Realism and the quantum-mechanical two-state oscillator

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We consider how a class of realistic models in which an individual system, while undergoing transitions, is at any instant definitely in one or the other state can mimic the prediction of quantum theory for two-state oscillations. We show explicitly the way in which the measurement must be invasive for such models, even at the microscopic level.

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In recent years, there has been considerable debate [1–6] about an argument given by Leggett and Garge [7,8] claiming to show that the conjunction of two general assumptions of macrorealism and noninvasive measurability leads to an inequality imposing constraints on time-separated joint probabilities pertaining to oscillations in two-state systems. This inequality is shown to be violated by the predictions of quantum mechanics as applied to rf superconducting-quantum-interference-device (SQUID) rings at the macroscopic level. The relevant debate has so far focused on the following issues: (i) whether the assumption of noninvasive measurability (that it is possible, at least in principle, to determine the state of an individual system with an arbitrarily small effect on its subsequent dynamics) is satisfied in the particular case of a measurement of the flux state of an rf SQUID and (ii) the validity of the assumption of noninvasive measurability, in general, at the macroscopic level. In this paper we use a specific class of realist model for the two-state oscillator, thus following Bell's exhortation [9] to "test general reasoning against simple models." We demonstrate that the incompatibility is not between quantum theory and realism, but occurs essentially with the additional assumption of noninvasive measurability. We give an explicit example of how quantum results may be recovered in a straightforward manner for a realist model by violation of noninvasive measurability.

We begin by considering a system, oscillating between two states $|A\rangle$ and $|B\rangle$ which are degenerate eigenstates of the Hamiltonian H_0:

$$H_0|A\rangle=E_0|A\rangle\ , \tag{1a}$$

$$H_0|B\rangle=E_0|B\rangle\ , \tag{1b}$$

and

$$\langle A|H_0|B\rangle=\langle B|H_0|A\rangle=0\ . \tag{1c}$$

Oscillatory transitions between $|A\rangle$ and $|B\rangle$ are induced by H' where

$$\langle A|H'|B\rangle=\langle B|H'|A\rangle=\Delta E \tag{2a}$$

and

$$\langle A|H'|A\rangle=\langle B|H'|B\rangle=E'\ . \tag{2b}$$

If we take, at $t=0$,

$$|\psi(0)\rangle=|A\rangle\ , \tag{3}$$

then from

$$(H_0+H')|\psi(t)\rangle=i\hbar\frac{d}{dt}|\psi(t)\rangle \tag{4}$$

we obtain

$$|\psi(t)\rangle=\exp[-i(E_0+E')t/\hbar]\times[\cos(\Delta Et/\hbar)|A\rangle-i\sin(\Delta Et/\hbar)|B\rangle]\ , \tag{5}$$

hence the following:

$$|\langle A|\psi(t)\rangle|^2=\tfrac{1}{2}(1-\cos\omega t)\ , \tag{6a}$$

$$|\langle B|\psi(t)\rangle|^2=\tfrac{1}{2}(1-\cos\omega t)\ , \tag{6b}$$

where $\omega=2\Delta E/\hbar$.

Equations (6a) and (6b) predict experimentally verifiable probabilities for finding the system in one or the other of the two states $|A\rangle$ and $|B\rangle$ at a given instant t. A key feature inherent in the quantum-mechanical calculation of these probabilities is that before any measurement the system is at any instant in a state that is neither $|A\rangle$ nor $|B\rangle$ but a superposition of these two states with time-dependent coefficients. Here the crux of the issue is: Are the quantum-mechanical results compatible with the "realist" assumption that an individual system, while undergoing oscillatory transitions between $|A\rangle$ and $|B\rangle$, is at all instants either in $|A\rangle$ or $|B\rangle$, independent of the act of measurement? Compatibility would require that the integrated behavior of an ensemble of such identical systems should yield the results (6a) and (6b).

We begin by noting that if $y_A(t)$ and $y_B(t)$ denote probabilities for finding the system in $|A\rangle$ and $|B\rangle$, respectively, at an instant t, then the realist description would imply the following rate equations for describing the oscillations of the system between $|A\rangle$ and $|B\rangle$:

$$y_A(t+dt)=y_a(t)(1-W_1dt)+y_B(t)W_2dt\ , \tag{7a}$$

$$y_B(t+dt)=y_B(t)(1-W_2dt)+y_A(t)W_1dt\ , \tag{7b}$$

where W_1 is the rate of transition (transition probability per unit time) from $|A\rangle$ to $|B\rangle$ and W_2 is the rate from $|B\rangle$ to $|A\rangle$. Note that these transition rates are defined essentially in the realist description and have no counterpart in the quantum-mechanical treatment; the underlying assumption is that there exist elements of reality that determine the values of these transition rates pertaining to an individual system. From Eqs. (7a) and (7b) we obtain the differential equations

$$\frac{dy_A}{dt}=-W_1y_A+W_2y_B\ , \tag{8a}$$

$$\frac{dy_B}{dt}=-W_2y_B+W_1y_A\ . \tag{8b}$$

The question of compatibility between quantum mechanics and realism in the context of the two-state oscillator, therefore, reduces to the following one: Consistent with the quantum-mechanical expressions for $y_A(t)$ and $y_B(t)$ given by Eqs. (6a) and (6b), respectively, is it possible to have physically reasonable W_1 and W_2 that satisfy Eqs. (8a) and (8b)?

A straightforward way to examine the above question is to write down the solutions of Eqs. (8a) and (8b) for y_A and y_B in terms of W_1 and W_2, subject to the initial conditions

$$y_a(t=0)=1,\quad y_B(t=0)=0\ . \tag{9}$$

If, for the moment, we assume that W_1 and W_2 are constants, this solution is

$$y_a=\{W_2+W_1\exp[-(W_1+W_2)t]\}/(W_1+W_2)\ , \tag{10a}$$

$$y_B=W_1\{1-\exp[-(W_1+W_2)t]\}/(W_1+W_2)\ . \tag{10b}$$

It is then obvious that there are no real values of W_1 and W_2 that make Eqs. (10) compatible with Eqs. (6). In passing, we note that consistency is possible if W_1 and W_2 are complex and if we regard the real parts of the expressions in (10) as the physical quantities. However, we reject this "solution" as being incompatible with the realist conception, because W_1 and W_2, even though not directly observable, have the same ontological status in this realist model as actually observable quantities (we could not make a classical system with this behavior).

We are therefore forced to drop the restriction that the W's are independent of time. Then we ask whether it is possible to find suitable expressions for W_1 and W_2 such that the solutions of Eqs. (8) agree with the right-hand sides of Eqs. (6). Thus we put

$$y_A(t)=|\langle A|\psi\rangle|^2=\tfrac{1}{2}(1+\cos\omega t) \tag{11a}$$

and

$$y_B(t)=|\langle B|\psi\rangle|^2=\tfrac{1}{2}(1-\cos\omega t)\ . \tag{11b}$$

Differentiating with respect to t gives

$$\dot{y}_A=-\tfrac{1}{2}\omega\sin\omega t \tag{12a}$$

and

$$\dot{y}_B=\tfrac{1}{2}\omega\sin\omega t\ . \tag{12b}$$

Inserting (12) into (8), we find that W_1 and W_2 must satisfy the following relation:

$$\omega\sin\omega t=W_1(1+\cos\omega t)-W_2(1-\cos\omega t)\ . \tag{13}$$

(There is only one equation here because both models ensure particle conservation.) We are interested in solutions of (13) for which both W_1 and W_2 are non-negative. Possible forms are

$$W_1=\omega\frac{\lambda+\sin\omega t}{1+\cos\omega t} \tag{14a}$$

and

$$W_2=\omega\frac{\lambda}{1-\cos\omega t} \tag{14b}$$

for any positive λ satisfying $\lambda+\sin\omega t>0$. For example, with $\lambda=1-\frac{1}{2}\sin\omega t$, we have

$$W_1=\omega\frac{1+\frac{1}{2}\sin\omega t}{1+\cos\omega t} \tag{15a}$$

and

$$W_2=\omega\frac{1-\frac{1}{2}\sin\omega t}{1-\cos\omega t}\ . \tag{15b}$$

With such time-dependent expressions for W_1 and W_2, we have a realistic model which agrees at all times with the predictions of quantum theory for the distribution of states A and B. What this means is that we could construct a large number of mechanical two-state devices such that the proportion in, say, state A at any given time is given by Eq. (6a), apart, of course, from statistical fluctuations. An example would be to use a ball bouncing about in a closed container with two parts (A and B) separated by a membrane with a hole in it. The time dependence of the transition rates could be modeled by having the sizes of the containers vary with time. That this is possible should not surprise us; it is simply the two-state analog of the hidden-variable model of de Broglie–Bohm. The fact that, here, the model is stochastic (it contains transition *probabilities*) rather than deterministic, as in de Broglie–Bohm, reflects the fact that in a two-state system only one "trajectory" is possible. Of course, the classical construction referred to above has an underlying origin for the stochasticity, namely, the random trajectories of the ball.

That the W_i's have to depend on time is also not surprising in view of the de Broglie–Bohm model. It should really be thought of as a dependence on the wave function, which also depends on time, and merely reflects the dependence of the quantum potential of the de Broglie–Bohm model on the wave function. The particular choices of Eqs. (14) and (15) make W_1 and W_2 infinite at particular times. It may be shown that this is a *general* requirement of Eq. (13) if one insists on W_1 and W_2 being continuous as well as non-negative. However, this does not have unphysical consequences since the relevant products in Eq. (8) remain finite.

It is this time-dependent feature of the macrorealistic model that causes some problems in trying to reconcile it with all predictions of quantum theory. As we have seen, by construction, there is no problem when we look at the probability of a single result. However, if we ask for the probability of obtaining, say, A at time t, given that we have measured A at time $t_1<t$, then at first sight the model seems to give a different result to quantum theory. It is this fact that is exploited in the Einstein-Podolsky-Roson-like proof of Leggett and Garg [7,8].

In orthodox quantum theory, according to the standard method of calculating joint probabilities, we start again with the state $|A\rangle$ at $t=t_1$, and therefore obtain

$$y_A=\tfrac{1}{2}[1+\cos\omega(t-t_1)] , \tag{16a}$$

$$y_B=\tfrac{1}{2}[1-\cos\omega(t-t_1)] \tag{16b}$$

[cf. Eq. (11)].

In our macrorealistic model the natural procedure is to solve Eqs. (8), with the same W_1 and W_2 as before, but with $y_A=1$ at $t=t_1$ (not at $t=0$). To see what this implies, we replace Eqs. (8) by

$$\dot{y}_A=(W_1+W_2)y_A+W_2 , \tag{17}$$

where we have used $y_A+y_B=1$. If we substitute

$$y_A=\tfrac{1}{2}(1+\cos\omega t)+F \tag{18}$$

in this equation, and use (13), we obtain

$$\dot{F}=-(W_1+W_2)F . \tag{19}$$

The general solution of (17) therefore has the form

$$y_A=\tfrac{1}{2}(1+\cos\omega t)+C\exp\left[-\int_{t_1}^{t}(W_1+W_2)dt\right] \tag{20}$$

for any C.

Given the above condition at $t=t_1$, we now have

$$y_A=\tfrac{1}{2}(1+\cos\omega t)+\tfrac{1}{2}(1-\cos\omega t_1)\exp\left[-\int_{t_1}^{t}(W_1+W_2)dt\right]. \tag{21}$$

The results here depend on the precise choice of W_1 and W_2. However, it is easy to see that *no* choice allows (21) to agree with (16a). To see this, we simply put the two expressions equal, which give

$$\cos\omega(t-t_1)\cos\omega t =(1-\cos\omega t_1)\exp\left[-\int_{t_1}^{t}(W_1+W_2)dt\right] . \tag{22}$$

Since, for some values of t, the left-hand side of (20) is negative, the equation clearly cannot be satisfied for all t (unless, of course, $t_1=2\pi n/\omega$ for integral n).

An alternative way of seeing that our macrorealistic model cannot reproduce the predictions of quantum mechanics is the following. First assume, as is required if the model is to reproduce the predictions of quantum mechanics, that both (11) and (16) are solutions of the rate equations (8). Putting (11) in (8) gives (13) and putting (16) in (8) gives

$$\omega\sin\omega(t-t_1)=W_1[1+\cos\omega(t-t_1)] -W_2[1-\cos\omega(t-t_1)] . \tag{23}$$

Now, (13) and (23) are simultaneous equations which can be solved for W_1+W_2 to give

$$W_1+W_2=\omega\cot\frac{\omega}{2}(2t-t_1) . \tag{24}$$

This solution is unsatisfactory since it takes negative values for some values of t, whereas only positive values of W_1 and W_2 are physically reasonable.

We have now established, with our simple model, the alleged discrepancy between macrorealism and quantum theory. It is here, however, that we have to introduce the idea of noninvasive measurability. The result at this stage refers to abstract mathematical quantities, so we need to ask whether they have significance for actual experiments. This means we have to "look" at our system to see whether it is in state A or state B, i.e., introduce some measurement procedure. To this end, we enlarge the system by supposing that, at any given time, we can send a "photon" through the oscillator. We also suppose that this photon will not interact with $|A\rangle$ but is, say, deflected through 90° by $|B\rangle$ (see Fig. 1). Then the linear combination $\alpha|A\rangle+\beta|B\rangle$ leads to the state

$$|\underline{\psi}\rangle=\alpha|A\rangle|\downarrow\rangle+\beta|B\rangle|\rightarrow\rangle \tag{25}$$

in an obvious notation.

The properties of the oscillator, according to orthodox quantum theory, are now obtained by considering the *mixture* of $|A\rangle$ with probability $|\alpha|^2$ and $|B\rangle$ with probability $|\beta|^2$. This is because, in a suitable measurement situation, the two photon states are orthogonal, and hence there is no interference between the two terms in Eq. (21). Thus if we have observed the state $|A\rangle$ at time t_1, i.e., we have seen the $|\downarrow\rangle$ photon, we must use only the $|A\rangle$ state at $t>t_1$. Hence we obtain the *effect* of collapse of the wave function, and have the previous result [Eq. (6)].

Note that we are assuming that the photon is a small perturbation on the "dynamics" of the quantum oscillator. In principle, this is always possible—certainly so if the oscillator is a macroscopic system. In *this sense,* the measurement may be said to be noninvasive.

We now consider the macrorealistic model. Again it may be natural to assume that the photon has little effect on the system, so we would still obtain the contradiction noted above. This would be the noninvasive case. However, we do not *have* to do this. We could instead design the model so that, if the photon at time t_1 leaves at P (see Fig. 1), it automatically resets the timing mechanism to

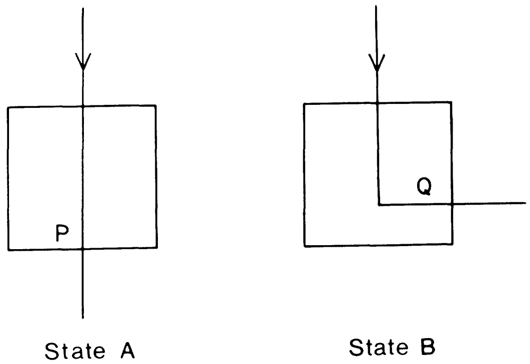

FIG. 1. How a photon is used to measure whether the system is in state A or state B.

start at time t_1. With such a mechanism, then the realistic model will again agree completely with the results of quantum theory, as is evident from Eq. (22). Similarly, if the photon leaves at Q, it resets the timing mechanism to start at time (π/ω). With such a mechanism, then the realistic model will again agree completely with the results of quantum theory.

We conclude then that we could actually *construct* an object, working on strictly realistic, classical, principles which would mimic the results of quantum theory. Clearly, the model does not have noninvasive measurability. Although the small exchange of energy with the photon might have a negligible effect on the basic system (i.e., on the ball in the container), it has an important effect on the mechanism that is responsible for the timing and that determines the values of W_1 and W_2. This is how the model gets around an interesting objection (Leggett and Garg [8]). They note that if the photons come sufficiently frequently, then they *forbid* the transition (this is the watched-pot or zero effect [10]), in which case there is no exchange of energy since the photon only interacts with $|A\rangle$. How, ask Leggett and Garg [8], in a realistic model can something that never interacts have any effect at all? The answer here is that, though forbidding the transition implies that y_A and y_B are not allowed to change, this type of apparently noninvasive measurement can still have an effect on other properties at the realist level, namely, the values of W_1 and W_2.

Whether we can regard a model of this type as a "reasonable" realistic model is, of course, a separate question. It appears more reasonable in the context of the de Broglie–Bohm model, where it happens automatically. This is because the quantum potential depends not only on the system but on the combined wave function of the system and the measuring apparatus, in this case the position of the photon. If this is in the $|\downarrow\rangle$ state (as associated with the $|A\rangle$ state), then the only relevant part of the wave function of Eq. (21) is $|A\rangle|\downarrow\rangle$. It is this that is used to calculate the quantum potential acting on the oscillator in the $|A\rangle$ state. It is, therefore, clear that the modification of the quantum potential accounts for the effects due to the apparently noninvasive measurement.

We conclude that the experimental corroboration of the quantum-mechanical predictions for two-state oscillations pertaining to macroscropic systems (such as an rf SQUID ring) will not refute macrorealism *per se,* even though such results would have an important bearing on the quantum measurement problem as emphasized by Leggett [11].

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