASSICAL EIGENSPINOR

The observed four-velocity and orientation of a particle is given by the special Lorentz transformation Λ which transforms the particle from its rest frame $\mathbf{K}_0$ to the observer's frame $\mathbf{K}$. The particle is considered *elementary* if and only if its motion in $\mathbf{K}$ at any point $x(\tau)$ on its world line can be described by a *single* Lorentz transformation $\Lambda(\tau)$ at the proper time τ. Such a particle cannot contain independent structures (such as several elementary particles) since it would then generally require different transformations Λ for each component part and would therefore not be elementary. If the particle contained separate structures which moved rigidly together, it could still be elementary, but would suffer the well-known conflict with causality of any rigid body when accelerated: superluminal signals would be required to keep the various parts rigidly together. It is therefore probably correct to characterize any elementary particle as *structureless,* but it does not have to be a point particle and the existence of spin is not precluded.

The transformation Λ can generally be written as the product of a rotation $\mathcal{R}$ and a boost $\mathcal{B}$: $\Lambda=\mathcal{B}\mathcal{R}$. The four-velocity of the particle is found by applying Λ to the rest four-velocity, which for a positive-energy particle (in units with $c=1$) is $u_{\text{rest}}=1$:

$$u=\Lambda u_{\text{rest}}\Lambda^\dagger=\Lambda\Lambda^\dagger=\mathcal{B}^2\ . \tag{7}$$

Note that u is independent of the rotation $\mathcal{R}$. More generally, however, Λ gives not only the four-velocity, but also the orientation of the particle. If the particle is observed in a different frame $\mathbf{K}'$, Λ must be replaced by the transformed

$$\Lambda\to\Lambda'=L\Lambda\ , \tag{8}$$

where L transforms quantities from $\mathbf{K}$ to $\mathbf{K}'$. This transformation behavior [12] is one way to define *spinors* in $\mathcal{P}$, and Λ may be called the *eigenspinor* of the particle. In a 2×2 matrix representation of the algebra, Λ is precisely Gürsey's "proper matrix" [2], each column of which is a common two-component spinor (see Sec. IV, below).

The eigenspinor plays a dual role: it is both an operator with which properties of the particle, such as its four-velocity (7), can be transformed from the rest frame to the observer's frame, and it is an operand on which other Lorentz transformations can act, as in (8), from the left. Note that the spinor transformation (8) is peculiar to Lorentz transformations like Λ which connect an *object* frame to an *observer* frame. The Lorentz transformation L affects only one of these: a *passive transformation* changes the observer's frame, whereas an *active transformation* changes the object frame. On the other hand, a transformation L_{12} connecting two frames $\mathbf{K}_1$ and $\mathbf{K}_2$, both of which are affected by L, transforms as $L_{12}\to LL_{12}\bar{L}$.

The motion of an elementary particle is determined from its initial velocity and distribution in space if the evolution of its eigenspinor is known. The time evolution of Λ from τ_0 to τ can be written

$$\Lambda(\tau)=L(\tau,\tau_0)\Lambda(\tau_0)\ , \tag{9}$$

where the classical time-evolution operator $L(\tau,\tau_0)$ is a unimodular Lorentz transformation obeying $L(\tau_0,\tau_0)=1$ and $L(\tau,\tau_0)=L(\tau,\tau_1)L(\tau_1,\tau_0)$. Both $\Lambda(\tau)$ and $L(\tau,\tau_0)$ obey the same equation of motion, which takes the form of a trivial identity

$$\dot{\Lambda}=G\Lambda\ ,\quad G\equiv\dot{\Lambda}\bar{\Lambda}\ , \tag{10}$$

where the dot indicates a derivative with respect to the proper time τ. Since Λ is unimodular, $\Lambda\bar{\Lambda}=1$ and $\dot{\Lambda}\bar{\Lambda}=-\Lambda\dot{\bar{\Lambda}}=-\overline{\dot{\Lambda}\bar{\Lambda}}$. As a result, G is the negative of its spatial reversal and is therefore a pure (complex) vector. Thus its scalar part vanishes: $1\cdot G=\dot{\Lambda}\cdot\bar{\Lambda}=0$, and consequently $\dot{\Lambda}$ is four-orthogonal to Λ.

By the spinor transformation (8) and the Lorentz invariance of τ, G must transform as a six-vector: $G\to LG\bar{L}$. In particular, G in the observer's frame is related to G_{rest} in the rest frame of the particle by

$$G=\Lambda G_{\text{rest}}\bar{\Lambda}\ . \tag{11}$$

As a consequence, the equation of motion (10) can also be written

$$\dot{\Lambda}=\Lambda G_{\text{rest}}=\Lambda(\mathbf{a}-i\mathbf{b})/2\ . \tag{10$'$}$$

The real part $\mathbf{a}$ and imaginary part $\mathbf{b}$ of $2G^\dagger_{\text{rest}}$ are, respectively, the instantaneous acceleration and the instantaneous rotation rate of the particle in its rest frame.

The eigenspinor Λ can transform any four-vector or six-vector property from the particle frame to the observer's laboratory frame. The transformation of the four-orthogonal basis four-vectors $\mathbf{e}_\mu$ gives the *Frenet tetrad* of four-vectors

$$\mathbf{u}_\mu=\Lambda\mathbf{e}_\mu\Lambda^\dagger\ , \tag{12}$$

as discussed by Gürsey [2]. Note that the subscripts here label different four-vectors, not components of a four-vector, and that the Frenet vectors are all four-orthogonal to each other. The timelike Frenet four-vector $\mathbf{u}_0$ is just the four-velocity of the particle:

$$u=\mathbf{u}_0=\Lambda\mathbf{e}_0\Lambda^\dagger=\Lambda\Lambda^\dagger \ . \tag{13}$$

Since the basis vectors $\mathbf{e}_\mu$ are constant, the proper-time derivative of any Frenet vector is simply [13]

$$\begin{aligned}\dot{\mathbf{u}}_\mu&=\dot\Lambda\mathbf{e}_\mu\Lambda^\dagger+\Lambda\mathbf{e}_\mu\dot\Lambda^\dagger\\&=G\mathbf{u}_\mu+\mathbf{u}_\mu G^\dagger \ .\end{aligned} \tag{14}$$

When $\mu=0$, this is exactly the form of the Lorentz-force equation in $\mathcal{P}$ [9]. A charge e of mass m in an external electromagnetic field $\mathbf{F}=\mathbf{E}+i\mathbf{B}$ is governed by (14) with G replaced by the six-vector $(e/2m)\mathbf{F}$, where $\mathbf{F}(\tau)$ is the field on the world line of the particle of proper time τ. The most general form of G which makes (14) consistent with the Lorentz-force equation is

$$G=\frac{e}{2m}\mathbf{F}(\tau)-2m\mathbf{S} \ , \tag{15}$$

where $\mathbf{S}$ is a six-vector constrained by $\mathbf{S}u+u\mathbf{S}^\dagger=\mathbf{0}$. In other words, the four-vector $s=-i\mathbf{S}u$ is real. This condition ensures that $\mathbf{S}$ does not influence the four-acceleration $\dot u$. In the rest frame, $u=1$ and the condition on $\mathbf{S}$ means that s_{rest} is a pure vector $s_{\text{rest}}\equiv\mathbf{s}_0$, and according to (10′), it contributes an angular velocity $4m\mathbf{s}_0$ to the rest-frame rotation rate $\mathbf{b}$.

Evidently the six-vector $\mathbf{S}$ and its "dual" s are associated with the particle spin, the magnitude and direction of which are not restricted by classical equations of motion. The factor $2m$ multiplying $\mathbf{S}$ is arbitrary; it has been chosen so that in Sec. V below, $\mathbf{S}$ corresponds directly to a spin of $\frac{1}{2}$ (in units with $c=\hbar=1$). The linear time-evolution equation (10) for the eigenspinor Λ, together with (15), has been used to simplify the solutions of a number of problems in classical electrodynamics [10]; for example, one sees immediately that the electric-field part $(e/m)\mathbf{E}$ of $2G$ generates boosts whereas the magnetic-field part $-(e/m)\mathbf{B}$, together with $-4im\mathbf{S}=4ms\bar{u}$, generates rotations.

The six-vector $\mathbf{S}$ in (15) can depend on the electromagnetic field $\mathbf{F}$, but the only six-vector expression of first order in $\mathbf{F}$ and $\mathbf{F}^\dagger$ which satisfies the constraint $\mathbf{S}u+u\mathbf{S}^\dagger=\mathbf{0}$ is

$$\mathbf{S}^{(1)}=x\mathbf{F}\bar{x}-u\bar{x}^\dagger\mathbf{F}^\dagger x^\dagger\bar{u} \ , \tag{16}$$

where x is some Pauli element. No matter how x is chosen, if expression (16) is nonzero, it depends on the four-velocity u and makes the time evolution (10) of the eigenspinor nonlinear in Λ. There is a reason to avoid such nonlinear equations of evolution, a reason beyond questions of beauty and mathematical simplicity.

The elementary particle considered so far is structureless, but not necessarily confined to a single point. It seems desirable to avoid the singularities associated with particles of zero dimension, and even classically, "point particles" usually have distributions reached as a limit of some finite distribution as a characteristic dimension approaches zero. Any particle of finite size, no matter how small, will have different bits moving at slightly different velocities whenever rotation or curved trajectories are involved. Any nonlinearity in the evolution equation for the eigenspinor will then yield different eigenspinors for different parts, and the particle will no longer be elementary. Thus elementary particles that are not isolated points require linear equations of motion.

To achieve the linearity of (10) with respect to Λ, contributions to $\mathbf{S}$ which are first order in the fields, as in (16), must vanish. Similar considerations appear to eliminate all six-vectors formed from odd powers of the electromagnetic field. However, there may exist a contribution of $\mathbf{S}$ whose rest-frame value $\mathbf{S}_{\text{rest}}=i\mathbf{s}_0$ is of the form of a Lorentz-invariant scalar times a constant pseudovector, and the scalar may contain some field dependence such as $\mathbf{F}^2$ or $u\cdot\bar{A}$ (see below). Its Lorentz invariance ensures that the equation of motion (10) with (15) is linear in Λ. In the absence of external fields, symmetry under time translations suggests that the spin vector $\mathbf{s}_0$ should be constant. The angular velocity of the classical spin in the rest frame is then $4m\mathbf{s}_0$ and is also constant.

IV. THE CLASSICAL DIRAC EQUATION

The "classical Dirac equation" is simply the spinor form of the equation relating the four-momentum and the four-velocity: $p=mu$. From (7) and the unimodularity of the eigenspinor Λ, this relation can be written in a form linear in Λ and $\bar\Lambda^\dagger$:

$$p\bar\Lambda^\dagger=m\Lambda \ . \tag{17}$$

Since (17) is valid if and only if $\Lambda\Lambda^\dagger/(\bar\Lambda^\dagger\Lambda^\dagger)$ is the four-velocity of the particle, any spinor Λ which satisfies the classical Dirac equation (17) must, within an arbitrary initial rotation and a real scalar multiplying factor, be the eigenspinor of the particle.

In a minimal (2×2) faithful matrix representation of (17), the two columns, say $\eta=\Lambda\binom{1}{0}$ and $\xi=\Lambda\binom{0}{1}$, of the eigenspinor Λ are acted upon independently. The equation may thus be written separately for both the right and left columns of (17). The spinor transformation (8) also holds independently for η and ξ, so that they may be identified as two-component spinors.

In order to express $\bar\Lambda^\dagger$ in terms of η and ξ, note that in the standard Pauli representation where $\mathbf{e}_\mu=\underline{\sigma}_\mu$,

$$\Lambda\equiv(\eta,\xi)=\Lambda^\mu\mathbf{e}_\mu=\begin{pmatrix}\Lambda^0+\Lambda^3 & \Lambda^1-i\Lambda^2\\ \Lambda^1+i\Lambda^2 & \Lambda^0-\Lambda^3\end{pmatrix} \tag{18}$$

and its spatial reversal is

$$\bar\Lambda=\overline{(\eta,\xi)}=\overline{\begin{pmatrix}\eta^1 & \xi^1\\ \eta^2 & \xi^2\end{pmatrix}}=\begin{pmatrix}\xi^2 & -\xi^1\\ -\eta^2 & \eta^1\end{pmatrix} \ . \tag{19}$$

The Hermitian conjugate of $\bar\Lambda$ is thus

$$\bar\Lambda^\dagger=\begin{pmatrix}\xi^{2*} & -\eta^{2*}\\ -\xi^{1*} & \eta^{1*}\end{pmatrix}\equiv(-\bar\xi^\dagger,\bar\eta^\dagger) \ , \tag{20}$$

where we have defined the two-spinor

$$\bar{\eta}^{\dagger}=\begin{bmatrix}-\eta^{2*}\\ \eta^{1*}\end{bmatrix}=-i\mathbf{e}_2\eta^* \ . \tag{21}$$

The right-hand column of (17) together with the Hermitian conjugate of the spatial reversal of the left-hand column of the same equation thus give

$$p\bar{\eta}^{\dagger}=m\xi \ , \quad \bar{p}\xi=m\bar{\eta}^{\dagger} \ . \tag{22}$$

Note that barring a two-component spinor is equivalent to transposing it and lowering its indices: $\bar{\eta}=(\eta_1,\eta_2)$, where the subscripted spinor components are given by the usual prescription $\eta_A=\epsilon_{AB}\eta^B$, with $\epsilon_{11}=\epsilon_{22}=0$ and $\epsilon_{21}=-\epsilon_{12}=1$. Note further that $(\bar{\eta})^{\dagger}=\overline{(\eta^{\dagger})}$ and that double barring of two-component spinor changes its sign: $\bar{\bar{\eta}}=\overline{(\bar{\eta}^{\dagger})}^{\dagger}=-\eta$. This relation shows that two-component spinors like η and ξ are not elements of $\mathcal{P}$ since for all $p\in\mathcal{P}$, spatial reversal is involutory: $\bar{\bar{p}}=p$.

The discussion to this point has concerned positive-energy particles, but for a complete comparison with the Dirac theory, negative-energy particles must also be considered. Such particles can be consistently introduced in a classical context. Each has a rest-frame four-momentum $p_{\text{rest}}=mu_{\text{rest}}=-m$, so that place of (7), one has $u=\Lambda u_{\text{rest}}\Lambda^{\dagger}=-\Lambda\Lambda^{\dagger}$ and the observed four-momentum is

$$p=mu=-m\Lambda\Lambda^{\dagger} \ . \tag{23}$$

The four-velocity u is $dx/d\tau$, where dx is the increment moved by the particle along its world line in proper times $d\tau$, and the relation $u=-\Lambda\Lambda^{\dagger}$ implies $u\cdot 1=-\Lambda\cdot\Lambda^{\dagger}=dt/d\tau\le -1$. Thus negative-energy particles move backward in time, and as in the usual Feynman-Stückelberg interpretations [13], they correspond to antiparticles with the opposite four-momentum, namely with $p(\text{antiparticle})=-p=\Lambda m\Lambda^{\dagger}$. Therefore the eigenspinor Λ of negative-energy particles is, within a complex phase factor and an initial rotation, just the Lorentz transformation of the corresponding antiparticle from its rest frame to the observer's frame. The eigenspinor for negative-energy particles must satisfy (23), which is consistent with (17) only if Λ is now *antiunimodular:* $\Lambda\bar{\Lambda}=-1$. One possibility is to identify $i\Lambda$ as the unimodular eigenspinor of the corresponding antiparticle.

When the two two-component spinors are combined into a four-component column spinor

$$\psi=\begin{bmatrix}\bar{\eta}^{\dagger}\\ \xi\end{bmatrix} , \tag{24}$$

the classical equations (17) or (22) take the usual quantum form

$$\gamma_{\mu}p^{\mu}\psi=m\psi \ , \tag{25}$$

where the 4×4 matrices γ_μ are defined in the Weyl (or spinor) representation [14]:

$$\gamma_0=\gamma^0=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} , \quad \gamma_k=-\gamma^k=\begin{bmatrix}0 & -\underline{\sigma}_k\\ \underline{\sigma}_k & 0\end{bmatrix} . \tag{26}$$

The choice (24) is not unique since one may also define, for example,

$$\psi'=\begin{bmatrix}-\bar{\xi}^{\dagger}\\ \eta\end{bmatrix} \tag{27}$$

to obtain the same equations (25). This and many other alternate choices are equivalent to (18) and (24) with an additional initial rotation: $\Lambda'=\Lambda\mathcal{R}$. In the case of ψ' (27), $\mathcal{R}=i\mathbf{e}_2=\exp(i\mathbf{e}_2\pi/2)$ and is a rotation by π about $-\mathbf{e}_2$. Similarly, another obvious choice for the antiparticle eigenspinor corresponding to the antiunimodular eigenspinor of negative-energy particles is $\Lambda\mathbf{e}_1$, which differs from $i\Lambda$ by a rotation of π about $-\mathbf{e}_1$. Since the axes chosen to specify the initial orientation of the rest frame are arbitrary, these various representations as well as any others related by an initial rotation are physically equivalent. In what follows, the identification (24) is adopted with $\Lambda=(\eta,\xi)$. Relations among some of the equivalent forms will be associated with parity, time-reversal, and charge-conjugation transformations in the next section.

The simple interpretation of the classical Dirac equation (17), (22), or (25) seems to have been missed by other authors: it merely expresses the relation between the four-momentum of a particle and its four-velocity. It corresponds most closely to the momentum representation of the quantum Dirac equations. The source of the considerable physical content of the quantum Dirac equation will be pursued in Sec. VI, but it is worthwhile first to consider further correspondences between the classical and quantum formalisms.

V. BILINEAR COVARIANTS AND SYMMETRY TRANSFORMATIONS

Bilinear covariants arise naturally in $\mathcal{P}$ as Lorentz transformations of rest-frame properties of the particle. The eigenspinor Λ enters in the role of the transformation from the rest frame $\mathbf{K}_0$ to the observer's frame $\mathbf{K}$. The Frenet four-vectors $\mathbf{u}_\mu$ (12) and the six-vectors formed from them

$$\mathbf{u}_{\mu}\bar{\mathbf{u}}_{\nu}=\Lambda\mathbf{e}_{\mu}\bar{\mathbf{e}}_{\nu}\bar{\Lambda} \tag{28}$$

are such bilinear covariants. These can be expressed in terms of the two-spinors η,ξ and compared to the quantum bilinear covariants. Consider, for example, the four-velocity $u=\mathbf{u}_0$, which for a positive-energy particle is simply $u_{\text{rest}}=1$ in $\mathbf{K}_0$. In $\mathbf{K}$,

$$u=\Lambda u_{\text{rest}}\Lambda^{\dagger}=\Lambda\Lambda^{\dagger}=\eta\eta^{\dagger}+\xi\xi^{\dagger} \ . \tag{29}$$

Its components u^μ can be expressed in terms of the classical four-spinor (24) by

$$u^{\mu}=\tfrac{1}{2}\bar{\psi}\gamma^{\mu}\psi \ , \tag{30}$$

where the barred four-spinor is defined as in the Dirac theory by $\bar{\psi}=\psi^{\dagger}\gamma^0=(\xi^{\dagger},\bar{\eta})$ and the scalar identity $\bar{\eta}\bar{\eta}^{\dagger}=\eta^{\dagger}\eta$ has been used. Both relations (29) and (30) pick up an overall minus sign for negative-energy particles. The bilinear form (30) is identical to that for the

TABLE I. Bilinear covariants of the classical eigenspinors Λ and ψ. Where there are two signs, the upper one refers to positive-energy particles and the lower one to negative-energy ones.

Type	$\mathcal{P}$-algebra form	Four-spinor form[a]	Interpretation
Scalar	$\Lambda\bar{\Lambda}=\bar{\Lambda}\Lambda=\pm 1$	$\bar{\psi}\psi=\pm 2$	Unimodularity or antiunimodularity
Four-vector	$\Lambda\Lambda^{\dagger}=\pm u$	$\bar{\psi}\gamma^{\mu}\psi=\pm 2u^{\mu}$	Four-velocity
Six-vector[b]	$-\frac{i}{2}\Lambda\mathbf{e}_3\bar{\Lambda}=\mathbf{S}$	$\frac{1}{2}\bar{\psi}\sigma^{\mu\nu}\psi=2S^{\mu\nu}$	Spin
Pseudovector	$-\frac{i}{2}\Lambda\mathbf{e}_3\Lambda^{\dagger}=is$	$\frac{1}{2}\bar{\psi}\gamma^5\gamma^{\mu}\psi=2s^{\mu}$	Spin dual
Pseudoscalar	$\Lambda\bar{\Lambda}=\bar{\Lambda}^{\dagger}\Lambda^{\dagger}=0$	$\bar{\psi}\gamma^5\psi=0$	

[a] $\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma^{\mu\nu}=\frac{i}{2}(\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu})$.

[b] The six-vector $\mathbf{S}$ is related to the antisymmetric spin tensor $S^{\mu\nu}$ by $\mathbf{S}=[S^{k0}-(i/2)\epsilon_{ijk}S^{ij}]\boldsymbol{\sigma}_k$ in the same way that the electromagnetic field $\mathbf{F}$ is related to $F^{\mu\nu}$ (Ref. [10]).

four-current density of quantum theory. As in the quantum Dirac theory, Eqs. (29) and (30) have the same form if we use different linear combinations of the two-component spinors $\xi,\bar{\eta}^{\dagger}$; only the representations of the γ matrices differ.

The correspondence of all the bilinear forms in Table I has been established by expanding the covariants in terms of the Weyl spinors $\bar{\eta}^{\dagger},\xi$. In the case of spin, the correspondence exists if the six-vector $\mathbf{S}$ is identified in terms of the Frenet vectors as $\mathbf{S}=\mathbf{u}_1\bar{\mathbf{u}}_2/2=-i\mathbf{u}_3\bar{\mathbf{u}}_0/2$. The invariant rest-frame vector $\mathbf{s}_0=-i\mathbf{S}_{\text{rest}}$ is therefore $-\mathbf{e}_3/2$. Since solutions to the classical Dirac equation (22) can include an arbitrary initial rotation, this choice of orientation is permissible; the fact that it yields a four-spinor form identical to the usual quantum one is related to the projectors $(1\pm\mathbf{e}_3)/2$, which isolate the two columns of Λ (see below in this section).

The time dependence of $\mathbf{S}$ is determined by the same equation that governs the evolution of any Frenet six-vector (28):

$$\dot{\mathbf{S}}=\dot{\Lambda}\bar{\Lambda}\mathbf{S}+\mathbf{S}\Lambda\dot{\bar{\Lambda}}=G\mathbf{S}+\mathbf{S}\bar{G}=G\mathbf{S}-\mathbf{S}G\ . \tag{31}$$

The pseudo-four-vector $is=\mathbf{S}u=i\Lambda\mathbf{s}_0\Lambda^{\dagger}$ is i times the dual of the spin tensor $S^{\mu\nu}$: $s^{\mu}=\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}S_{\nu\alpha}u_{\beta}$ where $\epsilon^{\mu\nu\alpha\beta}$ is the fourth-rank antisymmetric tensor with $\epsilon^{0123}=1$. It is a linear combination of the spacelike Frenet four-vectors and obeys their equation of motion (14):

$$\dot{s}=\dot{\Lambda}\mathbf{s}_0\Lambda^{\dagger}+\Lambda\mathbf{s}_0\dot{\Lambda}^{\dagger}=Gs+sG^{\dagger}\ . \tag{32}$$

With $G=(e/2m)\mathbf{F}(\tau)-2m\mathbf{S}$ as in (15) above, and with $\mathbf{S}=i\Lambda\mathbf{s}_0\bar{\Lambda}$ where $\mathbf{s}_0$ is constant, the equations of motion (31) and (32) for $\mathbf{S}$ and s are fully equivalent to the Bargmann, Michel, and Telegdi (BMT) equation [15] for a spin with g factor $g=2$ in an electromagnetic field $\mathbf{F}$. The value $g=2$ is a consequence of assuming a *linear* time development (10) for Λ. A more general, nonlinear time-development equation can accommodate an arbitrary g factor: if one puts $4mx=4m\bar{x}=\sqrt{e(2-g)}$ in (16) and uses this expression for the six-vector $\mathbf{S}$, the BMT equation is obtained for an arbitrary g factor, but as discussed in Sec. III, this choice leads to a nonlinear equation for the time evolution of the eigenspinor and is therefore unsatisfactory for an elementary particle which is not an isolated point.

In essence, the definition of an elementary particle as one whose motion is described by a single eigenspinor forces the orbital motion and spin precession of such a particle to proceed at the same rate in a magnetic field. [The result is not automatic since orbital motion can be caused by boosts as well as by rotations, but it does follow from the linearity of the time evolution (10) of Λ.] The ensuing equality of the Larmor and cyclotron frequencies—as in numerous earlier (but contested) claims [16]—makes $g=2$. The concept of an elementary particle is a key bridge from purely classical theories, where there are no elementary particles, where the spin and orbital motions are independent degrees of freedom, and where therefore the g factor is arbitrary, to quantum theories of elementary fermions where the spin is an inseparable property of the particle and where $g=2$ to within small QED corrections. (It should perhaps be added that the proof given here than at classical elementary particle must have a g factor of 2 is not dependent on the use of the Pauli algebra: the proof may also be given in terms of traditional Minkowski-space matrices.)

Equivalent expressions for the basic symmetry transformations **P**, **T**, and **C** can also be identified for the two forms of the eigenspinor. As in the case of the bilinear covariants, the four-spinors forms are identical to the usual quantum ones, but the corresponding forms for the eigenspinor Λ are relatively simpler and have transparent geometrical interpretations (see Table II). Note especially the **CPT** transformation, which simply multiplies Λ by i, and the **PT** transformation, which rotates the initial rest frame by π about $-\mathbf{e}_1$. It is the **CPT** transformation which relates the eigenspinor Λ for negative-energy particles to that for the corresponding antiparticles. The information in Table II can be used to determine how the bilinear covariants transform under the various symmetry operations. For example, the four-velocity is seen to be invariant under **C**, **PT**, and **CPT**, but to transform as $u\to\bar{u}$ under **P** or **T** whereas the spin dual s changes sign under either **C** or **PT**. Note that under **C** or **CPT**, unimodular eigenspinors become antiunimodular and vice versa. Furthermore, $C^2=P^2=1$, but $T^2=(CPT)^2=-1$.

Although all sixteen linearly independent bilinear co-

TABLE II. Symmetry transformations of the classical eigenspinors Λ and ψ.

Transformation	$\mathcal{P}$-algebra form	Four-spinor form[a]	Interpretation
P	$\Lambda \to \bar{\Lambda}^\dagger$	$\psi \to \gamma_0 \psi$	Spatial inversion
T	$\Lambda \to \bar{\Lambda}^\dagger e^{i\pi \mathbf{e}_1/2}$	$\psi \to -\Sigma_2 \psi^*$	Time reversal
PT	$\Lambda \to \Lambda e^{i\pi \mathbf{e}_1/2}$	$\psi \to -\gamma_0 \Sigma_2 \psi^*$	Initial rotation by π
C	$\Lambda \to \Lambda \mathbf{e}_1$	$\psi \to -i\gamma^2 \psi^*$	Initial reflection
CPT	$\Lambda \to i\Lambda$	$\psi \to i\gamma^5 \psi$	Antiparticle spinor

[a] $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\Sigma_2 = -i\gamma^1\gamma^3$.

variants of the form $\bar{\psi}\Gamma\psi$, where Γ is a linear combination of products of Dirac gamma matrices, have been given in Table I together with their equivalent expressions in terms of Λ, there are many other bilinear covariants of Λ, namely the other Frenet four-vectors (12) and six-vectors (28), which *cannot* be so expressed. The Frenet four-vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, for example, cannot be written in the form $\bar{\psi}\Gamma\psi$. This limitation is reasonable if there is at most one preferred direction (namely $\mathbf{e}_3$) in the rest frame of the elementary particle. Bilinear expressions for $\mathbf{u}_1, \mathbf{u}_2$ *can* be found in terms of ψ and $\psi^c = -i\gamma^2\psi^*$ (see Table II). Thus the components of $\mathbf{u}_1$ and $\mathbf{u}_2$ can be seen to be the real and imaginary parts of $\bar{\psi}\gamma^\mu\psi^c$ [17].

To strengthen further the association of classical and quantum spinors, consider the limiting expressions at high and low velocities. At low velocities, the eigenspinor approaches a pure rotation

$$\Lambda \equiv (\eta, \xi) = \mathcal{B}\mathcal{R} \approx \mathcal{R} = \bar{\mathcal{R}}^\dagger \approx \bar{\Lambda}^\dagger \equiv (\bar{\xi}^\dagger, -\bar{\eta}^\dagger) \tag{33}$$

so that $\bar{\eta}^\dagger \approx \xi$, $\bar{\xi}^\dagger \approx -\eta$, and the sum $\xi + \bar{\eta}^\dagger$ (corresponding to the "large" component of the Dirac tetrad in the Dirac-Pauli representation) is much larger than the difference $\bar{\eta}^\dagger - \xi$ (corresponding to the "small" component). Negative-energy eigenspinors, however, satisfy $\Lambda \approx -\bar{\Lambda}^\dagger$, which means that the roles of the small and large components are interchanged.

In the ultrarelativistic limit, when the energy $p_0 = E$ and momentum $\mathbf{p}$ become equal in magnitude: $|E| \approx |\mathbf{p}| \gg m$, the relative size of the two-component spinors depends on the helicity, and hence on the dot product of the unit vectors $\hat{\mathbf{p}} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}} = -\mathcal{R}\mathbf{e}_3\mathcal{R}^\dagger$ is the rotated spin direction. When $\hat{\mathbf{n}}$ is aligned along $\mathbf{p}$, one of the two classical two-component spinors ξ, η vanishes in the limit of high velocity. Thus for positive-energy particles or antiparticles, since the left and right columns of $\Lambda = \mathcal{B}\mathcal{R}$ are $(\eta, 0) = \Lambda(1+\mathbf{e}_3)/2$ and $(0, \xi) = \Lambda(1-\mathbf{e}_3)/2$, the ratio of their square magnitudes

$$\frac{\eta^\dagger\eta}{\xi^\dagger\xi} = \frac{[\mathcal{B}\mathcal{R}(1+\mathbf{e}_3)\mathcal{R}^\dagger\mathcal{B}]\cdot m}{[\mathcal{B}\mathcal{R}(1-\mathbf{e}_3)\mathcal{R}^\dagger\mathcal{B}]\cdot m} = \frac{p\cdot(1-\hat{\mathbf{n}})}{p\cdot(1+\hat{\mathbf{n}})} \approx \frac{1-\hat{\mathbf{p}}\cdot\hat{\mathbf{n}}}{1+\hat{\mathbf{p}}\cdot\hat{\mathbf{n}}} \tag{34}$$

becomes large if the helicity is negative ($\hat{\mathbf{n}} = -\hat{\mathbf{p}}$) and vanishes if it is positive ($\hat{\mathbf{n}} = \hat{\mathbf{p}}$). For massless fermions (neutrinos), these limits are exact, and one of η and ξ vanishes. The solutions representing the two helicities are then related by the interchange $\eta \leftrightarrow \xi$, which is exactly the transformation of charge conjugation (see Table II).

VI. QUANTUM AND CLASSICAL AMPLITUDES

A close association has been established between the Dirac four-spinor of relativistic quantum theory and the classical eigenspinor Λ. Since Λ gives both the velocity and the orientation of the particle, it is reasonable that it should be closely related to the wave function. Indeed, Λ satisfies a classical Dirac equation (17) whose form, when expressed in terms of a four-spinor ψ (24), is exactly the same as in the quantum theory; its time evolution follows the linear equations (9) and (10); and its bilinear covariants and symmetry transformations have the same expressions in terms of the two-spinors η, ξ as in quantum theory. Of course, there are still major differences between the classical theory discussed here and the first-quantized Dirac theory: the eigenspinor Λ has so far been expressed as a function of the proper time of the particle, whereas the quantum Dirac four-spinor is a function of local space-time and serves not only as a Lorentz transformation but also as a probability amplitude. Furthermore, the classical Dirac equation, as emphasized in Sec. IV, is almost trivial: the equation simply expresses the relation between the four-momentum and the four-velocity. The physical content of the quantum Dirac equation evidently lies largely in the operator form of the four-momentum. In this section, the prospects are investigated for pushing the classical theory still closer toward the quantum interpretation.

As defined in Sec. III, the classical eigenspinor is the Lorentz transformation of the particle from its rest frame to the observer's frame. It does not give directly the world line of the particle but rather its four-velocity and hence the tangent vector to the world line. Of course, if the initial position of the particle is known, the four-velocity can be integrated to calculate the world line itself, but by demanding that Eqs. (9) and (10) of time evolution be linear, we have avoided the need for confining the particle to a point. An elementary particle is structureless since it requires only one Lorentz transformation (its eigenspinor) to describe its motion, but it may be that its position is best represented by a distribution, say by a density ρ_0 in its rest frame. The Lorentz transformation

$$j = \Lambda\rho_0\Lambda^\dagger = \rho_0 u \tag{35}$$

of ρ_0 then gives the corresponding current density in the observer's frame. Comparing (35) to the bilinear convariants of Sec. V and to the usual quantum expression $j^\mu = \bar{\psi}\gamma^\mu\psi$ for the (probability) current density suggests that it is $(\rho_0/2)^{1/2}\Lambda$ which corresponds, within an arbitrary initial rotation, to the quantum Dirac spinor as usually normalized.

Since the basic equations (9), (10), and (17) obeyed by the classical eigenspinor Λ are linear in Λ and $\bar{\Lambda}^\dagger$, they are also satisfied by any real linear combination $\sum_n \alpha_n \Lambda_n$ of such eigenspinors, where the α_n are real scalars. Quantum theory allows *complex* linear combinations of wave functions, and multiplication by a complex phase does not change the state represented by the wave function. However, it is easy to verify that multiplication of the four-spinor ψ (24) by a phase factor $\exp(i\varphi)$ is equivalent to a rotation of the particle rest frame by 2φ about $\mathbf{e}_3$:

$$\psi \to e^{i\varphi}\psi = e^{i\varphi}\begin{bmatrix}\bar{\eta}^\dagger \\ \xi\end{bmatrix} \Longleftrightarrow \Lambda=(\eta,\xi)\to(e^{-i\varphi}\eta, e^{i\varphi}\xi) = \Lambda e^{-i\varphi\mathbf{e}_3} \ . \quad (36)$$

Therefore any complex linear combination of four-spinors ψ corresponds to a real linear combination of eigenspinors Λ.

One must still question whether such linear combinations are classically meaningful. In many cases, the physical significance of a linear combination is easy to discern. Consider, for example, two eigenspinors with the same boost component but with four-orthogonal rotations:

$$\Lambda_j = \mathcal{B}\mathcal{R}_j \ , \quad j=1,2$$
$$\bar{\mathcal{R}}_1 \cdot \mathcal{R}_2 = 0 \ . \quad (37)$$

Because their rotational parts are four-orthogonal, so are the eigenspinors themselves: $\bar{\Lambda}_1 \cdot \Lambda_2 = \bar{\mathcal{R}}_1 \cdot \mathcal{R}_2 = 0$. The eigenspinors are related by a 180° rotation:

$$\Lambda_2 = \Lambda_1 \mathcal{R}_{12} \ , \quad (38)$$

where the scalar part of the rotation

$$\mathcal{R}_{12} \equiv \bar{\mathcal{R}}_1 \mathcal{R}_2 = \exp(-i\boldsymbol{\theta}_{12}/2) \quad (39)$$

is

$$\cos(\theta_{12}/2) = 1 \cdot \mathcal{R}_{12} = \bar{\mathcal{R}}_1 \cdot \mathcal{R}_2 = 0 \ . \quad (40)$$

If the axis $\mathbf{n} \equiv \hat{\boldsymbol{\theta}}_{12}$ is aligned with $\mathbf{e}_3$, the corresponding four-spinors differ only by a phase factor, but if $\mathbf{n}$ lies in the $\mathbf{e}_1\mathbf{e}_2$ plane, the corresponding four-spinors are orthogonal. In either case, $\mathcal{R}_{12} = -i\mathbf{n}$ and the real linear combination

$$\Lambda = \Lambda_1 \cos\alpha/2 + \Lambda_2 \sin\alpha/2 = \Lambda_1(\cos\alpha/2 - i\mathbf{n}\sin\alpha/2) = \Lambda_1 \exp(-i\mathbf{n}\alpha/2) \quad (41)$$

produces an eigenspinor differing from Λ_1 by an initial rotation α about $\mathbf{n}$. The close correspondence of this superposition of eigenspinors with the superposition of quantum spin-$\frac{1}{2}$ states is evident. The correspondence with spin eigenstates is made precise by taking $\mathcal{R}_1 = 1$ ("spin up") and letting the rotation axis $\mathbf{n}$ be perpendicular to $\mathbf{e}_3$.

Normalized linear combinations of eigenspinors with different boost components are not as easy to interpret. Since boosts are always timelike real elements of $\mathcal{P}$, no two pure boosts can be four-orthogonal. Furthermore, a real linear combination of unimodular boosts may be antiunimodular if coefficients of differing sign are allowed. Evidently, linear combinations of boosts can generally be meaningful in the classical theory only if negative-energy particles are included.

If linear superpositions of eigenspinors are accepted into the classical theory, then a classical spinor field $\Psi(x)$ can be defined as such a summation of $(\rho_0/2)^{1/2}\Lambda$ from each contributing path. The simplest case is that of free particles. In Sec. III, it was seen that the eigenspinor of a free particle can spin at a constant rate $4m\mathbf{s}_0$ in its rest frame [see the discussion following (15)]. Classically, the constant spin vector $\mathbf{s}_0$ is undetermined, but we can take its direction to define the rest-frame axis $-\mathbf{e}_3$: $\mathbf{s}_0 = -s_0\mathbf{e}_3$. The four-momentum of the particle and hence [by (17)] its four-velocity is assumed constant. Its eigenspinor thus obeys

$$\Lambda(\tau) = \Lambda(0)\exp(i2ms_0\tau\mathbf{e}_3) \ . \quad (42)$$

In order to superimpose such eigenspinors, the Lorentz-invariant rotation angle $-4ms_0\tau\mathbf{e}_3$ needs to be expressed in terms of the space-time coordinates x. Since for a particle of constant four-momentum p, $m\tau = p\cdot\bar{x}$ where p is the four-momentum of the particle, (42) may be rewritten

$$\Lambda(x) = \Lambda(0)\exp(2is_0 p\cdot\bar{x}\mathbf{e}_3) \ . \quad (42')$$

The space-time dependence of the corresponding classical four-spinor is exactly that of the familiar plane wave of quantum-mechanical momentum eigenstates, but now the complex phase of the plane wave takes on a concrete geometrical meaning: it is twice the rotation angle of the rest frame about $-\mathbf{e}_3$. The scalar constant s_0 is evidently not fixed classically, but its magnitude determines the scale of quantum effects: by putting $s_0 \equiv \frac{1}{2}$ and thereby absorbing it into the units, one effectively sets the size of $\hbar$. In the resulting units, the rotation of the free eigenspinor is seen to be the *Zitterbewegung* frequency [4,5] $2m = 2mc^2/\hbar$.

The above discussion indicates how classical eigenspinors are natural amplitudes which can be superimposed and closely associated with quantum wave functions. Although some details must still worked out, many possibilities for establishing further intimate relationships are clear. Just as any wave function can be expanded in plane waves, so can any spinor field $\Lambda(x)$ be expanded in free eigenspinors of different momenta. Local gauge invariance in quantum theory corresponds to the invariance under a position-dependent rotation of the eigenspinor about the rest-frame $-\mathbf{e}_3$ axis. It can be ensured by the introduction of a *gauge field* $A(x)$ into the momentum term:

$$p\cdot\bar{x} \to \int (p+eA)\cdot d\bar{x} \quad (43)$$

so that a rotation by the angle $2\chi(x)$ about $\mathbf{e}_3$ is equivalent to the gauge transformation $eA \to eA + \partial\chi(x)$ of the vector potential A.

In terms of the four-spinors, the amplitudes being summed all have the form

$$\psi(x_2)=\psi(x_1)\exp\left[-i\int_1^2 d\bar{x}\cdot(p+eA)\right] . \tag{44}$$

The superposition required is a path integral of the sort introduced by Feynman [18] over four-spinors (44). Equation (44) and its equivalent differential form

$$i\partial\psi(x)=(p+eA)\psi(x) \tag{45}$$

is one relation which all contributing amplitudes obey, and since it is linear, all linear superpositions of eigenspinors including the Dirac wave function must also obey it. Relation (45) gives the full first-quantized quantum Dirac equation when combined with the classical Dirac equation (25) and the identification of $j^\mu=\bar{\psi}\gamma^\mu\psi$ as the probability current density.

Classical results are derived by the standard stationary-phase arguments: the action

$$-\int_1^2 d\bar{x}\cdot(p+eA)=-\int_1^2 d\tau\,\bar{u}\cdot(p+eA)=\int_1^2 dt\,\mathcal{L} , \tag{46}$$

of macroscopic objects is large enough that the rapidly oscillating phase in the linear superposition of four-spinors (44) cancels all contributions except for paths where the phase is stationary. Since $\mathcal{L}=-(m+e\bar{u}\cdot A)/\gamma$ is the Lagrangian for a charge e interacting with an external potential A, the stationary phase condition is equivalent to Hamilton's principle of least action [19].

The half-integer spin of solutions to the Dirac equation follow as usual from the behavior of $\psi(x)$ under active rotations: since the spinor rotation operator $\mathcal{R}=\exp(-i\theta/2)$ is unitary, both $\bar{\eta}^\dagger$ and ξ transform in the same way, and the four-spinor therefore transforms according to

$$\psi\to\begin{bmatrix}\mathcal{R} & 0\\ 0 & \mathcal{R}\end{bmatrix}\psi=\exp(-i\boldsymbol{\Sigma}\cdot\boldsymbol{\theta}/2)\psi , \tag{47}$$

where $\boldsymbol{\theta}\cdot\boldsymbol{\Sigma}$ is short for $\theta^k\Sigma_k$ (summed over $k=1,2,3$) with

$$\Sigma_k=\begin{bmatrix}\underline{\sigma}_k & 0\\ 0 & \underline{\sigma}_k\end{bmatrix} . \tag{48}$$

The total rotation of $\psi(x)$ involves not only the rotational mixing of the components, but also the backward rotation of the argument x [20]. Of course it is the orbital angular momentum $\mathbf{L}=-i\mathbf{r}\times\nabla$ which generates the transformation of the argument:

$$\psi(x-d\boldsymbol{\theta}\times\mathbf{x})=(1-i\mathbf{L}\cdot d\boldsymbol{\theta})\psi(x) \tag{49}$$

and the total generator of rotations in ψ is $\mathbf{L}+\boldsymbol{\Sigma}/2$, where $\boldsymbol{\Sigma}/2$ is seen by its commutation relations and its square $\boldsymbol{\Sigma}^2\psi=3\psi$ to represent an angular momentum of $\frac{1}{2}$ in units of $\hbar$.

Boost transformations can be written down immediately from the transformation law (8) $\Lambda\to\mathcal{B}(\mathbf{w})\Lambda$ for eigenspinors and the relations (18) and (24) between Λ and ψ:

$$\psi\to\begin{bmatrix}\bar{\mathcal{B}}\bar{\eta}^\dagger\\ \mathcal{B}\xi\end{bmatrix}=\begin{bmatrix}e^{-\mathbf{w}/2} & 0\\ 0 & e^{\mathbf{w}/2}\end{bmatrix}\psi=e^{\mathbf{w}\cdot\boldsymbol{\alpha}/2}\psi , \tag{50}$$

where in the Weyl representation

$$\boldsymbol{\alpha}=\gamma_0\boldsymbol{\gamma}=\begin{bmatrix}-\underline{\sigma} & 0\\ 0 & \underline{\sigma}\end{bmatrix} . \tag{51}$$

Now in the rest frame of the particle, Λ is simply a rotational spinor and obeys $\Lambda=\pm\bar{\Lambda}^\dagger$, where the two signs are for positive- and negative-energy particles (see Sec. V), and therefore the two-component spinors are simply related [see (33)]: $\xi=\pm\bar{\eta}^\dagger$. A boost along the spin direction $\hat{\mathbf{s}}=\Lambda\hat{\mathbf{s}}_0\Lambda^\dagger/(\Lambda\bar{\Lambda})$ (this is a three-vector when Λ is a rotational spinor) then gives, for positive-energy particles or antiparticles,

$$\begin{aligned}\Lambda\to\Lambda'&\equiv(\eta',\xi')=\mathcal{B}(w\hat{\mathbf{s}})\Lambda=\Lambda\mathcal{B}(w\hat{\mathbf{s}}_0)\\ &=(\eta,\xi)e^{-w\mathbf{e}_3/2}=(e^{-w/2}\eta,e^{w/2}\xi) .\end{aligned} \tag{52}$$

If the magnitude w of the boost parameter is large, η' becomes vanishing small compared to ξ'. On the other hand, boosts along $-\hat{\mathbf{s}}$ make η' dominant. This is, of course, consistent with (34). The sign of w is reversed in the last three equalities of (52) for negative-energy solutions.

The quantum Dirac equation can also be expressed in the Pauli algebra in terms of the eigenspinor $\Lambda(x)$, but then the momentum operator takes a different form from that in (45):

$$\overline{(p+eA)}\Lambda=-i\bar{\partial}\Lambda\mathbf{e}_3=2\bar{\partial}\mathbf{S}\Lambda , \tag{53}$$

where $\mathbf{S}=-(i/2)\Lambda\mathbf{e}_3\bar{\Lambda}$ is the six-vector spin (see Sec. V and Table I), so that the Dirac equation becomes

$$\bar{p}\Lambda=(2\bar{\partial}\mathbf{S}-e\bar{A})\Lambda=m\bar{\Lambda}^\dagger . \tag{54}$$

The imaginary i in the usual operator relation has thus been replaced by twice the spin six-vector $\mathbf{S}$. Note that $\mathbf{S}^2=\mathrm{s}\bar{\mathrm{s}}=-\frac{1}{4}$.

VII. CONCLUSIONS

In this paper, an algebraic spinor approach has been presented for studying to correspondence between relativistic classical theory and quantum theory. A close relationship has been established between the Dirac wave function and the classical eigenspinor, which is the Lorentz transformation from the instantaneous rest frame of the particle to the observer's frame. The classical Dirac equation has been shown here to be a trivial kinematic identify for the eigenspinor, and transformations of particle properties such as the four-velocity and spin from the rest frame to the laboratory frame give bilinear covariants fully analogous to those of the Dirac theory. The symmetry transformations of **P**, **T**, and **C** of the classical eigenspinor also have exactly the same form as in the Dirac theory. The classical equations of motion for the eigenspinor allow an arbitrary rest-frame spin, and for elementary particles which are not isolated points, they are consistent only with a g factor of 2.

The close relationship between the classical eigenspinor and quantum wave functions led us to consider superpositions of classical eigenspinors describing a spin at the *Zitterbewegung* frequency. The four-spinor forms of such free-particle eigenspinors are plane waves whose superposition can lead to interference and other quantum-like phenomena. The results support some of Hestenes's work [4] relating *Zitterbewegung* to the complex phase of the Dirac and Schrödinger wave functions, but the approach here seems more general. Unlike the quantum formulation of Hestenes or the classical models of Gürsey [2], Barut [6], and others, the present work does not require helical lightlike trajectories or a separation of mass and charge centers. Indeed, although the elementary particle is not necessarily a point particle, causality seems to preclude any identifiable internal structure, and the Dirac equation itself is seen as a statement that the four-velocity and four-momentum are parallel. The spin of the eigenspinor provides undulations like those of the "wave simplex," on which Tisza [7] based his non-Newtonian formulation of classical physics. However, whereas Tisza asserted that no simple mathematical theory can handle translation, rotation, and undulation simultaneously, the classical eigenspinor introduced here does in fact unify precisely these three kinematical modes.

Since the classical Dirac equation and other classical constraints are invariant under an arbitrary rotation of the rest frame, it may be questioned whether the "spin" of a free elementary particle corresponds to a physical rotation of some sort. The experimental evidence that charged fermions have magnetic moments and that their spins contribute to the total angular momentum, together with analyses of the rotational motion of quantum solutions to the Dirac equation like those of Huang [5] demonstrate a physical reality of the spinning motion.

Although clearly more work is needed to find new nonperturbative solutions to QED problems, the close correspondence established here in the Pauli-algebra approach between relativistic classical mechanics and the quantum Dirac theory lays the groundwork for future studies and suggests that it would be useful to extend the approach to formulations of such phenomena as radiation reaction, interactions with quantized electromagnetic fields, electroweak interactions, and many-body effects.

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