Phase-slip dynamics in one-dimensional distributed systems

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The phase-slip process, which occurs as the instantaneous variation of the phase of the order parameter on a multiple of 2π for one-dimensionally distributed systems, is studied in detail. The general analytical results are supplemented by a computer simulation of the phase-slip process for the case of the Ginzburg-Landau equation.

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A great variety of physical phenomena, such as certain problems of convective systems [1,2], superconductivity [3], and phase transitions [4], are described by the following type of equations:

$$\psi_t=\hat{D}\psi \,, \tag{1}$$

where $\psi(\mathbf{x},t)$ is a complex function of the space coordinate $\mathbf{x}$ and time t and $\hat{D}$ is a differential operator with respect to $\mathbf{x}$. It is supposed that Eq. (1) is completed by the periodic or rigid boundary conditions and that the operator $\hat{D}$ is invariant with respect to arbitrary translation in the $(\mathbf{x},t)$ plane.

By using two real functions, an amplitude R and a phase ϕ with the relation $\psi=R\exp(i\phi)$ instead of the complex function ψ, one obtains from Eq. (1) a pair of coupled equations of the form

$$R_t=\hat{D}_R(R,\phi) \,, \tag{2a}$$

$$\phi_t=\hat{D}_\phi(R,\phi) \,. \tag{2b}$$

In the one-dimensional case, Eq. (1) being formulated for a finite segment $-L<x<L$ has a weak conservation law: The total phase difference $\Phi=\phi(L,t)-\phi(-L,t)$ does not vary in time, as follows directly from Eq. (2b) and boundary conditions. This conservation law may be broken only at the so-called phase-slip moment when the amplitude R at a certain point $x_{\rm PS}$ vanishes. At this moment the phase $\phi(x_{\rm PS})$ is indefinite, so that its value may be shifted on any multiple of 2π. This process is well known and has been discussed in many papers (see, e.g., Refs. [3,5–8]). Nevertheless, some important details are yet unknown. For instance, while the phase change during one phase-slip process may in general be equal to any multiple of 2π, computer simulations [8] show that usually it equals its lowest value 2π. Another question is which of the phenomenon peculiarities are connected with the particular features of the concrete physical problem, and which ones are common for a great variety of problems? In the present Brief Report we develop the general analysis of the phase-slip process in order to fill this gap.

The main point of our consideration is the assumption that on the (x,t) plane, a phase-slip point $(t_{\rm PS},x_{\rm PS})$ is a singular one only if R and ϕ are taken as independent variables. If instead of them we introduce $u=\mathrm{Re}\psi$ and $v=\mathrm{Im}\psi$, the phase-slip point becomes a regular one.

In accordance with this, close to the phase-slip point the functions $u(x,t)$ and $v(x,t)$ may be expanded into powers of x and t as follows:

$$u=a_u t+b_u x+c_u x^2/2+\cdots \,, \tag{3a}$$

$$v=a_v t+b_v x+c_v x^2/2+\cdots \tag{3b}$$

[the translational invariance of the Eq. (1) on the (x,t) plane gives us rise to suppose without loss of a generality that $x_{\rm PS}=0$ and $t_{\rm PS}=0$]. Here $a_{u,v}$, $b_{u,v}$, and $c_{u,v}$ are constants.

It is natural to expect that close to the phase-slip moment a small vicinity of the point $x=x_{\rm PS}$ (phase-slip core) is a range of rapid spatiotemporal variations of the phase. In this case, studying the core structure, one may consider the range far away from the core as infinity, with boundary conditions of the form $\phi(x,t)\to\mathrm{const}_\pm$ at $x\to\pm\infty$.

If the underlying boundary-value problem possesses the invariance under the multiplication of ψ on $\exp(i\phi_0)$, where ϕ_0 is an arbitrary constant [9], it may be used for the elimination of the term $b_u x$ from Eq. (3b). As a result of such a transformation, the curve $u(x,t_{\rm PS})$ becomes tangent to the x axis. A simple analysis shows that in this case a variation of x from $x=-\infty$ to $x=\infty$ at any fixed t, depending on the sign of t, results either in (i) the phase shift of 2π, or (ii) gives no phase shift at all (see, e.g., Fig. 1).

Let us introduce now for case (i) two quantities $x_\pm(t)$ according to the relations $\phi(x_\pm,t)=\pm\pi/2$, so that the phase shift between these two points (equal to π) is the half of the total one (2π). A characteristic width of the phase-slip core $\Delta x_{\rm PS}$ may be determined now as the distance between x_+ and x_-; see Fig. 1(a). For case (ii) the same quantity may be introduced as the distance between the location points of the maximum and minimum of the curve $\phi(x)$ inside the phase-slip core; see Fig. 1(b). The characteristic width $\Delta x_{\rm PS}(t)$ for the phase slip in both

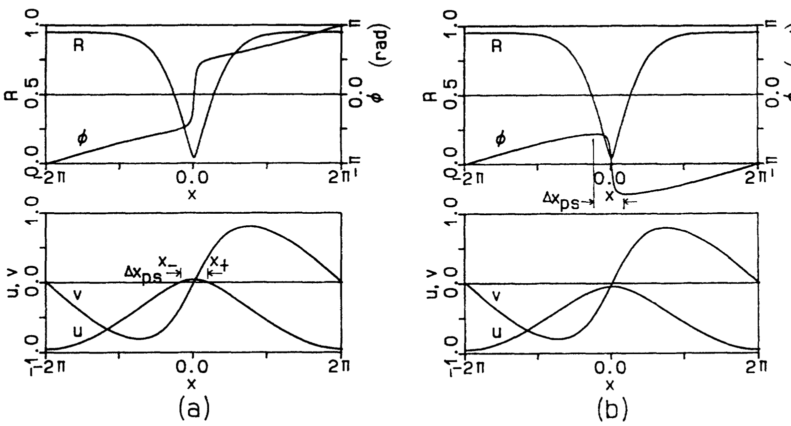

FIG. 1. Spatial variation of amplitude R and phase ϕ of the complex order parameter ψ obtained as a result of numerical integration of Eqs. (4)–(6). (a) $t=-0.2$ (just before the phase-slip moment), $\phi=2\pi$; (b) $t=0.2$ (just after the phase-slip moment), $\Phi=0$.

cases equals $|a_u t/c_u|^{1/2}$ independently of the sign of t.

A case in which the system has no invariance with respect to arbitrary rotations in the complex plane (see Ref. [9]), and hence the coefficient b_u does not equal zero, differs from the one discussed only by the fact that now the value of the phase at $x=-\infty$ cannot be considered as an arbitrary constant but must be fixed. It is evident that, at any fixed t, this results only in the shift of the curve $\phi(x)$ as a whole. In other words, all features of the phenomenon in this case remain the same.

Thus we determined that, in the general case, the phase-slip process gives rise to the phase shift equal to 2π. Multiples greater than that of 2π are possible only if at least two coefficients in Eq. (3) vanish simultaneously. Then the higher-order terms in the expansion must be taken into account.

To illustrate these general speculations, a numerical integration of the one-dimensional Ginzburg-Landau equation,

$$\psi_t=\psi_{xx}+(1-|\psi|^2)\psi \ , \tag{4}$$

for the finite segment $-2\pi<x<2\pi$ with periodic boundary conditions was carried out. According to the general procedure, instead of the complex function ψ, the pair of the real functions u and v was introduced. The computer simulation was done by the work station SUN3-260 (SUN Microsystems, Inc.) using the second-order Runge-Kutta method for the time derivative and the explicit difference scheme method for space.

The initial conditions were taken as follows:

$$u_0=R_s\cos\phi_s+u_1 \ , \tag{5a}$$

$$v_0=R_s\sin\phi_s \ , \tag{5b}$$

where

$$R_s=\{2[k^2+\alpha^2\tanh^2(\alpha x)]\}^{1/2} \ , \tag{6a}$$

$$\phi_s=kx+\tan^{-1}[(\alpha/k)\tanh(\alpha x)] \ , \tag{6b}$$

$$u_1=\sqrt{2}\epsilon k\,\mathrm{sech}(\alpha x) \ , \tag{6c}$$

$$\alpha=\{[(1-3k^2)/2]\}^{1/2} \ , \tag{6d}$$

with the following values of the constants: $\epsilon=-10^{-3}$, $k=1/3$, so that $\alpha=\sqrt{1/3}$, and the total phase difference between the edges of the segment equals 2π. This choice of the initial conditions corresponds to a slightly perturbed saddle-point solution of the Ginzburg-Landau equation (see, e.g., Refs. [5,10,11]). The perturbation, i.e., the term u_1 in Eq. (5a), is taken in such a form that for this saddle-point solution it has a nonzero projection on the unstable direction [11]. The results of this numerical analysis are shown in Figs. 1–3, and are in good agree-

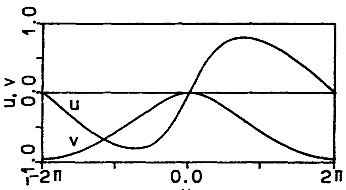

FIG. 2. Spatial variation of real (u) and imaginary (v) parts of ψ for Eqs. (4)–(6) at the phase-slip moment.

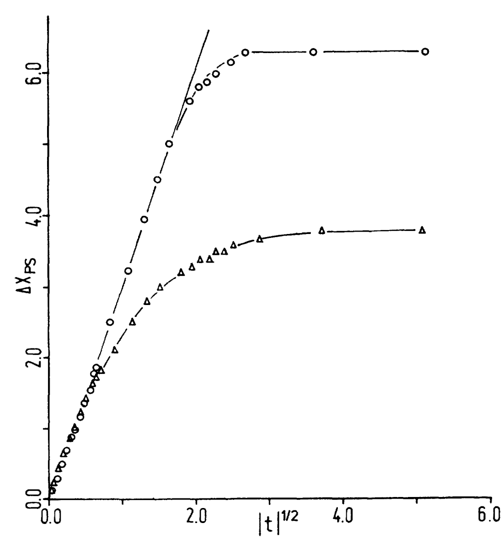

FIG. 3. Characteristic width (Δx_{PS}) of the phase-slip core as a function of time. Δ, $t<0$; $\bigcirc$, $t>0$.

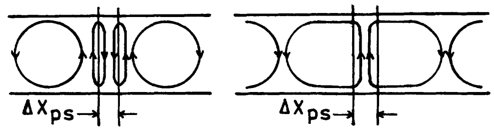

FIG. 4. Schematic drawing for the phase-slip process in convective rolls. Here Δx_{PS} describes the characteristic scale of the modulated rolls.

ment with the analytical treatment described above.

Note that the functions $u(x,t_{PS})$ and $v(x,t_{PS})$ have no singularities; see Fig. 2. The fact that the low $\Delta x_{PS} \sim \sqrt{|t|}$ at $t>0$ (after the phase-slip moment) is applied till much greater values of $|t|$ result than at $t<0$ (before the phase-slip moment) (see Fig. 3) probably is not connected with the phase-slip dynamics and is a particular property of the Ginzburg-Landau equation. The slowest process determining the approach to an equilibrium steady state for this equation is a phase diffusion. It results in the same law $\Delta x \sim \sqrt{|t|}$ as an intermediate asymptotic (until Δx is small in comparison with the system length $2L$). An overlapping of these two asymptotics may extend one at $t \to 0$ into an intermediate range.

Finally, we would like to stress that the dynamics of the phase-slip process discussed in the present Brief Report may be directly visible in real systems. For example, in convective fluids this process describes the annihilation (or nucleation) of a pair of convection rolls [12]. A qualitative picture corresponding to this case is shown in Fig. 4. More quantitative comparisons with real experiments in the electrohydrodynamic convections of liquid crystals [13] are planned to be reported elsewhere [14].

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