Operational approach to the phase of a quantum field

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We examine the problem of determining the phase difference between two optical fields, first for classical and later for quantum fields, by reference to two simple measurement schemes that yield the sine and/or cosine of the phase difference between classical fields. We show that certain difficulties exist even within the framework of semiclassical radiation theory when the field is very weak, and particularly when amplitude and phase fluctuations are correlated. We find that a clear distinction has to be made between the measured values of the sine or cosine and the values that can be inferred from a series of repeated measurements. A corresponding distinction can be made also for a quantum field, although the interpretation is not the same. The dynamical variables chosen to represent the cosine and sine that emerge from the discussion of the measurement schemes commute when the sine and cosine are obtained together, but not when the measurement yields one or the other. These sine and cosine operators have well-defined values only when there is a large dispersion of the photon number. We arrive at expressions for the moments of the measured and of the inferred sines and cosines that differ from most previous treatments. The expressions are applied to optical fields in several different quantum states. Only for the Fock state and for the so-called phase state, which was treated recently at some length by Pegg and Barnett [Phys. Rev. A **39**, 1665 (1989)], do the measured and the inferred moments coincide. Our analysis of the problem of phase measurement leads to the conclusion that the appropriate dynamical variables for the measured sine and cosine depend on the measurement scheme, and that different schemes correspond to different operators.

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I. INTRODUCTION

The problem of identifying the dynamical variable that corresponds to the phase of a quantized electromagnetic field has been debated ever since Dirac's first paper on the quantum theory of radiation [1]. In that paper Dirac concluded that the phase is canonically conjugate to the photon number, so that the phase has a well-defined value only when the photon number is very uncertain. However, as was pointed out by Louisell [2], the variable identified by Dirac as the phase operator is not strictly Hermitian. Later an alternative approach to the problem based on the use of Hermitian operators that are analogous to the sine and cosine was developed [3,4], which led to roughly the same conclusions as Dirac's for a well-defined phase. However, the sine and cosine operators do not commute, so that they do not both describe the same phase angle, and this has been widely regarded as an unsatisfactory feature of these operators.

Numerous attempts were made in the following years to introduce alternative dynamical variables to represent the phase of a quantum field [5–17]. In particular, Barnett and Pegg [11], in the course of a wide ranging discussion and comparison of different phase operators, introduced a so-called "measured phase operator" that is related to certain simple homodyne experiments. Later Pegg and Barnett [13,14] defined a Hermitian phase operator via a limiting procedure, from which the sine and cosine expressed in terms of unitary operators can also be derived. A number of authors have recently compared the properties of squeezed states in the framework of several alternative operator forms [16–21].

There have also been a few experiments dealing with the measurement of phase [22–25]. Some authors have attempted to test the validity of the different phase operators by making comparisons with the experiments [19,21,26–28], but no clear conclusion favoring one or another definition seems to have emerged. Despite the earlier introduction of the measured phase operator [11], most discussions of the phase have generally been more concerned with mathematical questions.

In the following we approach the phase problem in a somewhat different and more pragmatic way, by examining two typical experimental procedures that are known to yield the phase difference between strong classical fields. Although these schemes are not new and have been discussed before [25,29–36], and indeed the closely related topic of optical homodyning has a substantial literature [32–36], nevertheless the emphasis in these treatments has not been on the problem of extracting the phase.

We start by treating the problem of phase measurement in classical optics. We find that it is natural to make a distinction between the direct results of measurements which we refer to as the measured phases, and the results of calculations based on a long series of measurements, from which the correct phase ensemble may be inferred. Although it has often been taken for granted that the problems encountered in determining the phase are peculiar to the quantum domain, certain difficulties related to phase measurement appear even in classical optics. For example, the usual measurement procedure that yields the phase for strong fields gives the wrong answer for weak fields, even when the average is taken over many measurements. Moreover, there seems to be no procedure that is valid for inferring the phase of a very weak field, even in semiclassical radiation theory, when phase and intensity fluctuations are correlated. Some of the difficulties one encounters in the quantum domain are a reflection of similar problems in classical optics.

Because all measurements are related to differences of phase between two fields, we regard the phase difference as a more fundamental quantity than the absolute phase, as was also emphasized by Nieto [26]. We examine two closely related but distinct measurement schemes for determining phase differences between classical fields, that were also discussed by Loudon [29] and Walker [25,30] and show that they correspond to different quantum dynamical variables. The operators we choose to represent the cosine and sine of the phase difference are, in a sense, the primary variables, whereas the phase difference is derived from them. For the one measurement scheme in which either the sine or the cosine is measured, the corresponding sine and cosine operators do not commute, whereas for the other scheme that yields the sine and cosine together, the corresponding operators commute. In both cases the sine and cosine operators, however, commute with the total photon number. Just as in the classical domain, so also in the quantum domain one is led to make a distinction between measured sines and cosines and the ensemble of values that can be inferred from the measurements, and these lead to different expectations. We apply our formalism to several different quantum states, and calculate the means and dispersions of the cosine and sine. We find that the results are consistent with Dirac's conclusion that a well-defined phase requires a large dispersion of the photon number. In the special case of the coherent state, we present graphs for the expectation of the cosine and for the dispersions and show that they differ significantly from those based on the Susskind-Glogower and the Pegg-Barnett operators.

Our discussion of the phase problem leads us to the general conclusion that probably there is no one dynamical variable that universally represents the measured phase of the electromagnetic field, or even the sine or cosine. Rather it appears that the definition of measured phase ought not to be divorced from the measurement process that is used to determine it, and different measurement schemes lead naturally to different operators. This is the main distinction between our approach and most other discussions of the merits of different phase operators, which were generally concerned with other, more mathematical questions, and less with the relation to experiment.

II. PHASE MEASUREMENT OF A CLASSICAL FIELD

Before considering the problem of phase in the quantum domain, let us examine how one would determine phase differences in classical optics, by reference to some simple measurement schemes. The phase difference between two classical light fields is almost invariably determined from some kind of interference experiment, in which two light waves come together. Any phase difference then shows up in the resultant light intensity.

Let us consider the arrangement shown in Fig. 1 in which two incident quasimonochromatic light beams are combined by a symmetric 50%:50% beam splitter oriented at 45° to each beam, and the resultant light intensity emerging from each output port is measured by a photodetector. Let $V_1(t)$ and $V_2(t)$ be complex analytic signals representing the two TE-polarized input waves at two input planes which are equidistant from the beam splitter. Let us first focus on the simplest situation in which $\arg V_2 - \arg V_1 = \phi_2 - \phi_1$ is strictly constant in time. This will be the case if the two waves $V_1(t), V_2(t)$ are actually derived from a single wave with the help of another beam splitter, and in that case the light intensities $I_1(t)=|V_1(t)|^2$, $I_2(t)=|V_2(t)|^2$ are also strictly correlated in time. Let $V_3(t), V_4(t)$ represent the two output waves at the two detectors, which are also equidistant from the beam splitter. Let r,t be the complex-amplitude reflectance and transmittance of the beam splitter from one side and r',t' from the other side, with $|r|=|r'|=|t|=|t'|=1/\sqrt{2}$ and $\arg t + \arg t' - \arg r - \arg r' = \pm\pi$ [37]. We may relate V_3 and V_4 to V_1 and V_2 by writing

$$\begin{aligned} V_3 &= (tV_1 + r'V_2) , \\ V_4 &= (rV_1 + t'V_2) . \end{aligned} \tag{1}$$

It follows that the instantaneous light intensities at the two detectors are given by

$$\begin{aligned} I_3 &= |V_3|^2 \\ &= \tfrac{1}{2}[I_1 + I_2 - 2\sqrt{I_1 I_2}\sin(\phi_2 - \phi_1 + \theta_r' - \theta_t - \pi/2)] , \\ I_4 &= |V_4|^2 \\ &= \tfrac{1}{2}[I_1 + I_2 + 2\sqrt{I_1 I_2}\sin(\phi_2 - \phi_1 + \theta_t' - \theta_r + \pi/2)] , \end{aligned} \tag{2}$$

where $\theta_r = \arg r$, $\theta_t = \arg t$, etc. For a symmetric beam

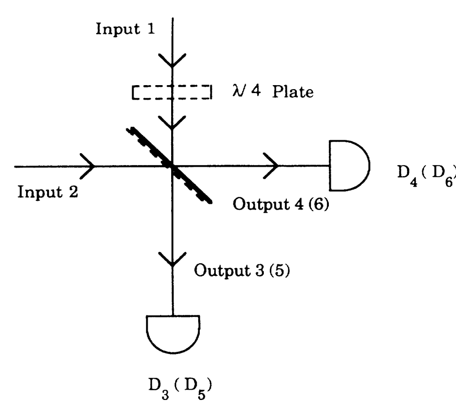

FIG. 1. Outline of a scheme for measuring the sine or cosine of the phase difference between two optical fields at input ports 1 and 2. When a quarter-wave phase shifter is inserted in channel 1 as shown, the output modes 3 and 4 are renamed 5 and 6.

splitter $\theta'_r-\theta_t-\pi/2=0$.

Let T be the measurement time interval. We may represent the photoelectric signals produced by detectors D_3 and D_4 during the time interval T by the quantities

$$W_3=\alpha\int_t^{t+T}I_3(t')dt' ,\qquad W_4=\alpha\int_t^{t+T}I_4(t')dt' . \tag{3}$$

α is the detector quantum efficiency, which we take to be equal for both detectors, for simplicity. If the detectors have different sensitivities, they can always be balanced in practice by placing an attenuator in front of the more sensitive one. We then have from Eqs. (2) and (3),

$$W_3=\tfrac{1}{2}[W_1+W_2-2W_{12}\sin(\phi_2-\phi_1+\theta'_r-\theta_t-\pi/2)] ,$$
$$W_4=\tfrac{1}{2}[W_1+W_2+2W_{12}\sin(\phi_2-\phi_1+\theta'_t-\theta_r+\pi/2)] , \tag{4}$$

where we have put

$$W_{12}\equiv\alpha\int_t^{t+T}[I_1(t')I_2(t')]^{1/2}dt' ,\qquad W_j\equiv\alpha\int_t^{t+T}I_j(t')dt' \quad (j=1,2,3,4) . \tag{5}$$

From these relations we can express the sine of the phase difference $\phi_2-\phi_1$ in the form

$$\sin[\phi_2-\phi_1+(\theta'_t-\theta_t+\theta'_r-\theta_r)/2]=\frac{W_4-W_3}{2W_{12}} . \tag{6}$$

Obviously, the constant $\theta'_t-\theta_t+\theta'_r-\theta_r$ vanishes for a symmetric beam splitter.

If the light intensities are fluctuating in time and T is not too long, then W_j and W_{12} may be different in successive measurement intervals T. On the other hand, when T is sufficiently long compared with all correlation times of the light, then W_j and W_{12} may be regarded as very good approximations to $\alpha\langle I_j\rangle_T T$ and $\alpha\langle[I_1(t)I_2(t)]^{1/2}\rangle_T T$. Here $\langle\ \rangle_T$ denotes the time average, which is equal to the average over the ensemble for a physical (ergodic) process. Under these circumstances W_j and W_{12} will not fluctuate significantly from measurement to measurement. In any case, whether T is long or short, $(W_4-W_3)/2W_{12}$ will have the same value every time when $\phi_2-\phi_1$ is constant.

Because the sine is double valued within the range from 0 to 2π, Eq. (6) still does not determine the phase difference completely. But with the help of a second measurement that yields $\cos[\phi_2-\phi_1+(\theta'_t-\theta_t+\theta'_r-\theta_r)/2]$, we can arrive at a unique value of $\phi_2-\phi_1$ within the interval 0 to 2π. If a quarter-wave phase plate is inserted in beam 1, as indicated in Fig. 1, immediately after the sine measurement, then ϕ_1 is effectively replaced by $\phi_1+\pi/2$ in each of the equations above. Let V_3,V_4 become V_5,V_6 when the phase plate is inserted. Then we obtain by the same argument as before, in place of Eqs. (4),

$$W_5=\tfrac{1}{2}[W_1+W_2+2W_{12}\cos(\phi_2-\phi_1+\theta'_r-\theta_t-\pi/2)] ,$$
$$W_6=\tfrac{1}{2}[W_1+W_2-2W_{12}\cos(\phi_2-\phi_1+\theta'_t-\theta_r+\pi/2)] , \tag{7}$$

from which it follows that

$$\cos[\phi_2-\phi_1+(\theta'_t-\theta_t+\theta'_r-\theta_r)/2]=\frac{W_5-W_6}{2W_{12}} . \tag{8}$$

Hence both the sine and cosine of the phase difference can be obtained from the two measurements, and so long as W_{12} is the same in both cases, as it will be for long T, this determines $\phi_2-\phi_1$ unambiguously. Although W_{12} may not be measured directly, we obtain, by squaring and adding Eqs. (6) and (8),

$$4W_{12}^2=(W_4-W_3)^2+(W_6-W_5)^2 , \tag{9}$$

so that

$$\sin(\phi_2-\phi_1)=\frac{W_4-W_3}{[(W_4-W_3)^2+(W_6-W_5)^2]^{1/2}} ,\qquad \cos(\phi_2-\phi_1)=\frac{W_5-W_6}{[(W_4-W_3)^2+(W_6-W_5)^2]^{1/2}} , \tag{10}$$

which expresses $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$ entirely in terms of the quantities W_j $(j=3,4,5,6)$. For simplicity we have taken the beam splitter to be symmetric, in which case the constant $\theta'_t-\theta_t+\theta'_r-\theta_r$ vanishes and can be discarded.

It is not strictly necessary to determine the sine and cosine from two separate measurements made in succession, as we indicated above. With the help of a slightly different arrangement shown in Fig. 2, they may be determined simultaneously [25,29,30]. By splitting both input waves with beam splitters BS_1 and BS_2, as shown, and then mixing the two beams at beam splitter BS_3 directly, and at beam splitter BS_5 with a quarter-wave plate inserted in one arm, as shown, we can determine $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$ together. This requires the use of four photodetectors D_3,D_4,D_5,D_6, and each measurement then yields the four quantities W_3,W_4,W_5,W_6 at once.

If all the beam splitters are identical and 50%:50%, and the output waves are denoted by V_3,V_4,V_5,V_6, reference to Fig. 2 shows that

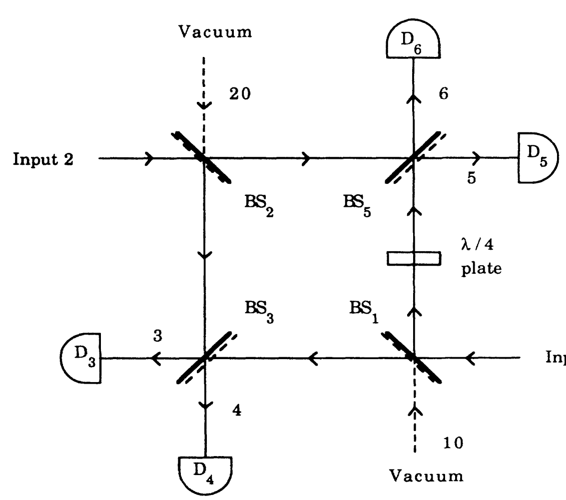

FIG. 2. Outline of a scheme for simultaneously measuring the sine and cosine of the phase difference between two fields at input ports 1 and 2; BS means beam splitter.

$$V_3=(tt'V_1+rr'V_2)\ ,$$
$$V_4=(r'tV_1+r'tV_2)\ ,$$
$$V_5=(irr'V_1+tt'V_2)\ ,$$
$$V_6=(irt'V_1+rt'V_2)\ ,$$

from which we have in place of Eqs. (4) and (7), since $|r|=|r'|=|t|=|t'|=1/\sqrt{2}$,

$$\begin{aligned} W_3&=\tfrac{1}{4}[W_1+W_2-2\sqrt{W_1W_2}\cos(\phi_2-\phi_1)]\ ,\\ W_4&=\tfrac{1}{4}[W_1+W_2+2\sqrt{W_1W_2}\cos(\phi_2-\phi_1)]\ ,\\ W_5&=\tfrac{1}{4}[W_1+W_2-2\sqrt{W_1W_2}\sin(\phi_2-\phi_1)]\ ,\\ W_6&=\tfrac{1}{4}[W_1+W_2+2\sqrt{W_1W_2}\sin(\phi_2-\phi_1)]\ . \end{aligned} \tag{11}$$

This time the phase angles $\theta_t,\theta_{t'},\theta_r,\theta_{r'}$ do not enter explicitly, because of the condition $\theta_t+\theta_{t'}-\theta_r-\theta_{r'}=\pm\pi$.

Apart from the absence of $\theta_{r'}-\theta_t-\pi/2$, the additional factor $\frac{1}{2}$ in front, and the interchange of W_3 and W_4 with W_5 and W_6, these equations are very similar to Eqs. (4) and (7) above, and they yield the same information. We merely need to interchange sin and cos and some signs in Eqs. (10). Hence the arrangements shown in Figs. 1 and 2 are effectively equivalent for the determination of the phase difference of two classical fields, so long as the phases and the light intensities are the same. In Sec. III below, we assume that the arrangement of Fig. 2 is used.

III. PHOTOELECTRIC COUNTING MEASUREMENTS

In the foregoing we have identified W_j $(j=3,4,5,6)$ with the photoelectric signal registered by detector j in time T. Actually when classical light of intensity I falls on a photodetector, it results in photoelectric emissions that occur at an average rate proportional to I, but at random times. The number of photoelectrons emitted from the photocathode in any time interval T is of course an integer, and it represents the best information about the light intensity available from the measurement. In a quantum treatment this number is closely related to the number of absorbed photons, but in a classical treatment we avoid any reference to photons. Let m_j $(j=3,4,5,6)$ be the number of photoelectric emissions registered by detector j in the time interval T, and let us express the intensity I_j in units such that αI_j gives the average rate of photoelectric emissions by detector j. Then m_j is actually an integer random variable that fluctuates from trial to trial even when W_j does not. m_j obeys a Poisson distribution with mean

$$\langle m_j\rangle=\alpha\int_t^{t+T}I_j(t')dt'=W_j\quad(j=3,4,5,6) \tag{12}$$

and standard deviation $[\langle(\Delta m_j)^2\rangle]^{1/2}=\sqrt{W_j}$. When the W_j's fluctuate also, we have to perform an additional average over the ensemble of W_j to obtain the moments of m_j.

Strictly speaking, the discrete numbers m_j, rather than W_j, constitute the "signal" registered by photodetector j. Of course when $W_j\gg 1$, so that the fractional fluctuations $[\langle(\Delta m_j)^2\rangle]^{1/2}/\langle m_j\rangle=1/\sqrt{W_j}$ are very small, then the measured numbers m_j $(j=3,4,5,6)$ of photoelectric counts are excellent representatives of W_j. We are then justified in replacing W_j by m_j in Eqs. (10) to a very good approximation and we may write [after making the changes in Eqs. (10) that apply to Fig. 2]

$$\begin{aligned} S_M&=\sin_M(\phi_2-\phi_1)\\ &=(m_6-m_5)/[(m_4-m_3)^2+(m_6-m_5)^2]^{1/2}\ ,\\ C_M&=\cos_M(\phi_2-\phi_1)\\ &=(m_4-m_3)/[(m_4-m_3)^2+(m_6-m_5)^2]^{1/2}\ . \end{aligned} \tag{13}$$

A single measurement of m_3,m_4,m_5,m_6 then determines the phase difference $\phi_2-\phi_1$. The subscript M serves to remind us that $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$ are obtained directly from the measurement, because sometimes the measured and the true values can be quite different, as we shall see.

The situation is quite different when the phase difference is not fixed but fluctuates, as it usually does, so that W_3,W_4,W_5,W_6 also fluctuate from one trial to another. There now exists a whole ensemble of phase differences, and one may be interested not only in the mean but also in the higher moments or the probability distribution of $\phi_2-\phi_1$. Moreover, if a single measurement of the phase is to be meaningful, the measurement time T cannot be arbitrarily long, but it must be shorter than the coherence time or the reciprocal bandwidth $1/\Delta\omega$ to ensure that $\phi_2-\phi_1$ does not change significantly during the measurement.

Let us suppose that the instantaneous values of the phases ϕ_1,ϕ_2 and the light intensities I_1,I_2 are not correlated as the field fluctuates. As W_3,W_4,W_5,W_6 are of the same order as W_1,W_2, it follows that the numbers m_3,m_4,m_5,m_6 are large whenever $W_1,W_2\gg 1$. So long as the numbers m_j are sufficiently large, they are accurate representatives of the W_j $(j=3,4,5,6)$, and we may use Eqs. (13) to determine the phase from the measured photoelectric counts, as before.

But for some optical fields $W_1,W_2\lesssim 1$, and then W_j and m_j are small also. An example is provided by a field from a typical thermal source, and it is not difficult to see why. It is implicit in our calculations that the optical fields look approximately like plane waves across the surfaces of the detectors. Hence the beam cross section must be smaller than the transverse coherence area, and we have already noted that the measurement time T must be shorter than the longitudinal coherence time. Hence the $\langle W_j\rangle$ must be smaller than the average photon number in a coherence volume, and this is known to be much less than 1 for typical thermal fields [38,39]. Under these conditions the measured numbers m_j are no longer representative of W_j, but fluctuate wildly from measurement to measurement. Then Eqs. (13) may lead to measured values for $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$ that are distortions of the true values, even on the average.

In order to understand why that is, we should note that when $W_1,W_2\ll 1$ the most likely values of m_3,m_4,m_5,m_6 resulting from a measurement are all zero, with an occasional 1 registered by one or another detector. The joint probability $p(m_3,m_4,m_5,m_6)$ is known to be given by the ensemble average product of four Poisson distributions [39]

$$p(m_3,m_4,m_5,m_6) = \left\langle \frac{W_3^{m_3}}{m_3!}e^{-W_3}\frac{W_4^{m_4}}{m_4!}e^{-W_4}\frac{W_5^{m_5}}{m_5!}e^{-W_5}\frac{W_6^{m_6}}{m_6!}e^{-W_6}\right\rangle , \tag{14}$$

where $\langle\ \rangle$ denotes the average over the ensemble of the fields. The most likely combination of outcomes $m_3=m_4=m_5=m_6=0$ has probability

$$p(0,0,0,0)=\langle \exp[-(W_3+W_4+W_5+W_6)]\rangle = \langle \exp[-(W_1+W_2)]\rangle , \tag{15a}$$

but as this leaves $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$, given by Eqs. (13), undefined, we shall discard these all-zero outcomes, and concentrate on the other possibilities. The corresponding probabilities then need to be renormalized by dividing $p(m_3,m_4,m_5,m_6)$, given by Eq. (14), by $\langle 1-\exp[-(W_1+W_2)]\rangle$. More generally, any combination $m_3=m_4$ and $m_5=m_6$ leaves S_M,C_M given by Eqs. (13) undefined, and these data should be discarded in the calculation of the moments of C_M,S_M. The probability of such a combination is given by

$$\sum_{m_3}\sum_{m_5} p(m_3,m_3,m_5,m_5) = \sum_{m_3}\sum_{m_5}\left\langle \frac{(W_3W_4)^{m_3}}{m_3!m_3!}\frac{(W_5W_6)^{m_5}}{m_5!m_5!}e^{-(W_1+W_2)}\right\rangle = \langle I_0(2\sqrt{W_3W_4})I_0(2\sqrt{W_5W_6})e^{-(W_1+W_2)}\rangle , \tag{15b}$$

where $I_0(z)$ is the zero-order modified Bessel function, and the renormalization factor then becomes

$$1-\langle I_0(2\sqrt{W_3W_4})I_0(2\sqrt{W_5W_6})e^{-(W_1+W_2)}\rangle .$$

For sufficiently small $\langle W_j\rangle$ this is indistinguishable from $1-\langle \exp(-W_1-W_2)\rangle$.

When $\langle W_j\rangle \ll 1$ ($j=3,4,5,6$), Eq. (14) yields approximately

$$\begin{aligned} p(1,0,0,0)&\approx\langle W_3\rangle ,\\ p(0,1,0,0)&\approx\langle W_4\rangle ,\\ p(0,0,1,0)&\approx\langle W_5\rangle ,\\ p(0,0,0,1)&\approx\langle W_6\rangle , \end{aligned} \tag{16}$$

with all other combinations of m_3,m_4,m_5,m_6 having much smaller probabilities. We then obtain, on averaging the right-hand side of Eqs. (13) with the help of Eqs. (16), renormalization and using Eqs. (11)

$$\begin{aligned} \langle C_M\rangle &= \left\langle \frac{m_4-m_3}{[(m_4-m_3)^2+(m_6-m_5)^2]^{1/2}}\right\rangle \\ &\approx \frac{\langle W_4\rangle-\langle W_3\rangle}{\langle 1-e^{-(W_1+W_2)}\rangle} \\ &\approx \frac{\langle\sqrt{W_1W_2}\rangle}{\langle W_1\rangle+\langle W_2\rangle}\langle\cos(\phi_2-\phi_1)\rangle ,\\ \langle S_M\rangle &= \left\langle \frac{m_6-m_5}{[(m_4-m_3)^2+(m_6-m_5)^2]^{1/2}}\right\rangle \\ &\approx \frac{\langle W_6\rangle-\langle W_5\rangle}{\langle 1-e^{-(W_1+W_2)}\rangle} \\ &\approx \frac{\langle\sqrt{W_1W_2}\rangle}{\langle W_1\rangle+\langle W_2\rangle}\langle\sin(\phi_2-\phi_1)\rangle . \end{aligned} \tag{17}$$

Now $\sqrt{W_1W_2}\le(W_1+W_2)/2$, so that the average yielded by an ensemble of measurements in the weak-field limit is smaller than the true average $\langle\sin(\phi_1-\phi_2)\rangle$ or $\langle\cos(\phi_1-\phi_2)\rangle$. Moreover, we readily find by a similar argument for the mean squares, that in this weak-field limit

$$\langle C_M^2\rangle = \left\langle \frac{(m_4-m_3)^2}{(m_4-m_3)^2+(m_6-m_5)^2}\right\rangle \approx \tfrac{1}{2} \approx \left\langle \frac{(m_6-m_5)^2}{(m_4-m_3)^2+(m_6-m_5)^2}\right\rangle = \langle S_M^2\rangle , \tag{18}$$

no matter what the correct ensemble average $\langle\cos^2(\phi_2-\phi_1)\rangle$ and $\langle\sin^2(\phi_2-\phi_1)\rangle$ may be. It follows that the ensemble of measured $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$ values generated by the use of Eqs. (13) may be quite different from the true ensemble. This appears to be the price one is obliged to pay for measuring a very weak field.

Nevertheless, there is a procedure for recovering the correct ensemble of $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$ and their moments from photoelectric counting measurements, even in a weak field. However, we cannot use Eqs. (13). Instead we return to the defining relations (11), from which it follows that, when the phases and the intensities are not correlated,

$$\begin{aligned} \langle C_I^r\rangle &= \langle\cos^r(\phi_2-\phi_1)\rangle = \frac{\langle (W_4-W_3)^r\rangle}{\langle (W_1W_2)^{r/2}\rangle} ,\\ \langle S_I^r\rangle &= \langle\sin^r(\phi_2-\phi_1)\rangle \\ &= \frac{\langle (W_6-W_5)^r\rangle}{\langle (W_1W_2)^{r/2}\rangle} ,\quad r=1,2,3,\ldots . \end{aligned} \tag{19}$$

The subscript I of S_I,C_I reminds us that we are dealing with the *inferred*, as distinct from the measured, values of the sine and cosine. Of course, the moments of W_j may be unknown, but with the help of Eq. (14) they can be related to the factorial moments of the counts m_j ($j=1,\ldots,6$), which can be measured, by [39]

$$\langle W_j^r\rangle=\langle m_j(m_j-1)(m_j-2)\cdots(m_j-r+1)\rangle$$
$$=\langle m_j^{(r)}\rangle\ ,\quad r=1,2,\ldots \tag{20}$$
$$\langle W_i^r W_j^s\rangle=\langle m_i^{(r)}m_j^{(s)}\rangle\ ,\quad i\neq j\ .$$

From Eqs. (19) and (20) we then obtain for the inferred moments of the ensemble of sines and cosines

$$\langle C_I^r\rangle=\sum_{s=0}^{r}\binom{r}{s}\langle m_4^{(r-s)}(-m_3)^{(s)}\rangle\Big/\langle (W_1W_2)^{r/2}\rangle\ ,$$
$$\langle S_I^r\rangle=\sum_{s=0}^{r}\binom{r}{s}\langle m_6^{(r-s)}(-m_5)^{(s)}\rangle\Big/\langle (W_1W_2)^{r/2}\rangle\ . \tag{21}$$

Provided $\langle (W_1W_2)^{r/2}\rangle$ is known, the true moments of the sine and cosine can be derived from measurements of m_3,m_4,m_5,m_6. In particular, when $r=2$, since $\langle S_I^2\rangle+\langle C_I^2\rangle=1$, we can express $\langle W_1W_2\rangle$ entirely in terms of the measured counts,

$$\langle W_1W_2\rangle=\langle m_4^{(2)}\rangle-2\langle m_4m_3\rangle+\langle m_3^{(2)}\rangle+\langle m_6^{(2)}\rangle$$
$$-2\langle m_6m_5\rangle+\langle m_5^{(2)}\rangle\ . \tag{22}$$

But for odd values of r it is less obvious how $\langle (W_1W_2)^{r/2}\rangle$ may be determined, and one would need to conduct an auxiliary measurement.

So far it has been assumed that the fluctuations of the phases and the intensities are not correlated. Let us briefly examine the situation when they are correlated. In that case, intensity and phase averages cannot be separated. If we average Eqs. (11) over the ensemble we obtain

$$\langle W_3\rangle=\tfrac{1}{4}[\langle W_1\rangle+\langle W_2\rangle-2|\Gamma_{12}|\cos(\arg\Gamma_{12})]\ ,$$
$$\langle W_4\rangle=\tfrac{1}{4}[\langle W_1\rangle+\langle W_2\rangle+2|\Gamma_{12}|\cos(\arg\Gamma_{12})]\ , \tag{23}$$

where $\Gamma_{12}=\alpha T\langle V_1^*V_2\rangle$ is the scaled mutual coherence, from which it follows that

$$\cos(\arg\Gamma_{12})=\frac{\langle W_4\rangle-\langle W_3\rangle}{|\Gamma_{12}|}\ , \tag{24a}$$

and similarly

$$\sin(\arg\Gamma_{12})=\frac{\langle W_6\rangle-\langle W_5\rangle}{|\Gamma_{12}|}\ . \tag{24b}$$

By using Eq. (20) we can replace $\langle W_j\rangle$ by the measured average $\langle m_j\rangle$, and by squaring and adding we can determine $|\Gamma_{12}|^2$, so that in terms of measured variables we may write

$$\cos(\arg\Gamma_{12})=\frac{\langle m_4\rangle-\langle m_3\rangle}{[(\langle m_4\rangle-\langle m_3\rangle)^2+(\langle m_6\rangle-\langle m_5\rangle)^2]^{1/2}}\ , \tag{25}$$

$$\sin(\arg\Gamma_{12})=\frac{\langle m_6\rangle-\langle m_5\rangle}{[(\langle m_4\rangle-\langle m_3\rangle)^2+(\langle m_6\rangle-\langle m_5\rangle)^2]^{1/2}}\ .$$

However, $\sin(\arg\Gamma_{12})$ is not related in any obvious way to the true or inferred value $\langle\sin(\phi_2-\phi_1)\rangle$, so that Eqs. (21) and (25) really refer to different problems. Of course in the special case $\langle W_1\rangle,\langle W_2\rangle\gg1$ when $C_I=C_M$, $S_I=S_M$, one can use Eqs. (13) directly to generate the ensemble of S_I and C_I. But when $\langle W_1\rangle$ or $\langle W_2\rangle$ is small there appears to be no way to disentangle the phases from the intensities nor to determine even the mean inferred values $\langle S_I\rangle$ and $\langle C_I\rangle$.

Let us briefly summarize what we have established so far. When the phase difference $\phi_2-\phi_1$ is constant, we can make the measurement time T as long as we wish. Then m_3,m_4,m_5,m_6 can be large and excellent representatives of W_3,W_4,W_5,W_6, and then Eqs. (13) yield the values of $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$ from a single measurement of m_3,m_4,m_5,m_6.

When the phase difference is fluctuating, the measurement time T needs to be shorter than the coherence time $1/\Delta\omega$, and this imposes limits on W_j ($j=1$ to 6). So long as $W_1,W_2\gg1$, Eqs. (13) again yield the values of $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$ at each trial, and a succession of trials generates the ensemble of $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$. But when $W_1\ll1$ or $W_2\ll1$, Eqs. (13) yield measured values that differ from the true sine and cosine, and the succession of measurements does not generate the true ensemble of $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$. In that case we need to make a distinction between the ensemble of measured values S_M and C_M and the true ensemble of $\sin(\phi_2-\phi_1)$ and $\cos(\phi_2-\phi_1)$, or the ensemble of inferred values S_I and C_I. Their moments can be quite different. We need to turn to Eqs. (21) for the inferred moments of S_I and C_I, and these contain information about the true ensemble, provided the intensities and phases are not correlated. With this restriction Eqs. (21) should hold for both weak and strong fields. Finally, when phases and intensities are correlated and $\langle W_1\rangle$ or $\langle W_2\rangle$ is small, there appears to be no way to extract even the mean values $\langle S_I\rangle$ and $\langle C_I\rangle$ of the phase ensemble from the measurements.

IV. PHASE MEASUREMENT OF A QUANTUM FIELD—SCHEME 1

We now analyze the same measurement processes that we have been discussing in classical terms for a quantum field. We start with the measurement scheme embodied in Fig. 1 and allow ourselves to be guided by correspondence with the classical treatment. For simplicity we limit ourselves to a single-field mode at each input and we label Hilbert space operators with a caret. Let $\hat a_1,\hat a_2$ be the photon annihilation operators that characterize the field at the two input ports, and let $\hat a_3,\hat a_4$ be the corresponding operators at the two output ports. These variables obey the commutation relations

$$[\hat a_i,\hat a_j]=0=[\hat a_i^\dagger,\hat a_j^\dagger]\ ,\quad i,j=1,2,3,4$$

and (26)

$$[\hat a_i,\hat a_j^\dagger]=\delta_{ij}\ ,\quad i,j=1,2\ \text{ or }\ i,j=3,4\ .$$

The mode amplitudes $\hat a_i$ are completely analogous to the classical field amplitudes V_i ($i=1,2,3,4$), and as before, inputs and outputs are related by

$$\hat{a}_3=(t\hat{a}_1+r'\hat{a}_2)\ ,$$
$$\hat{a}_4=(r\hat{a}_1+t'\hat{a}_2)\ , \qquad (27)$$

which are consistent with Eqs. (26). From Eqs. (27) we obtain

$$\hat{n}_3=\hat{a}_3^\dagger\hat{a}_3=(|t|^2\hat{n}_1+|r|^2\hat{n}_2+t^*r'\hat{a}_1^\dagger\hat{a}_2+tr'^*\hat{a}_2^\dagger\hat{a}_1)\ ,$$
$$\hat{n}_4=\hat{a}_4^\dagger\hat{a}_4=(|r|^2\hat{n}_1+|t|^2\hat{n}_2+r^*t'\hat{a}_1^\dagger\hat{a}_2+rt'^*\hat{a}_2^\dagger\hat{a}_1)\ . \qquad (28)$$

If a second measurement is made with a quarter-wave phase plate inserted in beam 1, we relabel the corresponding output operators $\hat{a}_5,\hat{a}_6$. These are related to $\hat{a}_1,\hat{a}_2$ by

$$\hat{a}_5=(it\hat{a}_1+r'\hat{a}_2)\ ,$$
$$\hat{a}_6=(ir\hat{a}_1+t'\hat{a}_2)\ , \qquad (29)$$

and they satisfy similar commutation relations. Equations (29) lead to the result

$$\hat{n}_5=(|t|^2\hat{n}_1+|r|^2\hat{n}_2-it^*r'\hat{a}_1^\dagger\hat{a}_2+itr'^*\hat{a}_2^\dagger\hat{a}_1)\ ,$$
$$\hat{n}_6=(|r|^2\hat{n}_1+|t|^2\hat{n}_2-ir^*t'\hat{a}_1^\dagger\hat{a}_2+irt'^*\hat{a}_2^\dagger\hat{a}_1)\ , \qquad (30)$$

and Eqs. (28) and (30) together are completely analogous to the classical Eqs. (4) and (7) when $|t|=|r|=1/\sqrt{2}$. The W_j are replaced by the photon numbers $\hat{n}_j$.

Let us suppose first that the photodetectors D_3,D_4 and D_5,D_6 count the photons at the appropriate port with 100% efficiency, for simplicity, so that the photon numbers n and the photoelectric counts m coincide. From Eqs. (30) the differences yield

$$\hat{n}_4-\hat{n}_3=e^{i(\theta_r-\theta_{t'})}\hat{a}_2^\dagger\hat{a}_1+e^{i(\theta_{t'}-\theta_r)}\hat{a}_1^\dagger\hat{a}_2 \qquad (31)$$

and

$$\hat{n}_6-\hat{n}_5=ie^{i(\theta_r-\theta_{t'})}\hat{a}_2^\dagger\hat{a}_1-ie^{i(\theta_{t'}-\theta_r)}\hat{a}_1^\dagger\hat{a}_2\ , \qquad (32)$$

and comparison with the classical relations (6) and (8) suggests that for this experiment we choose the former as representative of the sine of the phase difference and the latter as representative of the cosine, except for a scale factor. In the classical treatment, we argued that the scale factor should be the same in both cases if the sine and cosine measurements were made in rapid succession, so that neither the phases nor the field intensities had time to change in between. Such an assumption is not meaningful, however, in the quantum treatment, because a measurement of a quantum-mechanical variable in general changes the state of the field. Even if the state is the same, $\hat{n}_3,\hat{n}_4$ and $\hat{n}_5,\hat{n}_6$ now refer to quite separate measurements in general. All one can expect is that the operators chosen to represent the sine and cosine of the phase difference are of the form

$$\hat{S}=K_1(e^{i(\theta_r-\theta_{t'})}\hat{a}_2^\dagger\hat{a}_1+e^{i(\theta_{t'}-\theta_r)}\hat{a}_1^\dagger\hat{a}_2)\ ,$$
$$\hat{C}=K_2(ie^{i(\theta_r-\theta_{t'})}\hat{a}_2^\dagger\hat{a}_1-ie^{i(\theta_{t'}-\theta_r)}\hat{a}_1^\dagger\hat{a}_2)\ , \qquad (33)$$

where K_1,K_2 are constants.

It is interesting to note that from the definitions (33) the commutator

$$[\hat{C},\hat{S}]=2iK_1K_2(\hat{n}_2-\hat{n}_1)\neq 0\ , \qquad (34)$$

so that the sine and cosine operators given by Eqs. (33) do not commute. This may be regarded as a reflection of the fact that the experimental arrangement of Fig. 1 allows us to measure *either* the sine or the cosine, but not both at once. The failure of $\hat{S}$ and $\hat{C}$ to commute should not therefore be regarded as "unsatisfactory," but rather as a sign that the quantum description is consistent with the experiment. We also observe that

$$[\hat{S},\hat{n}_1+\hat{n}_2]=0=[\hat{C},\hat{n}_1+\hat{n}_2]\ , \qquad (35)$$

so that measurement of either the sine or the cosine of the phase difference is compatible with measurement of the total photon number, just as for the original Susskind-Glogower phase operators [3,4].

Finally, we note that $\hat{C},\hat{S}$ obey the relations

$$[\hat{S},\hat{n}_1]=i(K_1/K_2)\hat{C}\ ,$$
$$[\hat{S},\hat{n}_2]=-i(K_1/K_2)\hat{C}\ ,$$
$$[\hat{C},\hat{n}_1]=-i(K_2/K_1)\hat{S}\ ,$$
$$[\hat{C},\hat{n}_2]=i(K_2/K_1)\hat{S}\ , \qquad (36)$$

which are similar to the Susskind-Glogower relations [3,4] and imply that there are uncertainty relations between the sine and cosine and the photon numbers. However, because $\sin(\phi_2-\phi_1)$ or $\cos(\phi_2-\phi_1)$ separately do not determine $\phi_2-\phi_1$, we shall not pursue this measurement scheme and rather turn to the alternative scheme of Fig. 2.

V. PHASE MEASUREMENT OF A QUANTUM FIELD—SCHEME 2

We have seen that the measurement scheme illustrated in Fig. 2 allows both sine and cosine measurements of the classical phase difference to be made simultaneously. Let us now explore the quantum-mechanical implications of this scheme. We assume that identical 50%:50% beam splitters BS_1 and BS_2 are used to split the input fields $\hat{a}_1,\hat{a}_2$ in two, which are then combined by beam splitter BS_3 directly, and by beam splitter BS_5 after a quarter-wave phase plate is inserted in the one arm. Four detectors D_3,D_4 and D_5,D_6 count the photons simultaneously in each measurement interval T, and the quadruplet of photon numbers counted represents the outcome of one measurement. One significant difference between the quantum and the classical descriptions of the measurement scheme in Fig. 2 is that in the quantum treatment one cannot ignore the vacuum fields, represented by $\hat{a}_{10}$ and $\hat{a}_{20}$, that enter at the two unused input ports, as indicated.

Reference to Fig. 2 shows that at the beam splitters BS_1, BS_2 we generate the output fields

$$\hat{b}_1=(t\hat{a}_1+r'\hat{a}_{10})\ ,$$
$$\hat{c}_1=(r\hat{a}_1+t'\hat{a}_{10})\ ,$$
$$\hat{b}_2=(t'\hat{a}_2+r\hat{a}_{20})\ , \qquad (37)$$
$$\hat{c}_2=(r'\hat{a}_2+t\hat{a}_{20})\ ,$$

and at the output end

$$\begin{aligned}\hat{a}_3&=(t'\hat{b}_1+r\hat{c}_2)\ ,\\ \hat{a}_4&=(t\hat{c}_2+r'\hat{b}_1)\ ,\\ \hat{a}_5&=(ir'\hat{c}_1+t\hat{b}_2)\ ,\\ \hat{a}_6&=(it'\hat{c}_1+r\hat{b}_2)\ .\end{aligned}\tag{38}$$

Each $\hat{a}_i,\hat{b}_i,\hat{c}_i$ is a single-mode beam operator, satisfying the usual boson commutation relations. By combining Eqs. (37) and (38) we obtain

$$\begin{aligned}\hat{a}_3&=(tt'\hat{a}_1+rr'\hat{a}_2+r't'\hat{a}_{10}+rt\hat{a}_{20})\ ,\\ \hat{a}_4&=(r't\hat{a}_1+r't\hat{a}_2+r'^2\hat{a}_{10}+t^2\hat{a}_{20})\ ,\\ \hat{a}_5&=(rr'i\hat{a}_1+tt'\hat{a}_2+ir't'\hat{a}_{10}+rt\hat{a}_{20})\ ,\\ \hat{a}_6&=(irt'\hat{a}_1+rt'\hat{a}_2+it'^2\hat{a}_{10}+r^2\hat{a}_{20})\ ,\end{aligned}\tag{39}$$

and from these we readily find after some algebra, with $|r|=|r'|=|t|=|t'|=1/\sqrt{2}$,

$$\begin{aligned}\hat{n}_4-\hat{n}_3&=\tfrac{1}{2}[(\hat{a}_1^\dagger+e^{i(\theta_t-\theta_{r'})}\hat{a}_{10}^\dagger)\\ &\qquad\times(\hat{a}_2+e^{i(\theta_t-\theta_{r'})}\hat{a}_{20})+\text{H.c.}]\ ,\\ \hat{n}_6-\hat{n}_5&=+\tfrac{1}{2}[(-i\hat{a}_1^\dagger+ie^{i(\theta_t-\theta_{r'})}\hat{a}_{10}^\dagger)\\ &\qquad\times(\hat{a}_2-e^{i(\theta_t-\theta_{r'})}\hat{a}_{20})+\text{H.c.}]\ .\end{aligned}\tag{40}$$

As always, we allow the classical treatment to serve as guide in the choice of the corresponding quantum-mechanical operator. Comparison with the classical Eqs. (11) indicates that these quantities should be related to the cosine and sine of the phase difference, respectively, except for a scale factor.

Let us introduce the abbreviations

$$\begin{aligned}\hat{C}&\equiv\hat{n}_4-\hat{n}_3\ ,\\ \hat{S}&\equiv\hat{n}_6-\hat{n}_5\ .\end{aligned}\tag{41}$$

We readily find from Eqs. (40) and (41), that

$$[\hat{C},\hat{S}]=0\ .\tag{42}$$

We may now construct operators $\hat{C}_M,\hat{S}_M$ corresponding to the measured cosine and sine, exactly as in Eqs. (13) above, by normalizing by $(\hat{C}^2+\hat{S}^2)^{1/2}$ to ensure that $\hat{C}_M^2+\hat{S}_M^2=1$. Because $\hat{C}$ and $\hat{S}$ commute with each other, $\hat{C}$ also commutes with $(\hat{C}^2+\hat{S}^2)^{1/2}$ and so does $\hat{S}$. We therefore write

$$\begin{aligned}\hat{C}_M&=\tfrac{1}{2}[(\hat{a}_1^\dagger+\eta\hat{a}_{10}^\dagger)(\hat{a}_2+\eta\hat{a}_{20})+\text{H.c.}](\hat{C}^2+\hat{S}^2)^{-1/2}\ ,\\ \hat{S}_M&=\tfrac{1}{2}[(-i\hat{a}_1^\dagger+i\eta\hat{a}_{10}^\dagger)(\hat{a}_2-\eta\hat{a}_{20})+\text{H.c.}]\\ &\qquad\times(\hat{C}^2+\hat{S}^2)^{-1/2}\ ,\quad \eta\equiv e^{i(\theta_t-\theta_{r'})}\end{aligned}\tag{43}$$

for the operators corresponding to the measured cosine and sine, with the normalization factor standing on the left or on the right. Evidently from the definitions $\hat{C}_M^2+\hat{S}_M^2=1$ and

$$[\hat{C}_M,\hat{S}_M]=0\ ,\tag{44}$$

which may be taken as a reflection of the fact that the measured cosine and sine are compatible this time, and that the values are obtained simultaneously.

From the definitions (40) we may readily show that $\hat{C}$ and $\hat{S}$ commute with the total input photon number $\hat{n}_1+\hat{n}_2+\hat{n}_{10}+\hat{n}_{20}$, and it therefore follows also that

$$[\hat{C}_M,\hat{n}_1+\hat{n}_2+\hat{n}_{10}+\hat{n}_{20}]=0=[\hat{S}_M,\hat{n}_1+\hat{n}_2+\hat{n}_{10}+\hat{n}_{20}]\ .\tag{45}$$

Hence measurements of $\hat{C}_M,\hat{S}_M$ are compatible with measurements of the total number of photons at the input. The role played by the vacuum operators $\hat{n}_{10},\hat{n}_{20}$ in Eqs. (45) is a minor one, for we may write

$$\begin{aligned}[\hat{C}_M,\hat{n}_1+\hat{n}_2]&=-[\hat{C}_M,\hat{n}_{10}+\hat{n}_{20}]\ ,\\ [\hat{S}_M,\hat{n}_1+\hat{n}_2]&=-[\hat{S}_M,\hat{n}_{10}+\hat{n}_{20}]\ ,\end{aligned}\tag{46}$$

and although the right-hand sides are not zero, their expectations are zero. The corresponding uncertainty relations

$$\begin{aligned}&\langle(\Delta\hat{C}_M)^2\rangle\langle[\Delta(\hat{n}_1+\hat{n}_2)]^2\rangle\geq 0\ ,\\ &\langle(\Delta\hat{S}_M)^2\rangle\langle[\Delta(\hat{n}_1+\hat{n}_2)]^2\rangle\geq 0\ ,\end{aligned}\tag{47}$$

therefore impose no restrictions on the dispersions of $\hat{C}_M,\hat{S}_M,\hat{n}_1+\hat{n}_2$, just as if the right-hand sides of Eqs. (46) were zero.

From the definitions of $\hat{C}$ and $\hat{S}$ we find after a little algebra, that

$$\begin{aligned}\hat{C}^2+\hat{S}^2&=\hat{n}_1\hat{n}_2+\hat{n}_1+\hat{n}_2+\hat{n}_2\hat{n}_{10}+\hat{n}_{20}+\hat{n}_{10}+\hat{n}_1\hat{n}_{20}+\hat{n}_{10}\hat{n}_{20}+\eta\hat{a}_1^{\dagger 2}\hat{a}_2\hat{a}_{20}+\eta^3\hat{a}_{10}^\dagger\hat{a}_1^\dagger\hat{a}_{20}^2+\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_{10}\hat{a}_{20}\\ &\quad+\eta\hat{a}_{10}^\dagger\hat{a}_1^\dagger\hat{a}_2^2+\eta^3\hat{a}_{10}^{\dagger 2}\hat{a}_2\hat{a}_{20}+\hat{a}_{10}^\dagger\hat{a}_{20}^\dagger\hat{a}_2\hat{a}_1+\eta^2\hat{a}_{10}^\dagger\hat{a}_2^\dagger\hat{a}_1\hat{a}_{20}+\eta^*\hat{a}_{20}^\dagger\hat{a}_2^\dagger\hat{a}_1^2+\eta^*\hat{a}_2^{\dagger 2}\hat{a}_1\hat{a}_{10}\\ &\quad+\eta^{*2}\hat{a}_{20}^\dagger\hat{a}_1^\dagger\hat{a}_2\hat{a}_{10}+\eta^{*3}\hat{a}_{20}^{\dagger 2}\hat{a}_1\hat{a}_{10}+\eta^{*3}\hat{a}_{20}^\dagger\hat{a}_2^\dagger\hat{a}_{10}^2\ ,\quad \eta=e^{i(\theta_t-\theta_{r'})}\ ,\\ &=\hat{\mathcal{D}}+\hat{\mathcal{F}}\ ,\end{aligned}\tag{48}$$

$\hat{\mathcal{D}} \equiv \hat{n}_1\hat{n}_2+\hat{n}_1+\hat{n}_2$ and $\hat{\mathcal{F}}$ stands for the sum of the remaining 17 terms, all of which have zero expectation by virtue of the fact that modes 10 and 20 are vacuum modes.

Although Eqs. (43) define the operators corresponding to the measured cosine and sine of the phase difference, in analogy with the classical Eq. (13), the presence of the irrational normalization factor $(\hat{C}^2+\hat{S}^2)^{-1/2}$ can make it difficult to evaluate expectations. We shall therefore describe one method for doing this. We recall that $\hat{C}_M$ and $\hat{S}_M$ and powers thereof are all functions of $\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6$, which we represent collectively by $\{\hat{n}\}$, so that for any integer r

$$\hat{C}^r_M = f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6) = f(\{\hat{n}\}) , \tag{49}$$

and similarly for $\hat{S}^r_M$. Now if $P(\{n\})$ is the joint probability of the set of eigenvalues n_3,n_4,n_5,n_6 in a given quantum state, we can always determine the quantum expectation of $f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)$ from

$$\langle f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)\rangle = \sum_{\{n\}} f(\{n\})P(\{n\}) , \tag{50}$$

where the joint probability $P(\{n\})$ is expressible in the form [40–42]

$$P(\{n\}) = \left\langle : \prod_{j=3}^{6} \frac{(\hat{a}^\dagger_j\hat{a}_j)^{n_j} e^{-\hat{a}^\dagger_j\hat{a}_j}}{n_j!} : \right\rangle . \tag{51}$$

After combining Eqs. (50) and (51), substituting for $\hat{a}_j,\hat{a}^\dagger_j$ $(j=3,4,5,6)$ from Eqs. (39), and recalling that modes 10 and 20 are vacuum modes and that $\hat{n}_3+\hat{n}_4+\hat{n}_5+\hat{n}_6-\hat{n}_{10}-\hat{n}_{20}=\hat{n}_1+\hat{n}_2$, we obtain

$$\begin{aligned}\langle f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)\rangle &= \sum_{\{n\}} f(n_3,n_4,n_5,n_6)\left\langle : \frac{(\hat{a}^\dagger_1-\hat{a}^\dagger_2)^{n_3}(\hat{a}_1-\hat{a}_2)^{n_3}}{4^{n_3}n_3!}\,\frac{(\hat{a}^\dagger_1+\hat{a}^\dagger_2)^{n_4}(\hat{a}_1+\hat{a}_2)^{n_4}}{4^{n_4}n_4!}\right. \\ &\quad \left.\times\frac{(i\hat{a}^\dagger_1+\hat{a}^\dagger_2)^{n_5}(-i\hat{a}_1+\hat{a}_2)^{n_5}}{4^{n_5}n_5!}\,\frac{(-\hat{a}^\dagger_1-i\hat{a}^\dagger_2)^{n_6}(-\hat{a}_1+i\hat{a}_2)^{n_6}}{4^{n_6}n_6!}e^{-(\hat{n}_1+\hat{n}_2)}:\right\rangle .\end{aligned} \tag{52}$$

This expresses $\langle f(\{\hat{n}\})\rangle$, which stands for $\langle \hat{C}^r_M\rangle$ or $\langle \hat{S}^r_M\rangle$, as the expectation of a normally ordered operator power series in $\hat{a}_1,\hat{a}_2,\hat{a}^\dagger_1,\hat{a}^\dagger_2$, without irrational operators. It therefore avoids some of the difficulties associated with the use of Eqs. (43) directly.

As in the corresponding classical problem, we have to take note of the fact that $f(n_3,n_4,n_5,n_6)$ is undefined when $n_3=n_4$ and $n_5=n_6$. We therefore exclude this combination from the sum in Eq. (52). By using Eq. (51) we may show as in Sec. III, that the probability for $n_3=n_4$ and $n_5=n_6$ is given by

$$\mathcal{P} \equiv \left\langle : I_0\{[(\hat{a}^{\dagger 2}_1-\hat{a}^{\dagger 2}_2)(\hat{a}^2_1-\hat{a}^2_2)/2]^{1/2}\} I_0\{[(\hat{a}^{\dagger 2}_1+\hat{a}^{\dagger 2}_2)(\hat{a}^2_1+\hat{a}^2_2)/2]^{1/2}\} e^{-(\hat{n}_1+\hat{n}_2)} : \right\rangle . \tag{52'}$$

Hence the sum in Eq. (52) should be renormalized by dividing by $1-\mathcal{P}$. The correction is unimportant in a strong field, but becomes increasingly important in a weak field, when $\mathcal{P}\approx\langle:\exp(-\hat{n}_1-\hat{n}_2):\rangle$. Henceforth this modification of Eq. (52) will be assumed to have been made. We shall use the modified Eq. (52) below to calculate the moments of $\hat{C}_M$ and $\hat{S}_M$ in certain quantum states.

Finally, let us consider the situation when the expectation $\langle f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)\rangle$ is to be compared with measurements of the counts m_3,m_4,m_5,m_6 from which $\langle f(m_3,m_4,m_5,m_6)\rangle$ is derived, when the detectors have quantum efficiency α which is not necessarily unity. m_j now stands for the number of *detected* photons. The joint probability $P(\{m\})$ can be obtained from Eq. (51) by replacing n_j by m_j and $\hat{a}^\dagger_j\hat{a}_j=\hat{n}_j$ by $\alpha\hat{n}_j$. The net effect is that when $\langle f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)\rangle$ is replaced by $\langle f(m_3,m_4,m_5,m_6)\rangle$ on the left of Eq. (52), then each matrix element on the right is multiplied by $\alpha^{m_3+m_4+m_5+m_6}$, and $\exp(-\hat{n}_1-\hat{n}_2)$ becomes $\exp(-\alpha\hat{n}_1-\alpha\hat{n}_2)$. The same substitution is called for in the probability $\mathcal{P}$ for $m_3=m_4$ and $m_5=m_6$, and in addition the argument of each Bessel factor I_0 is multiplied by α.

VI. APPLICATION TO THE WEAK FIELD

Suppose that the incident field is so weak that $\langle\hat{n}_1\rangle,\langle\hat{n}_2\rangle \ll 1$. Then in the calculation of $\langle\hat{C}^r_M\rangle$, for example, after terms with $n_3=n_4=n_5=n_6=0$ are discarded, the dominant contributions to the sum obviously come from $n_3=1$, $n_4=0$, $n_5=0$, $n_6=0$ and $n_3=0$, $n_4=1$, $n_5=0$, $n_6=0$. Hence, to a first approximation we drop all remaining terms in the sum, and we obtain from Eqs. (52) and (52′)

$$\langle \hat{C}_M \rangle = \left\langle \frac{\hat{n}_4 - \hat{n}_3}{[(\hat{n}_4-\hat{n}_3)^2+(\hat{n}_6-\hat{n}_5)^2]^{1/2}} \right\rangle$$
$$= \frac{\langle :(\hat{n}_1+\hat{n}_2+\hat{a}_1^\dagger\hat{a}_2+\hat{a}_2^\dagger\hat{a}_1)e^{-(\hat{n}_1+\hat{n}_2)}:\rangle}{4\langle :1-\exp(-\hat{n}_1-\hat{n}_2):\rangle} - \frac{\langle :(\hat{n}_1+\hat{n}_2-\hat{a}_1^\dagger\hat{a}_2-\hat{a}_2^\dagger\hat{a}_1)e^{-(\hat{n}_1+\hat{n}_2)}:\rangle}{4\langle :1-\exp(-\hat{n}_1-\hat{n}_2):\rangle}$$
$$\approx \frac{\langle \hat{a}_1^\dagger\hat{a}_2+\hat{a}_2^\dagger\hat{a}_1\rangle}{2(\langle\hat{n}_1\rangle+\langle\hat{n}_2\rangle)} + \cdots , \quad (53)$$

where the ellipsis represents terms of higher order in $\langle\hat{n}_1\rangle, \langle\hat{n}_2\rangle$. Similarly

$$\langle \hat{S}_M \rangle = \left\langle \frac{\hat{n}_6 - \hat{n}_5}{[(\hat{n}_4-\hat{n}_3)^2+(\hat{n}_6-\hat{n}_5)^2]^{1/2}} \right\rangle \approx \frac{-i\langle \hat{a}_1^\dagger\hat{a}_2-\hat{a}_2^\dagger\hat{a}_1\rangle}{2(\langle\hat{n}_1\rangle+\langle\hat{n}_2\rangle)} . \quad (54)$$

The second moments can be evaluated in a similar manner, and we find

$$\langle \hat{C}_M^2 \rangle = \left\langle \frac{(\hat{n}_4 - \hat{n}_3)^2}{[(\hat{n}_4-\hat{n}_3)^2+(\hat{n}_6-\hat{n}_5)^2]} \right\rangle$$
$$\approx \frac{\langle :(\hat{n}_1+\hat{n}_2+\hat{a}_1^\dagger\hat{a}_2+\hat{a}_2^\dagger\hat{a}_1)e^{-(\hat{n}_1+\hat{n}_2)}:\rangle}{4\langle :1-\exp(-\hat{n}_1-\hat{n}_2):\rangle} + \frac{\langle :(\hat{n}_1+\hat{n}_2-\hat{a}_1^\dagger\hat{a}_2-\hat{a}_2^\dagger\hat{a}_1)e^{-(\hat{n}_1+\hat{n}_2)}:\rangle}{4\langle :1-\exp(-\hat{n}_1-\hat{n}_2):\rangle}$$
$$\approx \tfrac{1}{2} , \quad (55)$$

and also

$$\langle \hat{S}_M^2 \rangle \approx \tfrac{1}{2} . \quad (56)$$

Now we have the general operator relation

$$|\langle \hat{a}_1^\dagger\hat{a}_2\rangle| \le \sqrt{\langle\hat{n}_1\rangle\langle\hat{n}_2\rangle} \le \tfrac{1}{2}(\langle\hat{n}_1\rangle+\langle\hat{n}_2\rangle) ,$$

so that from Eqs. (53) and (54)

$$\langle \hat{C}_M\rangle, \langle\hat{S}_M\rangle \le \tfrac{1}{2} . \quad (57)$$

This places a lower limit on the dispersions of $\hat{C}_M$ and $\hat{S}_M$, because

$$\langle(\Delta\hat{C}_M)^2\rangle = \langle\hat{C}_M^2\rangle - \langle\hat{C}_M\rangle^2 \ge \tfrac{1}{4} ,$$
$$\langle(\Delta\hat{S}_M)^2\rangle = \langle\hat{S}_M^2\rangle - \langle\hat{S}_M\rangle^2 \ge \tfrac{1}{4} . \quad (58)$$

It follows therefore that the relative fluctuations of $\hat{C}_M$ and $\hat{S}_M$ are always large, because from Eqs. (57) and (58)

$$[\langle(\Delta\hat{C}_M)^2\rangle]^{1/2}/\langle\hat{C}_M\rangle \ge 1 ,$$
$$[\langle(\Delta\hat{S}_M)^2\rangle]^{1/2}/\langle\hat{S}_M\rangle \ge 1 . \quad (59)$$

Hence both the measured sine and the cosine of the phase difference are ill defined and are largely unobtainable quantities from one measurement by this process in a weak field.

VII. INFERRED PHASES

In the classical treatment given in Sec. III above we showed that, although the ensemble of cosines and sines generated by successive measurements may be different from the true ensemble for the underlying optical field, it is nevertheless possible to infer the moments of the true ensemble from a long series of measurements. We labeled the corresponding moments the inferred moments $\langle C_I^r\rangle, \langle S_I^r\rangle$.

The situation is different in the quantum domain. For a quantum-mechanical observable cannot have an "underlying value" that differs from its measured value, and indeed it is only as a result of the measurement process that it acquires a value at all. Nevertheless, from the mathematical point of view there may exist an ensemble of values of an observable for the quantum state, irrespective of the measurement. We shall refer to the values that are derived by a procedure analogous to that adopted in the classical treatment as the inferred values, and label them by the subscript I. The significance of the inferred values will appear as we apply the formalism to particular quantum states.

We now attempt to derive expressions for $\langle\hat{C}_I^r\rangle$, and $\langle\hat{S}_I^r\rangle$. From the form of the semiclassical Eqs. (21) it is apparent that when the moments of W_j ($j=3,4,5,6$) are expressed in terms of moments of the measured photoelectric counts m_j, then from Eqs. (20) moments of W_j become factorial moments of m_j. When the n_j are replaced by their operator equivalents in the quantum treatment, this feature is expected to be preserved. Recalling that factorial moments of the photon number operator $\hat{n}$ are equivalent to moments in normal order [42],

$$\hat{n}^{(r)} = :\hat{n}^r: , \quad (60)$$

we are led to make the identification

$$\langle\hat{C}_I^r\rangle = \frac{\langle :(\hat{n}_4-\hat{n}_3)^r:\rangle}{\langle(\hat{W}_1\hat{W}_2)^{r/2}\rangle} = \frac{\langle :[(\hat{a}_2^\dagger+\eta^*\hat{a}_{20}^\dagger)(\hat{a}_1+\eta^*\hat{a}_{10})+\text{H.c.}]^r:\rangle}{2^r\langle(\hat{W}_1\hat{W}_2)^{r/2}\rangle} ,$$
$$\langle\hat{S}_I^r\rangle = \frac{\langle :(\hat{n}_6-\hat{n}_5)^r:\rangle}{\langle(\hat{W}_1\hat{W}_2)^{r/2}\rangle} = \frac{\langle :[(\hat{a}_2^\dagger-\eta^*\hat{a}_{20}^\dagger)(i\hat{a}_1-i\eta^*\hat{a}_{10})+\text{H.c.}]^r:\rangle}{2^r\langle(\hat{W}_1\hat{W}_2)^{r/2}\rangle} ,$$
$$\eta \equiv e^{i(\theta_t-\theta_r)} \quad (61)$$

for the inferred operator moments, provided $\langle(\hat{W}_1\hat{W}_2)^{r/2}\rangle \neq 0$, except that the operator $(\hat{W}_1\hat{W}_2)^{r/2}$ in the denominator still needs to be identified. Reference to Eq. (20) suggests that when $r=2$ we take

$$\hat{W}_1\hat{W}_2 = \hat{n}_1\hat{n}_2 , \quad (62)$$

and this choice is made even more plausible when we observe with the help of Eqs. (40), after some algebra, that

$$\langle :(\hat{n}_4-\hat{n}_3)^2:\rangle+\langle :(\hat{n}_6-\hat{n}_5)^2:\rangle=\langle \hat{n}_1\hat{n}_2\rangle , \quad (63)$$

because the modes labeled 10 and 20 are vacuum modes. For this reason the $\hat{a}_{10}$ and $\hat{a}_{20}$ operators play a negligible role in the calculation of normally ordered moments, and we have from Eqs. (61)

$$\langle \hat{C}_I^2\rangle=\frac{\langle :(\hat{a}_2^\dagger\hat{a}_1+\hat{a}_1^\dagger\hat{a}_2)^2:\rangle}{4\langle(\hat{n}_1\hat{n}_2)\rangle} ,$$
$$\langle \hat{S}_I^2\rangle=\frac{\langle :(i\hat{a}_2^\dagger\hat{a}_1-i\hat{a}_1^\dagger\hat{a}_2)^2:\rangle}{4\langle(\hat{n}_1\hat{n}_2)\rangle} , \quad (64)$$

provided $\langle \hat{n}_1\hat{n}_2\rangle$ does not vanish. Evidently from the definition $\langle \hat{C}_I^2\rangle+\langle \hat{S}_I^2\rangle=1$. However, when $r=1$ or some other odd number in Eqs. (61), it is not so obvious what the normalization factor in the denominator should be. We have no relation analogous to Eq. (63) to guide us, and there is, in general, no unique method for identifying the operator that corresponds to a given c number like $\sqrt{W_1W_2}$.

We shall now illustrate the formalism we have developed by applying the equations to optical fields in several different quantum states.

VIII. APPLICATION TO THE PHASE STATE

As a first application of the ideas introduced in Secs. V–VII, let us consider the state of definite phase $|\theta\rangle$ discussed by several authors [13,29] previously, particularly by Pegg and Barnett [13]. For a single-mode field this state is defined by the Fock expansion, for infinite s

$$|\theta\rangle=\frac{1}{\sqrt{s+1}}\sum_{n=0}^{s}e^{in\theta}|n\rangle . \quad (65)$$

Strictly speaking, the limit $s\to\infty$ does not exist and we shall follow Pegg and Barnett and take it as understood that s is finite, but the limit $s\to\infty$ is to be taken at the end of the calculation of expectations. We readily find from the definition that

$$\langle\theta|\hat{a}|\theta\rangle=\frac{1}{s+1}\sum_{n=0}^{s}\sum_{m=0}^{s}e^{i(n-m)\theta}\sqrt{n}\,\langle m|n-1\rangle$$
$$=\frac{e^{i\theta}}{s+1}\sum_{n=0}^{s}\sqrt{n}$$
$$\to\tfrac{2}{3}s^{1/2}e^{i\theta} , \quad (66)$$

$$\langle\theta|\hat{a}^2|\theta\rangle=\frac{1}{s+1}\sum_{n=0}^{s}\sum_{m=0}^{s}e^{i(n-m)\theta}\sqrt{n(n-1)}\langle m|n-2\rangle$$
$$=\frac{e^{2i\theta}}{s+1}\sum_{n=0}^{s}\sqrt{n(n-1)}$$
$$\to\tfrac{1}{2}se^{2i\theta} , \quad (67)$$

$$\langle\theta|\hat{n}|\theta\rangle=\frac{1}{s+1}\sum_{n=0}^{s}n\to\tfrac{1}{2}s ,$$
$$\langle\theta|\hat{n}^{1/2}|\theta\rangle=\frac{1}{s+1}\sum_{n=0}^{s}\sqrt{n}\to\tfrac{2}{3}s^{1/2} , \quad (68)$$

$$\langle\theta|\hat{a}^\dagger\frac{1}{\sqrt{\hat{n}}}|\theta\rangle\to e^{-i\theta} . \quad (69)$$

In each case the discrete sums over s have been approximated by integrals, which should be excellent approximations for large s.

Now suppose that the input to the apparatus shown in Fig. 2 is the product state $|\theta_1\rangle_1|\theta_2\rangle_2|\mathrm{vac}\rangle_{10}|\mathrm{vac}\rangle_{20}$. We start by considering the measured quantities $\hat{C}_M,\hat{S}_M$ given Eqs. (43), with $\hat{C}^2+\hat{S}^2$ given by Eq. (48). Because the photon numbers $\hat{n}_1,\hat{n}_2$ are effectively infinite on the average in the phase state, we shall retain only the dominant term $\hat{n}_1\hat{n}_2$ in Eq. (48), and we therefore replace $(\hat{C}^2+\hat{S}^2)^{-1/2}$ in Eqs. (43) by $\hat{n}_1^{-1/2}\hat{n}_2^{-1/2}$.

With the help of Eqs. (69) we then find at once that

$$\langle\hat{C}_M\rangle=\tfrac{1}{2}\langle\theta_1|\hat{a}_1^\dagger\hat{n}_1^{-1/2}|\theta_1\rangle\langle\theta_2|\hat{a}_2\hat{n}_2^{-1/2}|\theta_2\rangle+\text{c.c.}$$
$$=\cos(\theta_2-\theta_1) , \quad (70)$$
$$\langle\hat{S}_M\rangle=\tfrac{1}{2}\langle\theta_1|i\hat{a}_1^\dagger\hat{n}_1^{-1/2}|\theta_1\rangle\langle\theta_2|\hat{a}_2\hat{n}_2^{-1/2}|\theta_2\rangle+\text{c.c.}$$
$$=\sin(\theta_2-\theta_1) .$$

Similarly, we obtain for the second moments, with the same approximations as before,

$$\langle\hat{C}_M^2\rangle=\cos^2(\theta_2-\theta_1) ,$$
$$\langle\hat{S}_M^2\rangle=\sin^2(\theta_2-\theta_1) , \quad (71)$$

so that the dispersions of $\hat{C}_M$ and $\hat{S}_M$ vanish, as befits a state of definite phase.

If we turn to Eqs. (64) for the inferred second moments we obtain

$$\langle\hat{C}_I^2\rangle=\frac{\langle\theta_1|\langle\theta_2|\hat{a}_2^{\dagger 2}\hat{a}_1^2+\hat{a}_1^{\dagger 2}\hat{a}_2^2+2\hat{a}_2^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_2|\theta_1\rangle|\theta_2\rangle}{4\langle\theta_1|\hat{n}_1|\theta_1\rangle\langle\theta_2|\hat{n}_2|\theta_2\rangle}$$
$$=\cos^2(\theta_2-\theta_1) , \quad (72)$$

with the help of Eqs. (67) and (68). Similarly

$$\langle\hat{S}_I^2\rangle=\sin^2(\theta_2-\theta_1) . \quad (73)$$

If we attempt to construct $\langle\hat{C}_I\rangle$ and $\langle\hat{S}_I\rangle$ from Eqs. (61) by taking the normalization constant corresponding to $(\hat{W}_1\hat{W}_2)^{1/2}$ to be $\hat{n}_1^{1/2}\hat{n}_2^{1/2}$, we also find

$$\langle\hat{C}_I\rangle=\cos(\theta_2-\theta_1) ,$$
$$\langle\hat{S}_I\rangle=\sin(\theta_2-\theta_1) . \quad (74)$$

The moments of the inferred and measured quantities therefore coincide for the phase state, which is of course a special situation. As $\langle(\Delta\hat{C}_M)^2\rangle$ vanishes and $\langle(\Delta\hat{n}_i)^2\rangle$ $(i=1,2)$ is infinite for the phase state after we let $s\to\infty$, the uncertainty product is not defined.

Because the state $|\theta\rangle$ defined by Eq. (65) has infinite energy, it is not a physical state that can be realized experimentally. States that correspond to a minimum phase dispersion subject to some restriction on the average energy have been discussed recently [16,17,43], but we shall not consider them here.

IX. APPLICATION TO THE COHERENT STATE

Next we suppose that the input state to the apparatus in Fig. 2 is given by the product $|v_1\rangle|v_2\rangle|\text{vac}\rangle_{10}|\text{vac}\rangle_{20}$, where $|v_1\rangle$, $|v_2\rangle$ are both coherent states labeled by the complex numbers

$$v_1=|v_1|e^{i\theta_1}\ ,$$
$$v_2=|v_2|e^{i\theta_2}\ .$$

Let us start with the measured moments of $\hat{C}_M$ and $\hat{S}_M$, and make use of Eq. (52), which is particularly suitable for calculating coherent-state averages, because the operator on the right is in normal order. Recalling that for a normally ordered operator $:F(\hat{a}_1,\hat{a}_1^\dagger;\hat{a}_2,\hat{a}_2^\dagger):$ [40,45,46],

$$\langle v_1|\langle v_2|:F(\hat{a}_1,\hat{a}_1^\dagger;\hat{a}_2,\hat{a}_2^\dagger):|v_1\rangle|v_2\rangle=F(v_1,v_1^*;v_2,v_2^*)\ , \tag{75}$$

we obtain immediately from Eq. (52), on setting $f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)=\hat{C}_M$,

$$\langle v_2|\langle v_1|f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)|v_1\rangle|v_2\rangle=\sum_{\{n\}}f(n_3,n_4,n_5,n_6)\frac{|(v_1-v_2)/2|^{2n_3}}{n_3!}\frac{|(v_1+v_2)/2|^{2n_4}}{n_4!}$$
$$\times\frac{|(-iv_1+v_2)/2|^{2n_5}}{n_5!}\frac{|(-v_1+iv_2)/2|^{2n_6}}{n_6!}e^{-(|v_1|^2+|v_2|^2)}\ . \tag{76}$$

As before, it is understood that the terms with $n_3=n_4$ and $n_5=n_6$ are to be excluded from the sum, and that the answer is to be renormalized by dividing by [cf. Eq. (15b)]

$$1-I_0(|v_1^2-v_2^2|/2)I_0(|v_1^2+v_2^2|/2)\exp(-|v_1|^2-|v_2|^2)\ .$$

Rather than attempting to evaluate the average in general, we shall consider the special cases when $|v_1|,|v_2|\gg1$ and when $|v_1|,|v_2|\ll1$.

(a) $|v_1|,|v_2|\gg1$. When $|v_1|,|v_2|$ are large and unequal, each of the Poisson distributions in n_3,n_4,n_5,n_6 in Eq. (76) can be well approximated by a continuous probability distribution $G(n;M)$ in the variable n with known mean M and variance $\sigma^2=M$. We therefore rewrite Eq. (76) as a multiple integral,

$$\langle v_1|\langle v_2|f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)|v_1\rangle|v_2\rangle$$
$$=\int\cdots\int dn_3dn_4dn_5dn_6f(n_3,n_4,n_5,n_6)G(n_3;|v_1-v_2|^2/4)G(n_4;|v_1+v_2|^2/4)$$
$$\times G(n_5;|-iv_1+v_2|^2/4)G(n_6;|-v_1+iv_2|^2/4)\ , \tag{77}$$

in which the random variables n_3,n_4,n_5,n_6 are all independent. As the n's are large numbers in general, we can expand $f(\{n\})$ in a Taylor series in each of the variables n_j about its mean value $\langle n_j\rangle$. Thus

$$f(\{n\})=f(\{\langle n\rangle\})+\Delta n_i\frac{\partial f(\{\langle n\rangle\})}{\partial n_i}+\frac{\Delta n_i\Delta n_j}{2!}\frac{\partial^2 f(\{\langle n\rangle\})}{\partial n_i\partial n_j}+\cdots\ . \tag{78}$$

When this is substituted in Eq. (77) and the integrations are carried out we obtain, because the terms in Δn_i average to zero,

$$\langle v_1|\langle v_2|f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)|v_1\rangle|v_2\rangle$$

$$=f\left[\frac{|v_1-v_2|^2}{4},\frac{|v_1+v_2|}{4},\frac{|-iv_1+v_2|^2}{4},\frac{|-v_1+iv_2|^2}{4}\right]$$

$$+\tfrac{1}{2}\frac{|v_1-v_2|^2}{4}\frac{\partial^2 f}{\partial n_3^2}\left[\frac{|v_1-v_2|^2}{4},\frac{|v_1+v_2|^2}{4},\frac{|-iv_1+v_2|^2}{4},\frac{|-v_1+iv_2|^2}{4}\right]$$

$$+\tfrac{1}{2}\frac{|v_1+v_2|^2}{4}\frac{\partial^2 f}{\partial n_4^2}\left[\frac{|v_1-v_2|^2}{4}\frac{|v_1+v_2|^2}{4},\frac{|-iv_1+v_2|^2}{4},\frac{|-v_1+iv_2|^2}{4}\right]$$

$$+\tfrac{1}{2}\frac{|-iv_1+v_2|^2}{4}\frac{\partial^2 f}{\partial n_5^2}\left[\frac{|v_1-v_2|^2}{4},\frac{|v_1+v_2|^2}{4},\frac{|-iv_1+v_2|^2}{4},\frac{|-v_1+iv_2|^2}{4}\right]$$

$$+\tfrac{1}{2}\frac{|-v_1+iv_2|^2}{4}\frac{\partial^2 f}{\partial n_6^2}\left[\frac{|v_1-v_2|^2}{4},\frac{|v_1+v_2|^2}{4},\frac{|-iv_1+v_2|^2}{4},\frac{|-v_1+iv_2|^2}{4}\right]$$

$$+\cdots \,. \qquad (79)$$

If we now identify $f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)$ with $\hat{C}_M$ given by $(\hat{n}_4-\hat{n}_3)/[(\hat{n}_4-\hat{n}_3)^2+(\hat{n}_6-\hat{n}_5)^2]^{1/2}$, we have

$$f\left[\frac{|v_1-v_2|^2}{4},\frac{|v_1+v_2|^2}{4},\frac{|-iv_1+v_2|^2}{4},\frac{|-v_1+iv_2|^2}{4}\right]=\frac{\frac{1}{2}(v_1^*v_2+v_1v_2^*)}{|v_1v_2|}=\cos(\theta_2-\theta_1)\,, \qquad (80)$$

$$\frac{\partial^2 f}{\partial n_j^2}\left[\frac{|v_1-v_2|^2}{4},\frac{|v_1+v_2|^2}{4},\frac{|-iv_1+v_2|^2}{4},\frac{|-v_1+iv_2|^2}{4}\right]=-\frac{3\cos(\theta_2-\theta_1)\sin^2(\theta_2-\theta_1)}{|v_1v_2|^2}\,,\quad j=3,4$$

$$=\frac{3\cos(\theta_2-\theta_1)[\sin^2(\theta_2-\theta_1)-\frac{1}{3}]}{|v_1v_2|^2}\,,\quad j=5,6\,. \qquad (81)$$

After substituting in Eq. (79) we arrive at

$$\langle \hat{C}_M\rangle=\cos(\theta_2-\theta_1)\left[1-\tfrac{1}{4}\left[\frac{1}{|v_1|^2}+\frac{1}{|v_2|^2}\right]\right]+O\left[\frac{1}{|v|^4}\right]. \qquad (82)$$

Next we identify $f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)$ with $\hat{C}_M^2$ and proceed in the same way. We find

$$f(\langle\hat{n}_3\rangle,\langle\hat{n}_4\rangle,\langle\hat{n}_5\rangle,\langle\hat{n}_6\rangle)=f\left[\frac{|v_1-v_2|^2}{4},\frac{|v_1+v_2|^2}{4},\frac{|-iv_1+v_2|^2}{4},\frac{|-v_1+iv_2|^2}{4}\right]$$

$$=\cos^2(\theta_2-\theta_1)\,, \qquad (83)$$

$$\frac{\partial^2 f}{\partial n_j^2}\left[\frac{|v_1-v_2|^2}{4},\frac{|v_1+v_2|^2}{4},\frac{|-iv_1+v_2|^2}{4},\frac{|-v_1+iv_2|^2}{4}\right]=-\frac{8\sin^2(\theta_2-\theta_1)[\cos^2(\theta_2-\theta_1)-\frac{1}{4}]}{|v_1v_2|^2}\,,\quad j=3,4$$

$$=\frac{8\cos^2(\theta_2-\theta_1)[\sin^2(\theta_2-\theta_1)-\frac{1}{4}]}{|v_1v_2|^2}\,,\quad j=5,6 \qquad (84)$$

so that from Eq. (79)

$$\langle\hat{C}_M^2\rangle=\cos^2(\theta_2-\theta_1)$$

$$-\tfrac{1}{2}\left[\frac{1}{|v_1|^2}+\frac{1}{|v_2|^2}\right]\cos2(\theta_2-\theta_1)+\cdots\,. \qquad (85)$$

Finally, after combining Eqs. (82) and (85) we obtain for the dispersion of $\hat{C}_M$

$$\langle(\Delta\hat{C}_M)^2\rangle=\tfrac{1}{2}\left[\frac{1}{|v_1|^2}+\frac{1}{|v_2|^2}\right]\sin^2(\theta_2-\theta_1)+O\left[\frac{1}{|v|^4}\right], \qquad (86)$$

and similarly we may show that

$$\langle(\Delta\hat{S}_M)^2\rangle=\frac{1}{2}\left[\frac{1}{|v_1|^2}+\frac{1}{|v_2|^2}\right]\cos^2(\theta_2-\theta_1)+O\left[\frac{1}{|v|^4}\right]. \tag{87}$$

The reason for the dependence of the dispersions of $\langle(\Delta\hat{C}_M)^2\rangle$ on $\sin(\theta_2-\theta_1)$ and of $\langle(\Delta\hat{S}_M)^2\rangle$ on $\cos(\theta_2-\theta_1)$ can be understood as follows. It is well known that the real and imaginary parts of $\hat{a},\hat{a}^\dagger$ have Gaussian distributions in the coherent state $|v\rangle$, resulting in uncertainties of the phase angle. The uncertainties of the corresponding sine and cosine must depend on the cosine and sine of the phase angle, because for phase angles near zero the cosine is rather insensitive to angle fluctuations, and for phase angles near $\pi/2$ the sine is rather insensitive to angle fluctuations.

Finally, it is interesting to note that if we focus on the special case in which $|v_1|^2=\beta|v_2|^2=\langle\hat{n}_1\rangle=\langle(\Delta\hat{n}_1)^2\rangle$, $(\beta<1)$, we can combine Eqs. (86) and (87) in the form

$$[\langle(\Delta\hat{C}_M)^2\rangle+\langle(\Delta\hat{S}_M)^2\rangle]^{1/2}[\langle(\Delta\hat{n}_1)^2\rangle]^{1/2}=\sqrt{(1+\beta)/2}\quad(\langle\hat{n}_1\rangle\gg1). \tag{88}$$

It is clear from this that the greater the dispersion of the photon number, the smaller is the dispersion of the phase.

(b) $|v_1|,|v_2|\ll1$. When $\langle\hat{n}_1\rangle,\langle\hat{n}_2\rangle\ll1$ we can make use of the general results for the weak field given by Eqs. (53)–(59). We then obtain immediately

$$\langle\hat{C}_M\rangle=\frac{|v_1v_2|}{|v_1|^2+|v_2|^2}\cos(\theta_2-\theta_1)=\frac{\cos(\theta_2-\theta_1)}{|v_1/v_2|+|v_2/v_1|}, \tag{89}$$

$$\langle\hat{C}_M^2\rangle=\frac{1}{2}, \tag{90}$$

and similarly for $\langle\hat{S}_M\rangle,\langle\hat{S}_M^2\rangle$. In this case $\langle\hat{C}_M\rangle$ and $\langle\hat{S}_M\rangle$ do not average to the expected values $\cos(\theta_2-\theta_1)$, $\sin(\theta_2-\theta_1)$, but to some other values depending on $|v_1/v_2|$. We find

$$\langle(\Delta\hat{C}_M)^2\rangle+\langle(\Delta\hat{S}_M)^2\rangle=1-\frac{1}{(|v_1/v_2|+|v_2/v_1|)^2}\geq\frac{3}{4}, \tag{91}$$

which is consistent with the idea that the phase difference is poorly defined when the photon numbers are small.

It can be shown that if the calculation is carried to the next approximation, involving the detection of two photons, i.e., to $O(|v|^2)$, one obtains in place of Eqs. (89)–(91),

$$\langle\hat{C}_M\rangle=\frac{\cos(\theta_2-\theta_1)}{|v_1/v_2|+|v_2/v_1|}\left[1-\left[\frac{1}{8}-\frac{1}{2\sqrt{2}}\right](|v_1|^2+|v_2|^2)-\frac{1}{4}\frac{|v_1|^2|v_2|^2}{|v_1|^2+|v_2|^2}+\cdots\right], \tag{92}$$

$$\langle\hat{C}_M^2\rangle=\frac{1}{2}\left[1+\frac{1}{4}\frac{|v_1|^2|v_2|^2}{|v_1|^2+|v_2|^2}\cos2(\theta_2-\theta_1)+\cdots\right], \tag{93}$$

and

$$\langle(\Delta\hat{C}_M)^2\rangle+\langle(\Delta\hat{S}_M)^2\rangle=1-\frac{1}{(|v_1/v_2|+|v_2/v_1|)^2}\left[1-\left[\frac{1}{4}-\frac{1}{\sqrt{2}}\right](|v_1|^2+|v_2|^2)-\frac{1}{2}\frac{|v_1|^2|v_2|^2}{|v_1|^2+|v_2|^2}+\cdots\right]. \tag{94}$$

In particular, when $|v_1|^2=|v_2|^2=\frac{1}{2}\langle N\rangle$, where $\langle N\rangle$ is the mean number of photons, then

$$\langle(\Delta\hat{C}_M)^2\rangle+\langle(\Delta\hat{S}_M)^2\rangle=\frac{3}{4}-0.083\langle N\rangle+\cdots. \tag{95}$$

Figures 3 and 4 show plots of $\langle\hat{C}_M\rangle$ and of $\langle(\Delta\hat{C}_M)^2+\langle(\Delta\hat{S}_M)^2\rangle$ as a function of mean photon number $\langle\hat{n}_1\rangle=|v_1|^2$ for different ratios $|v_2|^2/|v_1|^2$. It is apparent from these that $\langle\hat{C}_M\rangle$ does note always coincide with $\cos(\theta_2-\theta_1)$, and that the phase difference becomes increasingly ill defined as the mean photon numbers $\langle\hat{n}_1\rangle,\langle\hat{n}_2\rangle\to0$. On the other hand, $\langle\hat{C}_M\rangle$ does not tend to zero and $\langle(\Delta\hat{C}_M)^2\rangle+\langle(\Delta\hat{S}_M)^2\rangle$ does not tend to unity as $\langle\hat{n}_1\rangle,\langle\hat{n}_2\rangle\to0$, so that the phase difference does not become completely random, and some phase information remains even in the limit. This conclusion may seem strange, but it is a reflection of the fact that in the present measurement scheme all-zero outcomes $(m_3=m_4=m_5=m_6=0)$ associated with the vacuum

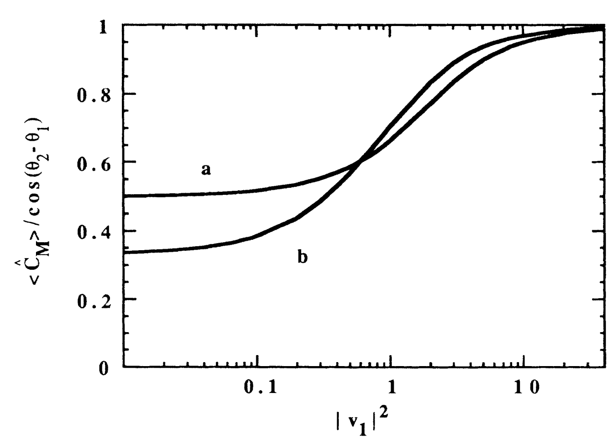

FIG. 3. The expected variation of $\langle\hat{C}_M\rangle/\cos(\theta_2-\theta_1)$ with mean photon number $\langle\hat{n}_1\rangle=|v_1|^2$ for an input field in a two-mode coherent state $|v_1,v_2\rangle$ for different intensity ratios (a) $|v_2|^2/|v_1|^2=1$; (b) $|v_2|^2/|v_1|^2=8$.

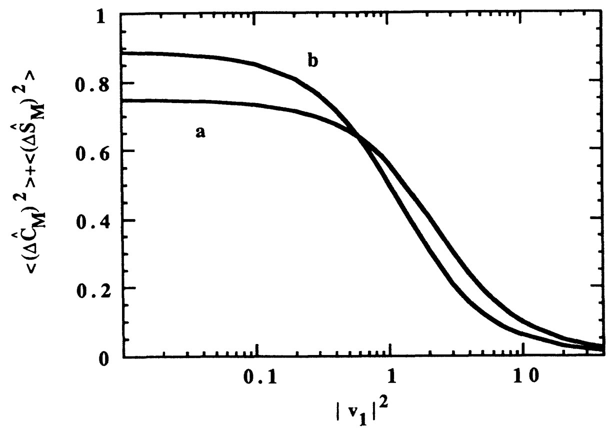

FIG. 4. The expected variation of the sum of the dispersions $\langle(\Delta\hat{C}_M)^2\rangle+\langle(\Delta\hat{S}_M)^2\rangle$ with mean photon number $\langle\hat{n}_1\rangle=|v_1|^2$ for an input field in the two-mode coherent state $|v_1,v_2\rangle$ for different intensity ratios (a) $|v_2|^2/|v_1|^2=1$, (b) $|v_2|^2/|v_1|^2=8$.

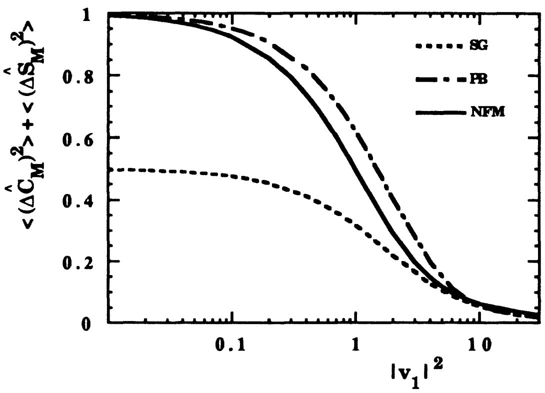

FIG. 6. Comparison of the expected variation of $\langle(\Delta\hat{C}_M)^2\rangle+\langle(\Delta\hat{S}_M)^2\rangle$ in a two-mode coherent state $|v_1\rangle_1|v_2\rangle_2$ with mean photon number $|v_1|^2=\langle\hat{n}_1\rangle$ when $|v_2|^2=50$ for the present theory (NFM), the Susskind and Glogower operators, and the Pegg-Barnett phase operator.

state are discarded. Needless to say, there are very few discarded outcomes $(m_3=m_4, m_5=m_6)$ when one of the two incoming fields is intense and has a definite phase, as is often assumed.

For comparison we show in Figs. 5 and 6 the corresponding curves for $\langle\hat{C}_M\rangle/\cos(\theta_2-\theta_1)$ and $\langle(\Delta\hat{C}_M)^2\rangle+\langle(\Delta\hat{S}_M)^2\rangle$ in the coherent state calculated for the Susskind-Glogower operators [3,4] (which do not commute), and for the Pegg-Barnett phase operator [13,20]. Evidently there are differences from our predictions, which are most pronounced when $|v|^2\lesssim 1$, and should show up in experiments. The results of some recent measurements [44] covering a range of $|v|^2$ from 10 down to 10^{-2} are found to be in very good agreement with our theory. The results differ from those predicted through use of the Susskind-Glogower and Pegg-Barnett phase operators, because the experiments evidently do not measure precisely these operators.

When we come to the inferred values given by the general relations (61) and (64), we find immediately

$$\langle\hat{C}_I^2\rangle=\cos^2(\theta_2-\theta_1),\quad \langle\hat{S}_I^2\rangle=\sin^2(\theta_2-\theta_1). \tag{96}$$

However, when $r=1$, if $(\hat{W}_1\hat{W}_2)^{1/2}$ is interpreted as $(\hat{n}_1\hat{n}_2)^{1/2}$, then the values $\langle\hat{C}_I\rangle$ and $\langle\hat{S}_I\rangle$ given by Eqs. (61) can exceed unity and do not make sense. Only on interpreting $(\hat{W}_1\hat{W}_2)^{1/2}$ as a normally ordered operator

$$\langle(\hat{W}_1\hat{W}_2)^{1/2}\rangle\to\langle:(\hat{n}_1\hat{n}_2)^{1/2}:\rangle=|v_1||v_2|, \tag{97}$$

do we obtain the inferred means

$$\langle\hat{C}_I\rangle=\cos(\theta_2-\theta_1),\quad \langle\hat{S}_I\rangle=\sin(\theta_2-\theta_1), \tag{98}$$

which makes the inferred dispersions $\langle(\Delta\hat{C}_I)^2\rangle=0=\langle(\Delta\hat{S}_I)^2\rangle$, just as for the phase state. However, the measured moments clearly distinguish between the coherent state and the phase state.

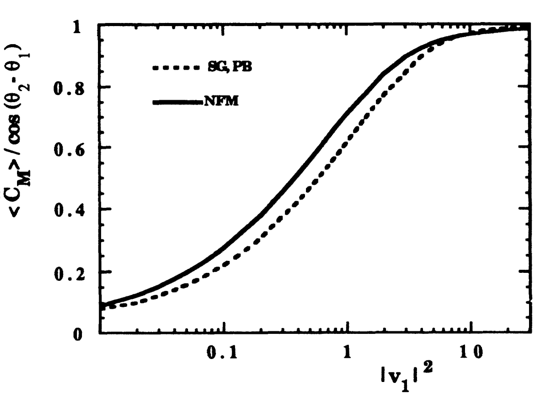

FIG. 5. Comparison of the expected variation of $\langle\hat{C}_M\rangle/\cos(\theta_2-\theta_1)$ in a two-mode coherent state $|v_1\rangle_1|v_2\rangle_2$ with mean photon number $|v_1|^2=\langle\hat{n}_1\rangle$ when $|v_2|^2=50$ for the present theory (NFM), the Susskind and Glogower operators, and the Pegg-Barnett phase operator.

X. AN ENSEMBLE OF COHERENT STATES

Let us suppose that the incident state for modes 1 and 2 has a density operator of the form

$$\hat{\rho}=\int\phi(v_1,v_2)|v_1v_2\rangle\langle v_1,v_2|d^2v_1d^2v_2. \tag{99}$$

It has been shown [40,45,46] that there is a sense in which any density operator $\hat{\rho}$ can be expressed in the form (99) if sufficiently generalized functions of $\phi(v_1v_2)$ are admitted. Here we limit ourselves to ordinary weight functions $\phi(v_1v_2)$ that either depend only on $|v_1|,|v_2|$ or reduce to a product of functions of $|v_1|,|v_2|$, and of θ_1,θ_2.

Then we may readily show that when $f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)$ stands for any moment of $\hat{C}_M$ or $\hat{S}_M$ as before, we have in place of Eq. (76)

$$\langle f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)\rangle=\sum_{\{n\}} f(n_3,n_4,n_5,n_6)\left\langle \frac{|(v_1-v_2)/2|^{2n_3}}{n_3!}\frac{|(v_1+v_2)/2|^{2n_4}}{n_4!}\frac{|(-iv_1+v_2)/2|^{2n_5}}{n_5!} \times\frac{|(-v_1+iv_2/2)/^{2n_6}}{n_6!}e^{-(|v_1|^2+|v_2|^2)}\right\rangle_\phi , \tag{100}$$

where $\langle\ \rangle_\phi$ denotes an average over v_1,v_2 with respect to the weight function $\phi(v_1,v_2)$. The exclusion of the cases $n_3=n_4$ and $n_5=n_6$ from the sum and the consequent renormalization of the average is again to be understood. It follows that all the previous results for $\langle \hat{C}_M^r\rangle,\langle \hat{S}_M^r\rangle$ in the coherent state $|v_1,v_2\rangle$ remain valid after we introduce the additional averaging operation $\langle\ \rangle_\phi$. Thus we have in place of Eq. (82),

$$\langle \hat{C}_M\rangle=\langle\cos(\theta_2-\theta_1)\rangle_\phi\left[1-\frac{1}{4}\left\langle\frac{1}{|v_1|^2}+\frac{1}{|v_2|^2}\right\rangle_\phi\right]+\cdots \tag{101}$$

and in place of Eq. (85)

$$\langle \hat{C}_M^2\rangle=\langle\cos^2(\theta_2-\theta_1)\rangle_\phi-\frac{1}{2}\left\langle\left[\frac{1}{|v_1|^2}+\frac{1}{|v_2|^2}\right]\right\rangle_\phi\langle\cos 2(\theta_2-\theta_1)\rangle_\phi , \tag{102}$$

and similarly for $\langle \hat{S}_M\rangle,\langle \hat{S}_M^2\rangle$ and for the dispersions $\langle(\Delta\hat{C}_M)^2\rangle,\langle\Delta\hat{S}_M)^2\rangle$.

XI. APPLICATION TO THE FOCK STATE $|n_1,n_2\rangle$

Next we apply the foregoing to calculate the first two moments of $\hat{C}_M,\hat{S}_M$ when the incoming field is in the two-mode Fock state $|n_1,n_2\rangle$. From Eq. (52) with $f(\hat{n}_3,\hat{n}_4,\hat{n}_5,\hat{n}_6)=\hat{C}_M=\hat{C}(\hat{C}^2+\hat{S}^2)^{-1/2}$, we obtain

$$\begin{aligned}\langle \hat{C}_M\rangle=&\sum_{\{n\}}\sum_{r_1,r_2}\frac{n_4-n_3}{[(n_4-n_3)^2+(n_6-n_5)^2]^{1/2}}\\&\times\langle n_1,n_2|(\hat{a}_1^\dagger-\hat{a}_2^\dagger)^{n_3}(\hat{a}_1^\dagger+\hat{a}_2^\dagger)^{n_4}(i\hat{a}_1^\dagger+\hat{a}_2^\dagger)^{n_5}(-\hat{a}_1^\dagger-i\hat{a}_2^\dagger)^{n_6}\hat{a}_1^{\dagger r_1}\hat{a}_2^{\dagger r_2}\hat{a}_2^{r_2}\hat{a}_1^{r_1}\\&\times(-\hat{a}_1+i\hat{a}_2)^{n_6}(-i\hat{a}_1+\hat{a}_2)^{n_5}(\hat{a}_1+\hat{a}_2)^{n_4}(\hat{a}_1-\hat{a}_2)^{n_3}|n_1,n_2\rangle\frac{(-1)^{r_1+r_2}}{r_1!r_2!}\\&\times\frac{1}{4^{n_3+n_4+n_5+n_6}n_3!n_4!n_5!n_6!}\\=&\sum_{\{n\}}\sum_{r_1,r_2}\sum_{s_3,s_4,s_5,s_6}\sum_{s_3',s_4',s_5's_6'}\frac{n_4-n_3}{[(n_4-n_3)^2+(n_6-n_5)^2]^{1/2}}\\&\times\begin{bmatrix}n_3\\s_3\end{bmatrix}\begin{bmatrix}n_4\\s_4\end{bmatrix}\begin{bmatrix}n_5\\s_5\end{bmatrix}\begin{bmatrix}n_6\\s_6\end{bmatrix}\begin{bmatrix}n_6\\s_6'\end{bmatrix}\begin{bmatrix}n_5\\s_5'\end{bmatrix}\begin{bmatrix}n_4\\s_4'\end{bmatrix}\begin{bmatrix}n_3\\s_3'\end{bmatrix}\\&\times\langle n_1|(\hat{a}_1^\dagger)^{s_3+s_4+s_5+s_6+r_1}(\hat{a}_1)^{s_6'+s_5'+s_4'+s_3'+r_1}|n_1\rangle\\&\times\langle n_2|(\hat{a}_2^\dagger)^{n_3-s_3+n_4-s_4+n_5-s_5+n_6-s_6+r_2}(\hat{a}_2)^{n_6-s_6'+n_5-s_5'+n_4-s_4'+n_3-s_3'+r_2}|n_2\rangle\\&\times\frac{(i)^{s_5-s_5'+s_6-s_6'}(-1)^{s_3+s_3'+s_6+s_6'+r_1+r_2}}{4^{n_3+n_4+n_5+n_6}r_1!r_2!n_3!n_4!n_5!n_6!} .\end{aligned} \tag{103}$$

It is again to be understood that the case $n_3=n_4$ and $n_5=n_6$ is excluded from the sum $\sum_{\{n\}}$, and that we divide by the renormalization factor $1-\mathcal{P}$, with $\mathcal{P}$ given by Eq. (52'), whenever it differs significantly from unity. The second equation follows from the first one after binomial expansions of the eight factors $(\hat{a}_1^\dagger-\hat{a}_2^\dagger)^{n_3}$, etc. Now the matrix elements under the sum evidently vanish unless

$$s_3+s_4+s_5+s_6=s_3'+s_4'+s_5'+s_6' , \tag{104}$$

so that

$$s_5-s_5'+s_6-s_6'=s_3'-s_3+s_4'-s_4 .$$

Hence the combination on the left or right must be even because $\langle \hat{C}_M\rangle$ is real.

We now make the transformations

$$s_3 \leftrightarrow s_4\ ,\quad s_3' \leftrightarrow s_4'\ ,$$
$$s_5 \leftrightarrow s_6\ ,\quad s_5' \leftrightarrow s_6'\ ,$$
$$n_3 \leftrightarrow n_4\ ,$$
$$n_5 \leftrightarrow n_6 \tag{105}$$

in Eq. (103). Then the right-hand side merely changes sign, because the factor $(-1)^{s_3+s_3'+s_6+s_6'} = (-1)^{s_3-s_3'+s_6-s_6'}$ changes to $(-1)^{s_4+s_4'+s_5+s_5'} = (-1)^{s_4'-s_4+s_5'-s_5}$ and the two factors are equal by virtue of Eq. (104). It follows that

$$\langle \hat{C}_M \rangle = 0\ , \tag{106}$$

and in the same way we may show that $\langle \hat{S}_M \rangle = 0$. More generally, it is easy to see by a similar argument that

$$\langle \hat{C}_M^r \rangle = \langle \hat{S}_M^r \rangle = 0 \quad \text{when } r \text{ is odd}\ . \tag{107}$$

Next we calculate $\langle \hat{C}_M^2 \rangle$ and $\langle \hat{S}_M^2 \rangle$ by the same technique. The equation for $\langle \hat{C}_M^2 \rangle$ looks just like Eq. (103), but with the factor

$$\frac{n_4 - n_3}{[(n_4-n_3)^2 + (n_6-n_5)^2]^{1/2}}$$

replaced by

$$\frac{(n_4-n_3)^2}{(n_4-n_3)^2+(n_6-n_5)^2}\ ,$$

and the equation for $\langle \hat{S}_M^2 \rangle$ has $(n_6-n_5)^2$ in the numerator instead. This time we make the transformations

$$s_3 \leftrightarrow s_5\ ,\quad s_3' \leftrightarrow s_5'\ ,$$
$$s_4 \leftrightarrow s_6\ ,\quad s_4' \leftrightarrow s_6'\ ,$$
$$n_3 \leftrightarrow n_5\ ,$$
$$n_4 \leftrightarrow n_6 \tag{108}$$

in the equation for $\langle \hat{C}_M^2 \rangle$. The factor $(-1)^{s_3+s_3'+s_6+s_6'}$ does not change for the same reason as before and $(i)^{s_5-s_5'+s_6-s_6'}$ does not change either, because the exponent is even. The only change in the equation for $\langle \hat{C}_M^2 \rangle$ is that $(n_4-n_3)^2$ in the numerator becomes $(n_6-n_5)^2$, which turns it into the equation for $\langle \hat{S}_M^2 \rangle$. Hence we conclude that

$$\langle \hat{C}_M^2 \rangle = \langle \hat{S}_M^2 \rangle = \tfrac{1}{2}\ , \tag{109}$$

and a similar argument shows that $\langle \hat{C}_M^r \rangle = \langle \hat{S}_M^r \rangle$ for any even r.

Equations (107) and (109) are consistent with the situation in which the phase difference at the two inputs is completely random, but they do not prove it. Actually, for small numbers n_1, n_2 the phase difference is not completely random. It is not difficult to show that for the one-photon state $|1\rangle_1|0\rangle_2$ we have $\langle \hat{C}_M^r \rangle = \langle \hat{S}_M^r \rangle = \frac{1}{2}$ for all even r and 0 for all odd r, so that the probability density of the phase difference effectively consists of the sum of four δ functions centered at $\theta_2 - \theta_1 = 0, \pi/2, \pi, 3\pi/2$. This is a reflection of the fact that the phase measurement in that case has only four possible outcomes. As the photon numbers increase, the probability density becomes increasingly uniform over the range 0 to 2π.

Finally, we have from Eqs. (64) for the second moments of the inferred phases

$$\langle n_1, n_2 | \hat{C}_I^2 | n_1, n_2 \rangle = \frac{2n_1 n_2}{4 n_1 n_2} = \tfrac{1}{2}$$
$$= \langle n_1, n_2 | \hat{S}_1^2 | n_1, n_2 \rangle\ , \tag{110}$$

and

$$\langle \hat{C}_I \rangle = 0 = \langle \hat{S}_I \rangle\ , \tag{111}$$

no matter how the normalization factor in Eq. (61) is interpreted. This time the measured and inferred moments coincide.

XII. APPLICATION TO THE "SPLIT PHOTON"

We consider the situation illustrated in Fig. 7, in which a single photon falls on a beam splitter BS_0, and the two beam-splitter outputs serve as the two inputs $\hat{a}_1, \hat{a}_2$ to the phase-measuring system of Fig. 2. This situation possibly represents the simplest easily realizable example of the phase-difference measurement of a nonclassical field, and it exhibits some interesting features. If $\mathcal{R}, \mathcal{T}$ are the complex amplitude reflectivity and transmissivity of BS_0, with $|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1$, then the input state $|\psi\rangle$ to the measuring system has the form

$$|\psi\rangle = \mathcal{R}|1\rangle_1|0\rangle_2 + \mathcal{T}|0\rangle_1|1\rangle_2\ . \tag{112}$$

We now use this state in Eq. (52), with $f(\{\hat{n}\})$ identified with $\hat{C}_M, \hat{C}_M^2, \hat{S}_M, \hat{S}_M^2$ in turn, as before. As the diagonal matrix elements

$${}_2\langle 1|_1\langle 0|\hat{C}_M|0\rangle_1|1\rangle_2\ ,\quad {}_2\langle 0|_1\langle 1|\hat{C}_M|1\rangle_1|0\rangle_2\ ,$$

etc., were already evaluated in Sec. XI, there remains only the problem of calculating the off-diagonal matrix elements. This is easily done by expansion of the normally ordered product of binomial and exponential factors in Eq. (52), because only the lowest-order terms contribute. As the procedure is similar to that used in Sec. XI, we shall not go into details here. It is not difficult to show that

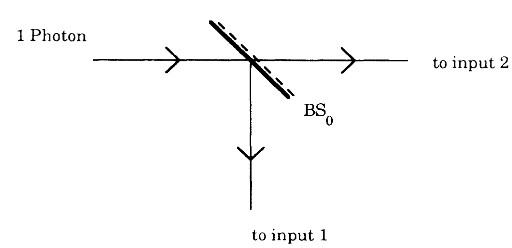

FIG. 7. Illustrating the generation of the two interferometer inputs as ports 1 and 2 of Fig. 2 with beam splitter BS_0 from one photon.

$$ {}_2\langle 0|_1\langle 1|\hat{C}_M|0\rangle_1|1\rangle_2=\tfrac{1}{2}={}_2\langle 1|_1\langle 0|\hat{C}_M|1\rangle_1|0\rangle_2 \ , \tag{113}$$

and when this is combined with Eq. (106) for the Fock state, we obtain

$$\langle\psi|\hat{C}_M|\psi\rangle=\tfrac{1}{2}(\mathcal{R}\mathcal{J}^*+\mathcal{R}^*\mathcal{J})=\mathrm{Re}(\mathcal{R}\mathcal{J}^*) \ . \tag{114}$$

Similarly we find

$$\langle\psi|\hat{S}_M|\psi\rangle=\mathrm{Im}(\mathcal{R}\mathcal{J}^*) \ . \tag{115}$$

For the off-diagonal contribution to $\langle\hat{C}_M^2\rangle$ and $\langle\hat{S}_M^2\rangle$ we can show that

$$ {}_2\langle 0|_1\langle 1|\hat{C}_M^2|0\rangle_1|1\rangle_2=0={}_2\langle 0|_1\langle 1|\hat{S}_M^2|0\rangle_1|1\rangle_2 \ , \tag{116}$$

and when this is combined with Eq. (109) for the Fock state we have immediately

$$\langle\psi|\hat{C}_M^2|\psi\rangle=\tfrac{1}{2}=\langle\psi|\hat{S}_M^2|\psi\rangle \ . \tag{117}$$

Hence we obtain finally

$$\langle(\Delta\hat{C}_M)^2\rangle=\langle\hat{C}_M^2\rangle-\langle\hat{C}_M\rangle^2=\tfrac{1}{2}-\tfrac{1}{4}(\mathcal{R}\mathcal{J}^*+\mathcal{R}^*\mathcal{J})^2 \ ,$$

$$\langle(\Delta\hat{S}_M)^2\rangle=\langle\hat{S}_M^2\rangle-\langle\hat{S}_M\rangle^2=\tfrac{1}{2}+\tfrac{1}{4}(\mathcal{R}\mathcal{J}^*-\mathcal{R}^*\mathcal{J})^2 \ ,$$

and

$$\langle(\Delta\hat{C}_M)^2\rangle+\langle(\Delta\hat{S}_M)^2\rangle=1-|\mathcal{R}|^2|\mathcal{J}|^2 \ . \tag{118}$$

But for the state $|\psi\rangle$ given by Eq. (112),

$$\begin{aligned}\langle\hat{n}_1\rangle&=|\mathcal{R}|^2=\langle\hat{n}_1^2\rangle \ ,\\ \langle\hat{n}_2\rangle&=|\mathcal{J}|^2=\langle\hat{n}_2^2\rangle \ ,\end{aligned} \tag{119}$$

so that

$$\langle(\Delta\hat{n}_1)^2\rangle=|\mathcal{R}|^2|\mathcal{J}|^2=\langle(\Delta\hat{n}_2)^2\rangle \ . \tag{120}$$

Hence we may write

$$[\langle(\Delta\hat{C}_M)^2\rangle+\langle(\Delta\hat{S}_M)^2\rangle]+\tfrac{1}{2}[\langle(\Delta\hat{n}_1)^2\rangle+\langle(\Delta\hat{n}_2)^2\rangle]=1 \ . \tag{121}$$

This is a kind of uncertainty relation connecting the dispersions of the measured cosines and sines and of the photon numbers. Because both dispersions are bounded from above, the relation involves an uncertainty sum rather than uncertainty product. Nevertheless the usual conclusion applies; whenever the combined dispersions of $\hat{C}_M$ and $\hat{S}_M$ are as small as possible, the dispersions of the photon numbers are as large as possible, and vice versa.

Actually, these results for the split photon are already contained in Eqs. (92)–(94) for the weak two-mode coherent state $|v_1\rangle_1|v_2\rangle_2$ when $|v_1|^2,|v_2|^2\ll 1$, because

$$|v_1\rangle_1|v_2\rangle_2=|0\rangle_1|0\rangle_2+v_1|1\rangle_1|0\rangle_2+v_2|0\rangle_1|1\rangle_2 +O(|v|^2) \ . \tag{122}$$

When the all-zero counting outcomes are discarded, then this state is equivalent to order $|v|^2$ to $|\psi\rangle$ given by Eq. (112) if we identify $\mathcal{R},\mathcal{J}$ with $v_1/(|v_1|^2+|v_2|^2)^{1/2}$, $v_2/(|v_1|^2+|v_2|^2)^{1/2}$. Finally, we note that $\langle\psi|\hat{n}_1\hat{n}_2|\psi\rangle=0$, so that the concept of inferred phase discussed in Sec. VII is not applicable to the quantum state $|\psi\rangle$.

XIII. DISCUSSION

We have approached the problem of identifying the quantum-dynamical variable corresponding to the phase difference between two optical fields, not via some mathematical criterion, as is usually the case, but through the measurement process that is known to determine it in the classical regime. In particular, we have allowed ourselves to be guided by the correspondence between certain classical and quantum observables. Because experiments generally yield the sine and cosine of the phase difference, rather than the phase directly, we regard the measurement operators $\hat{C}_M,\hat{S}_M$ as the fundamental dynamical variables [26]. Although other measurement schemes are worth studying in detail, it is already apparent from the two experimental schemes we have analyzed that there is not a universal expression for $\hat{C}_M$ or $\hat{S}_M$; the two different measurement schemes lead naturally to different operators. For a scheme such as that shown in Fig. 2, in which values of the sine and cosine are obtained together, the corresponding operators $\hat{C}_M,\hat{S}_M$ commute. On the other hand, $\hat{C}_M$ and $\hat{S}_M$ do not commute when sine and cosine measurements are mutually exclusive alternatives, as in the scheme of Fig. 1. However, in both cases $\hat{C}_M$ and $\hat{S}_M$ commute with the total number of photons, which implies that phase measurements and total photon-number measurements are compatible.

Our analysis of the problem of phase measurement is in accord with the long-established conclusion of Dirac [1] that the phase difference is well defined only when there are large dispersions of the photon numbers. Conversely, the moments of the sine and cosine operators are consistent with the phase difference being completely uncertain when the photon numbers are definite.

In classical optics, it is necessary to make a distinction between the measured and the inferred values of the sine and cosine and their moments. In a weak field, in which the average photon numbers counted in a measurement are of order or less than 1, the ensemble of measured values may be quite different from the true ensemble. Nevertheless, classically it is possible to infer the correct moments of the ensemble from the measured moments of the photon counts, provided phase and amplitude fluctuations of the fields are not directly correlated. Corresponding relations can be derived for certain states of the quantum field. Some of the same difficulties of phase determination that one encounters in the classical domain carry over into the quantum domain. For example, when phases and amplitudes are correlated, there appears to be no way to measure the phase difference for either weak classical or weak quantum fields.

There has been a good deal of discussion in the past of the most appropriate dynamical variable to represent the phase of a quantum field, and many candidates have been studied. Our analysis suggests that this question may not have a general answer with respect to the measured phase operators, because different measurement schemes lead to different operators. As in many other quantum-mechanical problems, it seems that questions about the value of a dynamical variable cannot be divorced from the measurement process that generates the ensemble. In our view the proper choice of a phase operator ought to be based both on the measurement scheme and on the correspondence with classical optics, because the concept of optical phase arises and has a natural definition within the domain of classical optics.

ACKNOWLEDGMENTS

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