Using chaos to direct orbits to targets in systems describable by a one-dimensional map

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The sensitivity of chaotic systems to small perturbations can be used to rapidly direct orbits to a desired state (the "target"). We formulate a particularly simple procedure for doing this for cases in which the system is describable by an approximately one-dimensional map, and demonstrate that the procedure is effective even in the presence of noise.

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Recent work [1] has demonstrated that chaotic orbits can be rapidly brought to a desired state (the "target") by the application of tiny, judiciously chosen perturbations to an available system parameter. In this paper we point out that the technique is particularly simple for the case in which the chaotic dynamics can be described by a one-dimensional map. Experimental implementation should be especially feasible in this case. We demonstrate the practicality of the technique on a particular system in the presence of noise.

Suppose that we are given a chaotic, continuous time system whose dynamics are found to be approximately describable by a one-dimensional map. In practice, the reduction to the map,

$$X_{n+1}=F(p,X_n) , \tag{1}$$

where p is some parameter, might proceed by a combination of the delay coordinate and surface of section techniques. An approximately one-dimensional description usually applies if the continuous time chaotic attractor has a fractal dimension that is slightly above 2 (the Lorenz system is an example). We imagine that the parameter p can be varied by some small amount about its nominal value $\bar{p}$, $p=\bar{p}+\delta p$, and we seek a value for the small perturbation δp in the range, $-\Delta p \le \delta p \le \Delta p$, which will take us from a current state of the flow, χ_s, to the vicinity of a final target state, χ_t, in a short amount of time. Here the quantity Δp is the maximum allowed size of the perturbation. We trace the flow forward in time from χ_s until the first intersection with our surface of section and call that point X_s, as shown in Fig. 1. Likewise, we trace the flow backward in time from the target, χ_t, to the surface and call that intersection X_t. To reach X_t, we observe that the variation in the state after one iteration of our map due to the variation in p is

$$\delta X_1 = \left.\frac{\partial F}{\partial p}\right|_{(\bar{p},X_s)} \delta p . \tag{2}$$

Note that $|\delta p|$ is restricted to be less than or equal to Δp; this defines an interval ΔX_1. This interval will typically grow with each successive iteration of the map until it encompasses the desired point X_t as shown [2] in Fig. 2. This is sure to occur when the interval covers the entire

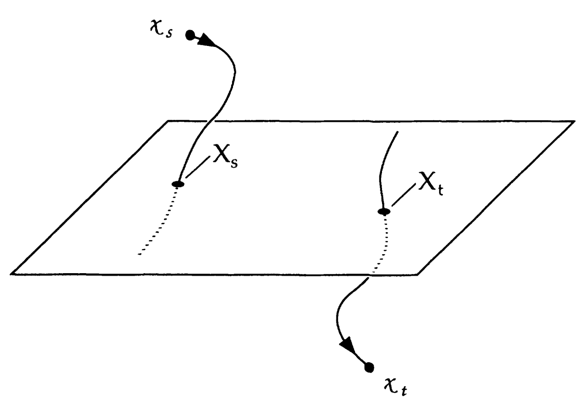

FIG. 1. Relation between states of flow and states of map.

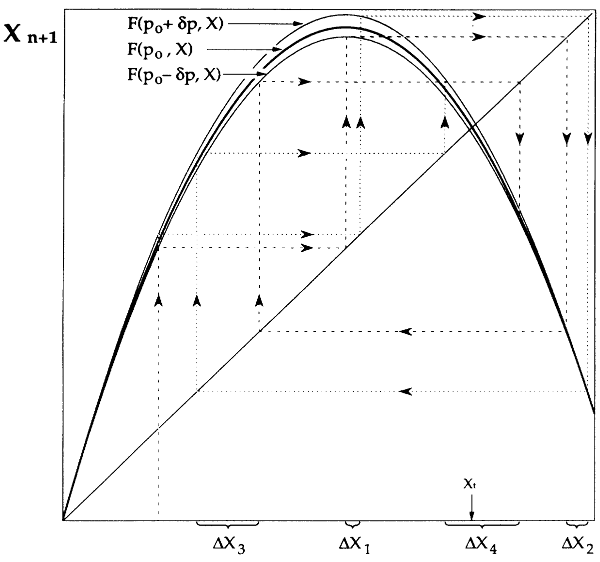

FIG. 2. Growth of successive iterations of chaotic map.

chaotic attractor of the map (1). Without loss of generality, we can take that size to be unity, and we find [3] that the number of iterations of the map required to encompass the desired neighborhood is typically

$$n_t \sim \frac{1}{\lambda}\ln(1/\Delta p) \ , \tag{3}$$

where λ is the Lyapunov exponent for the map. Once X_t is contained within the interval, we are all but done, for we know that some parameter value p_t between $p_{\min} = \bar{p} - \Delta p$ and $p_{\max} = \bar{p} + \Delta p$ will lead to X_t. All that remains is to refine the estimate of p_t. This is simply done. One rapid method is to repeatedly subdivide the parameter range in half and select the half range leading to an interval containing X_t. Thus we would map the original point, X_s, n times using a parameter value p_h halfway between $p_{\min}$ and $p_{\max}$. If X_t were contained in the interval $[F(p_{\min}, X_{n-1}), F(p_h, X_{n-1})]$, then we would repeat the process using a new p_h halfway between $p_{\min}$ and the old p_h. Otherwise, we would repeat the process using a new p_h halfway between the old p_h and $p_{\max}$. This procedure will give us the parameter value required to reach any arbitrarily small neighborhood [4] of X_t. If X_t is an unstable periodic point embedded in the attractor, then once in the neighborhood of X_t, we can maintain the system in the neighborhood [5].

As an example, we consider the Lorenz system [6], which has three degrees of freedom, and which, for the parameter values used by Lorenz, has an attractor whose dimension is slightly above 2. The Lorenz equations provide a leading-order description of the dynamics of a fluid contained in a thin vertically oriented torus with a heat source applied at the bottom [7]. We envision that the position of the heat source can be perturbed by moving it slightly to the left or the right in the plane of the torus. We characterize the size of this perturbation by the parameter p and use it to control the orbit. It can be shown

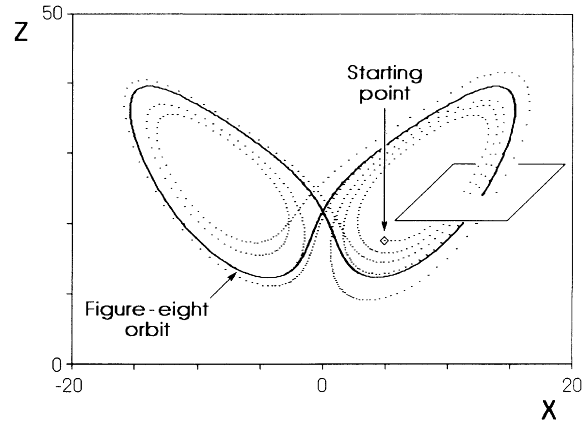

FIG. 3. Trajectory directed to figure-eight orbit from random point on Lorenz attractor.

that the flow subject to this control is well described by the addition of the term $p(t)$ to the second of the three Lorenz equations:

$$\begin{aligned} \dot{X} &= \sigma(Y - X) \ , \\ \dot{Y} &= -XZ + rX - Y + p(t) \ , \\ \dot{Z} &= XY - bZ \ , \end{aligned} \tag{4}$$

where we use Lorenz's values for the parameters: $\sigma = 10$, $r = 28$, $b = \frac{8}{3}$. To start with, we take the control function $p(t)$ to be constant in time. Later, when we discuss control in the presence of noise, we will alter the constant value periodically.

We can now apply our technique to reach a chosen periodic orbit on the attractor. Once we reach the periodic orbit, we reapply this technique to keep the trajectory on the periodic orbit. We choose a simple surface of section [8]: the half plane, $Z = 26.921$, $X > 8.0$. This surface of section is shown in Fig. 3, and the resulting approximately one-dimensional map is shown in Fig. 4. As an example, we assume that it is desired to reach and stabilize a fixed point of the resulting return map. In our case, the chosen fixed point of the map corresponds to a

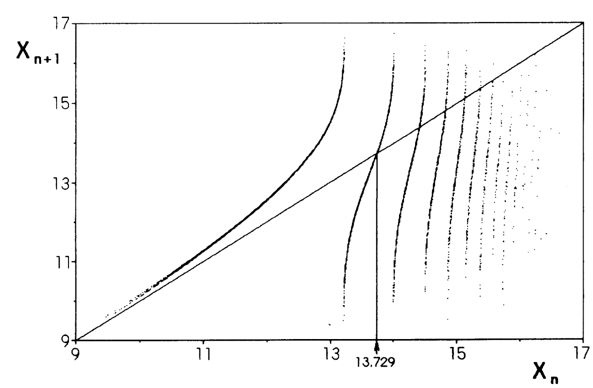

FIG. 4. Map resulting from given surface of section (10 000 points).

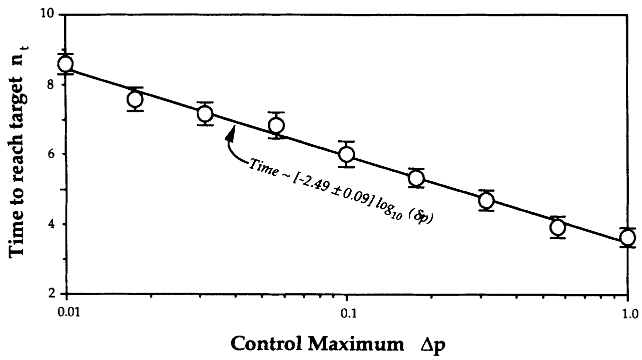

FIG. 5. Scaling behavior of technique.

figure-eight orbit of the flow [9], as shown in Fig. 3. In the figure, we have started from a point chosen at random on the attractor and used the technique [10] to reach and stabilize the desired orbit. We show the actual trajectory, recorded at every integration time step. In this example, we reach a small neighborhood $(X_t-0.008, X_t+0.008)$ of the desired orbit after only six map iterations. By comparison, without control we would typically require 200 iterations to reach the same neighborhood.

We remark that we are able to successfully apply our targeting technique despite the fact that the return map (Fig. 4) is manifestly discontinuous for this example. We excluded discontinuities from our parameter range by repeatedly reducing this range whenever a discontinuity was encountered [11]. Doubtless, other more efficient strategies are also possible.

In addition, we investigated the scaling behavior of the technique, which we have predicted should obey Eq. (3). To do so, we varied the maximum allowed parameter variation Δp, and directed trajectories to the figure-eight orbit of the flow from 25 different random starting points for each value of Δp. The result of this process is shown in Fig. 5, where we plot the number of surfaces of section piercings n_t before the target is reached versus Δp. The scaling is indeed consistent with Eq. (3). To appreciate the size of Δp relative to other terms, we note that the root-mean-squared time average, $\sqrt{\langle (dY/dt)^2 \rangle}$, for $\Delta p = 0$ is ~ 100. Thus for $\Delta p = 0.1$, the control at most produces a perturbation of the right-hand side of the dY/dt equation which is only of the order of $\frac{1}{1000}$ of the typical value of the right-hand side of this equation.

We can also use our technique in the presence of noise. This is particularly important because existing work [12] demonstrates the efficacy of a linearized method of stabilizing periodic points in chaotic systems. That method may become somewhat problematic, however, in the presence of noise. This is so because the noise may occasionally kick the orbit out of the small region within which the linearized stabilization procedure is effective. If p is kept fixed at the nominal value $\bar{p}$ when the orbit is outside of this small region, the orbit will wander chaotically until it reenters the small region and can again be captured by the control. If the region contains small measure, the time to reenter the region can be prohibitively long and the performance of the method will therefore be greatly degraded. Our technique provides a means for rapid recovery from these noise-induced bursts.

To target in the presence of noise, we must repeatedly apply our technique to compensate for wander of the system from the desired trajectory. As an example, we consider the prior targeting problem with white noise added to the system before each integration time step [13] (i.e., X, Y, and Z are independently changed by a small random amount with rms value δ). We chose an initial point on the attractor at random and targeted the figure-eight orbit as before, using $\Delta p = 0.1$. We again targeted this orbit after every 40 integration time steps (typically about one map cycle). We performed ten realizations of this process for successively larger amounts of noise, and found targeting was completely reliable for rms noise values [14] lower than $\delta = 0.1$, as shown in Fig. 6. To appreciate the size of the noise relative to other terms, consider the effect of the noise acting alone; that is, $dX/dt = \eta_x(t)$, $dY/dt = \eta_y(t)$, and $dZ/dt = \eta_z(t)$, where the noise $\eta_{x,y,z}(t)$ is temporally correlated according to $\langle \eta_{x,y,z}(t), \eta_{x,y,z}(t+\tau) \rangle = k\tau$. In this case, the action of the noise over a time interval T produces mean squared

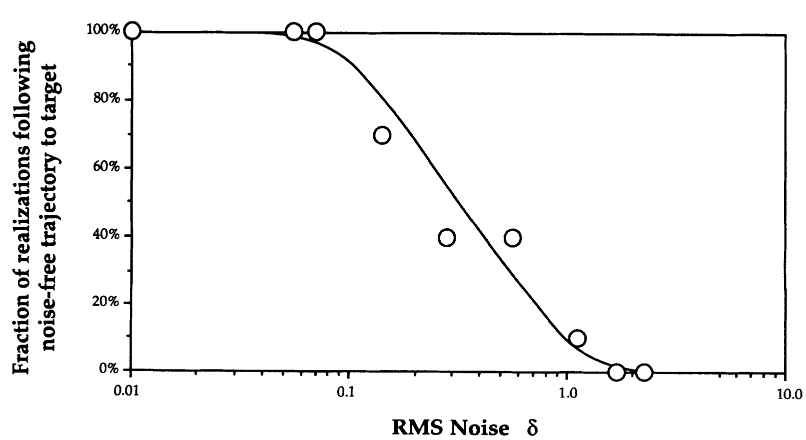

FIG. 6. Targeting effectiveness in the presence of noise.

deviation $\langle(\Delta X)^2\rangle=\langle(\Delta Y)^2\rangle=\langle(\Delta Z)^2\rangle=kT$. Choosing for T the mean time between surface of section piercings, we have $kT=1$ for $\delta=0.572$.

In conclusion, we have shown that the previously discussed [1] use of the sensitivity of chaotic flows to rapidly direct orbits to a desired state is particularly simple when the system is describable by a one-dimensional map. In this case, experimental implementation of targeting control should be especially effective. Moreover, the technique works well in the presence of small amplitude noise, and therefore it is well suited to be used in conjunction with control schemes which stabilize periodic orbits embedded in the attractor [6] and which may fail due to noise induced bursts. Some other works of interest in the study of control of complex dynamics are listed in Ref. [15].

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[1] T. Shinbrot, E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **65**, 3215 (1990).

[2] In the figure we show an arbitrary map and we imagine that we want to reach the fixed point of the map, i.e., $X_t=X^*$. The same procedure would work equally well for any other X_t also.

[3] For a more general derivation, see T. Shinbrot, E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **65**, 3215 (1990). The Lyapunov exponent can vary wildly in arbitrarily small intervals for systems such as the logistic map. In our estimate (3) we regard λ as the largest value of the Lyapunov exponent for the parameter p in the control range, $[p_{\min},p_{\max}]$. Alternatively, we can take λ in (3) to be the topological entropy of the attractor (this quantity is typically of order of the maximal exponent and varies smoothly with p).

[4] In our computations, following, we refined the estimate of δp 24 times, resulting in an accuracy of $1/2^{24}$, or better than 1 part in 10^7.

[5] The general problem of stabilizing an unstable periodic orbit embedded in a chaotic attractor is discussed by E. Ott, C. Grebogi, and J. A. Yorke [Phys. Rev. Lett. **64**, 1196 (1990)]. Experiments using this method are described in Ref. [12].

[6] E. N. Lorenz, J. Atmos, Sci. **20**, 130 (1963).

[7] J. A. Yorke, E. D. Yorke, and J. Mallet-Paret, Physica **24D**, 279 (1987).

[8] This surface was chosen purely for convenience [it contains the periodic point (13.729, 19.585, 26.921) and is horizontal—and hence intersections can be detected easily]. Any other suitable plane containing the desired state would work as well. We integrate Eqs. (4) by the fourth-order Runge-Kutta method and evaluate intersections with the surface by linear interpolation.

[9] This orbit might be of interest because it minimizes the variation in the flow variable Z. Control of this orbit has also been investigated by M. Ding (private communication).

[10] The parameter variation used was limited to be at most $|\delta p|=0.1$. This variation typically displaced the phase space location of the trajectory by 1 part in 10^5 after one Runge-Kutta time step.

[11] Discontinuities were detected by making use of the fact that discontinuities in the map result from nearby trajectories of the flow passing to opposite sides of the origin. Thus to detect a discontinuity, we had only to determine whether trajectories associated with the two parameter values struck the surfaces $Z=26.921$, $X<8.0$, and $Z=26.921$, $X>8.0$ in the same order.

[12] W. L. Ditto, S. N. Rauseo, and M. L. Spano, Phys. Rev. Lett. **65**, 3211 (1990); J. Singer, Y-Z. Wang, and H. H. Bau, *ibid.* **66**, 1123 (1991); E. R. Hunt, *ibid.* **67**, 1953 (1991).

[13] Cf. *Noise in nonlinear dynamical systems,* edited by F. Moss and P. V. E. McClintock (Cambridge University Press, Cambridge, 1989), Vol. I, pp. 267–276.

[14] The abscissa in Fig. 6 has been normalized so that a noise of 0.1 typically displaces the phase space location of the trajectory by 1 part in 10^5 after one Runge-Kutta time step.

[15] Some other works of interest include: C. H. Woo, Phys. Rev. A **29**, 2866 (1984); A. M. Bloch and J. E. Marsden, Theor. Comput. Fluid Dynamics **1**, 179 (1989); T. B. Fowler, IEEE Trans. Automatic Control **34**, 201 (1989); B. B. Plapp and A. W. Hübler, Phys. Rev. Lett. **65**, 2302 (1990); E. A. Jackson and A. W. Hübler, Physica D **44** 407 (1990); B. A. Huberman and E. Lumer, IEEE Trans. Circ. Syst. **37**, 547 (1990); E. A. Jackson, Phys. Lett. A **151**, 478 (1991); Elizabeth Bradley, in *Algebra Computing in Control,* edited by Francoise Lamnabhi-Lagarrigue and Gerard Jacob, Lecture Notes in Control and Information Sciences Vol. 165 (Springer-Verlag, Berlin, 1991); U. Dressler and G. Nitsche, Phys. Rev. Lett. **68**, 1 (1992).