Propagating solitons in damped ac-driven chains

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It is demonstrated that an external ac field may support stable propagation of solitons and cnoidal waves in a chain of interacting particles in the presence of friction, provided that the drive amplitude exceeds a certain threshold value. The effect is analyzed in the general form, and is then considered in detail both for the Toda lattice with weak friction and for a string of hard beads with arbitrary friction. The effects disappears in the continuum limit.

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I. INTRODUCTION

The subject of the present work is the propagation of collective excitations (solitons and cnoidal waves) in chains of interacting particles with friction and ac-driving fields. The general equation of motion for displacements y_n of particles in the chain is

$$m_n\ddot{y}_n=-\frac{\partial}{\partial y_n}[U_n(y_n-y_{n-1})+U_{n+1}(y_{n+1}-y_n)]-\alpha\dot{y}_n+\epsilon q_n\cos(\omega t)\ , \qquad (1)$$

where m_n are masses of the particles, α is the friction coefficient, $U_n(z)$ is the potential of the interparticle interaction (to simplify the notation, only the nearest-neighbor interaction is assumed, but this assumption is not essential), q_n are charges of the particles coupling them to the driving field, and ϵ and ω are, respectively, the amplitude and the frequency of the drive. It is assumed that all the quantities m_n, U_n, and q_n are periodic in n with some period n_0. The size of the elementary cell of the chain corresponding to this period will be designated a.

In some cases, the unperturbed equation (1) (with $\alpha=\epsilon=0$) can support solitary compression waves (solitons) propagating at a constant velocity V. The soliton is characterized by its degree of compression, i.e., the shift of y_n at $n=-\infty$ relative to $n=+\infty$. The well-known example is the Toda-lattice (TL) model [1] with $U_n(z)=\exp(-z)$ and $m_n\equiv1$. In the absence of the friction and drive, the TL soliton has the form

$$Y_n(t)=\ln[1+(\xi^{-2}-1)\xi^{2[n-\zeta(t)]}/(1+\xi^{2[n-\zeta(t)]})]\ , \qquad (2a)$$

where ξ is an arbitrary parameter taking values $|\xi|<1$, $\zeta(t)=Vt$ is the coordinate of the center of the soliton, and

$$V=\xi^{-1}(1-\xi^2)/\ln(\xi^{-2}) \qquad (2b)$$

is its velocity, which takes values $|V|>1$. At last, the soliton-mediated degree of compression of the Toda lattice is

$$\Delta y=\ln(\xi^{-2})\ . \qquad (2c)$$

As another example, one can consider a string of hard beads (HB) interacting only when they collide [2]. In this model, the "soliton" corresponds to the situation when, at any moment of time, only one particle is moving with the velocity V. Propagation of this soliton is provided by elastic collisions between the beads. However, in many cases solitary-wave solutions to Eq. (1) with $\alpha=\epsilon=0$, even in the case when all the masses and interparticle potentials do not depend on n, can only be found numerically (see, e.g., Ref. [3]).

It is necessary to note that, strictly speaking, only exactly integrable discrete models can support the stable propagation of solitons; if the model is nonintegrable, the moving soliton is gradually decelerated by radiative losses (see, e.g., Ref. [4]). However, in many cases the radiative braking proves to be extremely weak. For instance, the numerical simulations reported in Ref. [3] demonstrate a nearly stable propagation of solitons in nonintegrable chains. In fact, friction is a much more important problem than the radiative losses. The friction-induced damping of the TL soliton was studied in Refs. [5] and [6]. The aim of the present work is to demonstrate that the ac drive can compensate the dissipative losses and can thus support the propagation of solitons (and of periodic arrays of solitons) in the damped chain. The analysis to be developed below makes it obvious that the radiative losses can be compensated for as well, so that the stable propagation of solitons must also be possible in dissipationless ac-driven nonintegrable chains.

It will be demonstrated that these effects are specific for discrete models, and disappear in the continuum limit. They are also not possible if the masses and charges of all the particles in the chain are equal. Indeed, in this case the substitution $y_n=\tilde{y}_n+y_0(t)$, where $y_0(t)$ is a solution of the equation $\ddot{y}_0=-(\alpha/m)\dot{y}_0+(\epsilon/m)q\cos(\omega t)$, removes the driving term from Eq. (1) [7]. The ac-driven propagation of the solitons is possible if any masses and/or charges are different. To demonstrate this, the friction and drive terms in Eq. (1) will be treated as small perturbations, and it will be presumed that the soliton slowly evolves under their action.

The remainder of the paper is organized as follows.

The one-soliton case is considered in Sec. II. An effective equation of motion for the soliton is derived on the basis of the energy-balance analysis. It is demonstrated that this equation admits stationary regimes of the ac-driven motion of the soliton, provided the drive's amplitude exceeds a threshold value proportional to the dissipative constant α. The compensation of the dissipative (and radiative) losses is possible due to a resonance between the time-periodic drive and the periodic process of passage of the soliton through the sites of the underlying periodic chain. The resonance condition selects a discrete spectrum of the soliton's velocities. The ac-driven motion may be both stable and unstable, depending on the values of the parameters of the model. Next, we apply the general results to the two above-mentioned particular models,viz., the TL and the HB models. It is demonstrated that the ac-driven TL soliton is stable if it is not too broad. The propagation of the cnoidal waves (CW), i.e., periodic arrays of solitons, is analyzed in Sec. III (in particular, the collective excitations in a chain with periodic boundary conditions have the CW form). The analysis is again based on the energy balance. The resonant spectrum of velocities of the CW that admits the compensation of the losses by the ac drive is the same as for one soliton; the resonant spectrum of wavelengths of the CW's proves to be discrete as well. The general results obtained for the ac-driven propagation of the CW's are applied to the TL model. It is found that, in some region of the values of the drive's frequency ω, a hysteresis is possible in the TL model.

Concluding remarks are collected in Sec. IV. In particular, a collision of two ac-driven solitons moving at different velocities is briefly analyzed in that section. It is demonstrated that the collision may be both elastic and inelastic.

II. ac-DRIVEN MOTION OF ONE SOLITON

The simplest way to derive an equation of motion for one soliton within the framework of the perturbation theory is to analyze the balance of the soliton's energy in the presence of the dissipation and drive. If the form of the unperturbed soliton is $Y_n(t)$ [e.g., given by Eq. (2)], the energy dissipated in each elementary cell of the chain during the passage of the soliton is

$$E_d(V)=\alpha \sum_{n=1}^{n_0} \int_{-\infty}^{+\infty} dt [\dot{Y}_n(t)]^2 \tag{3}$$

(recall that n_0 is the number of the particles in the cell). Next, the total energy input $E_{\rm in}$ from the ac drive to each cell is

$$E_{\rm in}(V)=\epsilon \sum_{n=1}^{n_0} q_n \int_{-\infty}^{+\infty} dt\, \dot{Y}_n(t)\cos(\omega t) \ . \tag{4}$$

Proceeding to the adjacent cell, one has the same expression (3) for the energy loss, and the expression (4) for the energy input with

$$\dot{Y}_n(t) \rightarrow \dot{Y}_n(t+a/V) \tag{5}$$

(recall that a is the size of the cell). Making the substitution (5) in Eq. (4), one sees that, generally speaking, $E_{\rm in}$ oscillates from cell to cell, so that there is no mean influx of energy into the system. However, the input of energy is possible if V takes the resonant values

$$V_N=\omega a/2\pi N \ , \quad N=\pm 1\ , \pm 2, \pm 3, \ldots \ , \tag{6}$$

at which the oscillations of $E_{\rm in}$ are cancelled.

Thus, the ac-driven propagation of the soliton in the chain may be possible at the velocities (6). To analyze this possibility in more detail, let us write the balance equation for the soliton's energy $E(V)$:

$$\frac{d}{dt}E(V) \equiv E'(V)\frac{dV}{dt}=\epsilon \sum_{n=-\infty}^{+\infty} q_n \dot{Y}_n(t)\cos(\omega t) - \alpha \sum_{n=-\infty}^{+\infty} [\dot{Y}(t)]^2 \ , \tag{7}$$

cf. Eqs. (3) and (4). Since the right-hand side of Eq. (7) oscillates in time, and assuming the soliton's energy to be slowly varying in the spirit of the perturbation theory, it is natural to average Eq. (7) over a sufficiently long time, much larger than the time necessary for the soliton to travel a distance of its own size. The unperturbed soliton solution has the self-similar form $Y_n(t)=Y(n-Vt)$; see, e.g., Eq. (2a). Inserting this into Eq. (7) and averaging it over the long time, it is plain to see that the energy-balance equation takes the form

$$E'(V)\frac{dV}{dt}=Va^{-1}(E_{\rm in}-E_d) \ , \tag{7'}$$

$E_{\rm in}$ and E_d being the energies gained and lost in each cell of the chain, defined by Eqs. (3) and (4). The physical meaning of Eq. (7′) is quite clear: The averaged rate of change of the soliton's energy is equal to the net energy $(E_{\rm in}-E_d)$ supplied at one cell times the number of cells traveled by the soliton in a unit of time. This interpretation imposes limitations on the averaging time when proceeding from the exact equation (7) to the averaged one (7′): As mentioned above, the time of averaging must be much larger than the soliton's size divided by its velocity, but much smaller than a characteristic time at which the velocity of the soliton varies. Recall that the fundamental assumption adopted above is that, owing to the smallness of the perturbing parameters α and ϵ, the velocity changes sufficiently slowly. In what follows, the velocity $V(t)\equiv d\xi/dt$ in Eq. (7′) will be realized as a slowly varying function of time, $\xi(t)$ being the coordinate of the center of the soliton [see Eq. (2a)].

For an investigation of the stability of the ac-driven motion, we assume that $V(t)$ is close to a resonant value V_N, i.e.,

$$\xi(t)=V_N t+(a/2\pi N)\phi(t) \ , \tag{8}$$

where the slowly varying quantity $\phi(t)$ plays the role of the phase shift between the ac drive and the nearly periodic process of passage of the soliton through the cells of the chain. Inserting Eq. (8) into Eq. (4), one can perform the integration, treating $V=V_N+(a/2\pi N)\dot{\phi}$ as a constant:

$$E_{\rm in}(V)=\epsilon[E_1(V)\cos\phi+E_2(V)\sin\phi] . \quad (9)$$

The coefficients $E_1(V)$ and $E_2(V)$ depend on the particular form of the unperturbed soliton $Y_n(t)$, and on the values q_n. Finally, insertion of Eqs. (9) and (8) into Eq. (7′) yields the evolution equation for $\phi(t)$:

$$E'(V_N)\ddot{\phi}=\epsilon\omega[E_1(V_N)\cos\phi+E_2(V_N)\sin\phi] -\omega E_d(V_N)-[F_1(\phi)+F_2(\phi)]\dot{\phi} , \quad (10)$$

where

$$F_1(\phi)\equiv E_d(V_N)-\epsilon[E_1(V_N)\cos\phi+E_2(V_N)\sin\phi] , \quad (11)$$

and

$$F_2(\phi)\equiv\{E_d'(V_N)-\epsilon[E_1'(V_N)\cos\phi+E_2'(V_N)\sin\phi]\}V_N . \quad (12)$$

Equation (10) may be regarded as the equation of motion for a particle with the coordinate $\phi(t)$ and the mass $E'(V_N)$ in the biased harmonic potential

$$u(\phi)=\omega\epsilon[-E_1(V_N)\sin\phi+E_2(V_N)\cos\phi]+\omega E_d(V_N)\phi , \quad (13)$$

in the presence of two friction forces, with the position-dependent friction coefficients given by Eqs. (11) and (12). The potential (13) has two different types of equilibrium positions (maxima and minima), provided that the drive amplitude exceeds the threshold value

$$\epsilon_{\rm thr}=\omega E_d(V_N)[E_1^2(V_N)+E_2^2(V_N)]^{-1/2} . \quad (14)$$

The equilibrium corresponding to a local maximum of the potential (13) is unstable. The stability of the local minimum is determined by the friction forces. The friction coefficient (11) vanishes at the equilibrium positions, i.e., it does not affect the stability. Therefore, the potential minimum is stable if the friction coefficient (12) is positive at this point.

The stable stationary point of Eq. (10) corresponds to the stable ac-driven propagation of the soliton in the model (1). Let us now consider particular examples: the TL soliton (2) and the one in the HB string. In both cases, the simplest nontrivial coupling to the ac field corresponds to alternating positive and negative charges, i.e., $q_n=(-1)^n$ in Eq. (1) [8]; hence the size a of the elementary cell is double the chain spacing. The corresponding spectrum (6) of the resonant velocities decays into two branches (note that the velocity of the TL soliton must be limited from below, $|V_N|>1$):

$$V_N=\omega a/4\pi N ,$$

$$V_N=\omega a/2\pi(2N+1) . \quad (15)$$

Only the branch (15) admits a real input of energy (cancellation of the cell-to-cell oscillations) from the ac field.

For the TL model, straightforward calculations yield:

$$E_d=\alpha|\xi|^{-1}[(1+\xi^2)\ln(\xi^{-2})-2(1-\xi^2)] , \quad (16)$$

$$E_1=0 , \quad E_2=-\pi\,{\rm csch}[\pi\omega|\xi|/(1-\xi^2)] , \quad (17)$$

where ξ is the same as in Eqs. (2). Insertion of Eqs. (16) and (17) into Eq. (14) yields $\epsilon_{\rm thr}$ in an implicit form. Note that, in order to proceed to the continuum limit, one must take a very broad soliton, i.e., the one with $1-\xi^2\ll1$ (which implies $|V|-1\ll1$). According to Eqs. (17) and (14), in this case, $\epsilon_{\rm thr}$ is exponentially large, and the effect vanishes ($\epsilon_{\rm thr}=\infty$) in the continuum limit.

In the opposite limit $|V_N|\gg1$, i.e., $\omega\gg2\pi N$ [see Eq. (15)], the soliton is very narrow. In this case, $\epsilon_{\rm thr}$ can be found explicitly

$$\epsilon_{\rm thr}\approx2\alpha\omega\ln[\omega/\pi(2N+1)] . \quad (18)$$

At last, inserting Eqs. (16) and (17) into the stability condition $F_2>0$ [see Eq. (12)], one can see that the driven propagation of the TL soliton is unstable at $|V_N|-1\ll1$, and is stable at $|V_N|\gg1$.

The general analysis developed above, as well as its application to the TL soliton, implied that we dealt with the underdamped situation, when the friction and driving terms in Eq. (1) were treated as small perturbations. This approximation applies provided $\alpha\ll V/a$. With regard to Eq. (6), this applicability condition amounts to

$$\omega\gg2\pi N\alpha . \quad (19)$$

A more difficult problem is to find an ac-driven soliton in the *overdamped* case, when the friction term is of the same order or larger than the inertia term. In this case, the spectrum of the propagation velocities is given by the same equation (6), which has a purely kinematic origin and does not depend on details of the dynamics. The main difficulty is to guess the form of the soliton, since a soliton solution of the unperturbed model ($\alpha=\epsilon=0$) is no longer relevant as the zeroth approximation. Generally speaking, this problem can only be solved numerically. However, one can find a straightforward solution for the simplest model mentioned above, i.e., the HB string with $q_n=(-1)^n$. Elementary analysis of the equation of motion for a bead yields the drive's threshold amplitude, which proves to be independent of N, cf. Eq. (18):

$$\epsilon_{\rm thr}=\tfrac{1}{4}\alpha\omega a . \quad (20)$$

The expression (20) for the threshold is valid regardless of the condition (19), i.e., both for the underdamped and overdamped cases.

III. CNOIDAL WAVES

Let us proceed to periodic arrays of solitons, i.e., cnoidal waves (CW). Note that a soliton in a closed chain with periodic boundary conditions is equivalent to a CW. The straightforward analysis of the energy balance shows that the ac drive may support the propagation of the CW if its velocity takes the same resonant values (6), while its wavelength λ takes the values

$$\lambda=Ma/N , \quad (21)$$

where M is a new independent integer. The ac-driven CW cannot transfer mass, i.e., the mean velocity of each particle in the chain through which the CW travels is zero. The reason is that the ac drive in Eq. (1) may provide the energy input, but not the input of momentum. If the CW is regarded as a periodic array of solitons with the spacing λ, each soliton passing a particle shifts it by Δy [see, e.g., Eq. (2c)], so the mean velocity of the particle has to be $\bar{v}=(V/\lambda)\Delta y$; however, it is compensated by the motion of the chain as a whole with the constant velocity $-\bar{v}$.

Further analysis of the energy balance for the CW closely follows that for one soliton. In particular, the ac drive may support the propagation of the CW if its amplitude exceeds a certain threshold $\epsilon_{\rm thr}$ depending on both resonance indices N and M [see Eqs. (6) and (21)].

As an example, let us again take the TL model with $q_n=(-1)^n$. The resonant velocities of the driven CW are given by Eq. (15) with $a=2$. The unperturbed CW solution is [1]

$$\dot{y}_n=\mp 2KV\lambda^{-1}[Z(2\lambda^{-1}K(n\mp Vt)) -Z(2\lambda^{-1}K(n+1\mp Vt))] , \quad (22)$$

where Z is the Jacobi ζ function. The corresponding elliptic modulus k is related to λ and V by the dispersion equation

$$4(V/\lambda)^2[1/\mathrm{sn}^2(2K/\lambda)-1+E/K]=1 . \quad (23)$$

Here K and E are the complete elliptic integrals of the first and second kinds, respectively and sn is the Jacobian elliptic function. Inserting Eqs. (15) and (21) into Eq. (23) yields an equation to determine k as a function of ω, N, and M. This equation becomes relatively simple at $M=1$: $E(k)K(k)=(\pi/\omega)^2$. Further analysis demonstrates that in the case $\cos[\pi(2N+1)/M]<0$ the equation for k has one root in the region

$$\omega<\omega_{\max}=2M|\sin[\pi(2N+1)/2M]| , \quad (24)$$

and no root if $\omega>\omega_{\max}$. In the opposite case, there exists some $\tilde{\omega}_{\max}>\omega_{\max}$ such that the equation for k has one root for $\omega<\omega_{\max}$, no root for $\omega>\tilde{\omega}_{\max}$, and two roots for $\omega_{\max}<\omega<\tilde{\omega}_{\max}$. The existence of the two roots implies the hysteresis: At the same driving frequency ω, two different CW's with the same velocities and wavelengths may be supported.

The threshold value of the drive's amplitude can be found implicitly for $M=1$,

$$\epsilon_{\rm thr}=(\alpha/\pi)K^{3/2}E^{-1/2}[(2-k^2)K-2E]\sinh(\pi K'/K) , \quad (25)$$

where $K'\equiv K[(1-k^2)^{1/2}]$. As follows from Eqs. (24) and (25), it is easiest to drive the CW with $M=1$ in the case $k\ll 1$, i.e., when ω is close to $\omega_{\max}\equiv 2$: $\epsilon_{\rm thr}\approx 2\alpha[2(2-\omega)]^{1/2}$. For arbitrary M, the estimate $\epsilon_{\rm thr}\sim\alpha|\omega_{\max}-\omega|^{-(M-2)/2}$ can be obtained in the same limit.

IV. CONCLUSION

The existence of the stable ac-driven regimes of propagation of solitons in the damped chains opens the way for interesting new problems. One of them is the collision problem. Indeed, since the ac-driven solitons corresponding to different N in Eq. (6) have different velocities, they may collide. The collision can easily be analyzed in the underdamped case, provided the corresponding unperturbed model is exactly integrable (e.g., the TL model). As is well known [9], in the exactly integrable models the soliton-soliton collision is purely elastic, its only result being some shift $\Delta\zeta$ of the position of each soliton (the expression for $\Delta\zeta$ in the TL model can be found in Ref. [1]). In terms of the effective potential (13), this implies, in the first approximation, that the collision results in the instantaneous shift of the phase ϕ by $\Delta\phi=(2\pi N/a)\Delta\zeta$. Similarly, the collision of two solitons in the HB string shifts the phase of each soliton by $\Delta\phi=\pi$. Next, analyzing the shape of the potential (13), it is easy to see whether the shifted position remains trapped by the potential or escapes. In the former case, the corresponding ac-driven soliton survives the collision, while in the latter case it is kicked out from the driven state, and will eventually decay under the action of the dissipation. It is possible that both solitons, one of them, or none will survive the collision (formulas used to discern between the three cases prove to be rather cumbersome). In particular, the collision is almost always destructive for the soliton if ϵ lies slightly above the corresponding $\epsilon_{\rm thr}$, and it is almost always nondestructive if ϵ lies well above $\epsilon_{\rm thr}$. As concerns physical applications, the results obtained in the present work can be applied to predict propagating solitary-wave excitations, supported by the ac electric field, in ion-doped polymer molecules, and in quasi-one-dimensional ionic lattices.

Finally, it is relevant to mention that the ac drive can support propagation of solitons not only in the TL-like models, but also in those of the Frenkel-Kontorova type, i.e., chains of interacting particles placed in a periodic substrate potential. This problem has recently been considered elsewhere [10]. A preliminary version of the present work has been published in Ref. [11].

Note added in proof. Recently, direct numerical simulations of the ac-driven damped TL model, with $q_n=(-1)^n$, have corroborated the existence of the stable regimes of propagation of the driven solitons in the model with the periodic boundary conditions [T. Kuusela, J. Hietarinta, and B. A. Malomed (unpublished)].

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