Small-signal theory of pulse propagation in free-electron lasers

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We reconsider the theory of pulse propagation in free-electron lasers operating with arbitrary gain. We show that the dynamics can be accounted for by collective longitudinal excitations of supermode type and discuss their properties. We study the interplay between the optical-packet longitudinal dimension and the electron-beam energy spread, along with the induced gain depression. Semianalytical formulas for the gain, pulse length, etc., are presented.

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I. INTRODUCTION

The phenomenology connected to the pulse-propagation problem in free-electron lasers (FEL's) is rich and is one of the peculiar features of its dynamical behavior with respect to conventional lasers. As is well known, it was one of the puzzles contained in the first experimental results [1]. It soon became clear, however, that the so-called lethargy, with the consequent dependence of the gain and output laser power on the cavity mismatch, was just a by-product of the dynamical behavior of FEL's operating with short electron pulses.

The longitudinal FEL dynamics is a fairly well-established subject, both for single-pass and recirculated (storage-ring) operation. Within this framework a particular importance can be ascribed to the supermodes [2], which provide the proper expansion basis for the pulse FEL dynamics [3,4]. The supermodes are the eigenvectors of the FEL integral equation accounting for the field evolution after one cavity round trip and, from the physical point of view, can be understood as collections of longitudinal self-reproducing modes which, under some circumstances, can be expressed in terms of harmonic-oscillator functions (see Refs. [3,4]). The practical importance of supermodes as a tool to investigate the experimental results has been stressed in Ref. [5], where it was also pointed out that early predictions on, e.g., the operating laser bandwidth [6], were confirmed by the experiment. Although, as already stressed, longitudinal dynamics is now well understood, there is still the necessity of a simple semianalytical approach to the problem, capable of incorporating, with a relatively modest mathematical effort, as many features of an actual experimental configuration as possible. It would indeed be desirable to have, e.g., a gain formula accounting for energy spread and emittance degradation, nonlinear contributions, slippage, diffraction, focusing, etc. Partial attempts to include the above-quoted effects were accomplished in Ref. [7] and, within a more general framework, in Refs. [8,9].

In this paper we analyze the FEL small-signal pulse-propagation equation and pay particular attention to "collective longitudinal excitations" of supermode type. We study the FEL pulse evolution in both the hypotheses of quasicontinuous and bunched electron beams. We discuss the coupling mechanism between supermodes, and the reasons leading to gain depression, optical bunch dispersion, and longitudinal deceleration.

The paper is organized in four sections. In Sec. II we discuss the pulse problem in the quasicontinuous-electron-beam limit and study the amplification of an input mode-locked field, along with the modification induced by the FEL interaction. Particular attention is devoted to the gain depression due to the combined effects of the slippage and of the input optical bunch rms length. Furthermore, we discuss a simple argument leading to the understanding of the supermode gain correction. The case of finite-electron-beam limit is analyzed in Sec. III, where we discuss the quasiparticle nature of the collective longitudinal modes (CLM's) and the mode coupling induced by the electron bunch longitudinal shape. Section IV is devoted to concluding remarks where we clarify some physical aspects of the CLM dynamics and the role played by the electron-bunch structure in the high-gain correction.

II. FEL PULSE-PROPAGATION AND QUASICONTINUOUS-ELECTRON-BEAM LIMIT

The evolution of the optical field complex amplitude $a(Z,\tau)$ in a FEL operating with a longitudinally shaped electron beam is accounted for, in the small-signal regime and in the slowly-varying-amplitude approximation, by the equation [10]

$$\frac{\partial}{\partial\tau}a(Z,\tau) = -i\pi g_0 f(Z+\mu_E\tau)\int_0^\tau d\tau'\tau' e^{i\nu\tau'}a(Z+\mu_E\tau',\tau-\tau') , \tag{2.1}$$

where g_0 is the small-signal gain coefficient, ν is the gain detuning parameter [specifically defined as $\nu=2\pi N(\omega_0-\omega)/\omega_0$, $\omega_0=2\pi c/\lambda_0$], and with τ and Z representing the time and longitudinal coordinates, respectively, normalized to the interaction time and to the width σ_E of the input optical field. The parameter μ_E gives the ratio of the slippage distance $\Delta=N\lambda_0$ to σ_E (N is the number of undulator periods and λ_0 is the central emission frequency):

$$\mu_E=\frac{\Delta}{\sigma_E} . \tag{2.2}$$

Finally, the electron-beam distribution is described by the function $f(Z)$. Expanding the field $a(Z,\tau)$,

$$a(Z,\tau)=\sum_{n=0}^{\infty} a_n(\tau)u_n(Z) , \tag{2.3}$$

in terms of the Hermite function $u_n(Z)$ defined as [11]

$$u_n(Z)=\frac{1}{(n!2^n\sqrt{\pi})^{1/2}}H_n(Z)e^{-Z^2/2} , \tag{2.4}$$

with H_n being the nth Hermite polynomial, Eq. (2.1) turns into the following set of coupled equations for the coefficients $a_n(\tau)$ of the expansion (2.3):

$$\frac{d}{d\tau}a_n(\tau)=-i\pi g_0 \sum_{n'=0}^{\infty}\int_0^{\tau}d\tau'\tau' e^{i\nu\tau'} \times f_{nn'}(\tau,\tau')a_{n'}(\tau-\tau') , \tag{2.5}$$

with the initial conditions being specified by the input field through the overlap integrals:

$$a_n(0)=\int_{-\infty}^{+\infty}dZ\, a(Z,0)u_n(Z) . \tag{2.6}$$

The matrix elements $f_{nn'}$ defined as (for the details of the derivation, see the Appendix)

$$f_{nn'}(\tau,\tau')=\int_{-\infty}^{+\infty}dZ\, u_n(Z)f(Z+\mu_E\tau)u_{n'}(Z+\mu_E\tau') \tag{2.7}$$

represent a kind of current form factor and account for the coupling between the basis functions $u_n(Z)$, as a consequence of the slippage phenomenon, the electron-beam longitudinal distribution providing a further coupling source.

Equation (2.5) can be treated within the context of a perturbative analysis which uses the gain coefficient g_0 as an expansion parameter. Writing, indeed [12],

$$a_n(\tau)=\sum_{k=0}^{\infty} g_0^k a_n^{(k)}(\tau) \tag{2.8}$$

and inserting it into (2.5) yields the following recursive relations:

$$\frac{d}{d\tau}a_n^{(k)}(\tau)=-i\pi\sum_{n'=0}^{\infty}\int_0^{\tau}d\tau'\tau' e^{i\nu\tau'}f_{nn'}(\tau,\tau') \times a_{n'}^{(k-1)}(\tau-\tau') ,$$
$$a_n^{(-k)}(\tau)=0,\quad k=0,1,\ldots \tag{2.9}$$

with the initial conditions

$$a_n^{(k)}(0)=a_n(0)\delta_{k0} . \tag{2.10}$$

Consequently, the optical field can be expressed as a series of contributions at the various orders in g_0 as

$$a(Z,\tau)=\sum_{k=0}^{\infty} g_0^k a^{(k)}(Z,\tau) , \tag{2.11}$$

with

$$a^{(k)}(Z,\tau)=\sum_{n=0}^{\infty} a_n^{(k)}(\tau)u_n(Z) . \tag{2.12}$$

Correspondingly, the gain defined as

$$G(\nu)=\int_{-\infty}^{+\infty}|a(Z,1)|^2dZ-1 \tag{2.13}$$

(for the sake of simplicity we have assumed an input field normalized to unity) can be expressed in the form of a power series in g_0 with coefficients specified by the functions $a_n^{(k)}$, evaluated at $t=1$, according to

$$G(\nu)=\sum_{k,l=1}^{\infty} g_0^{k+l}\sum_{n=0}^{\infty}a_n^{(l)}(1)a_n^{*(k)}(1) . \tag{2.14}$$

In particular, for an initially Gaussian-shaped optical field, the above expression becomes

$$G(\nu)=g_0 2\,\mathrm{Re}a_0^{(1)}(1)+g_0^2\left[\sum_{n=0}^{\infty}|a_n^{(1)}(1)|^2+2\,\mathrm{Re}a_0^{(2)}(1)\right]+g_0^3\left[2\,\mathrm{Re}a_0^{(3)}(1)+2\,\mathrm{Re}\sum_{n=0}^{\infty}a_n^{(1)}(1)a_n^{*(2)}(1)\right]+O(g_0^4) , \tag{2.15}$$

where only terms up to g_0^3 have been explicitly reported.

As a preliminary analysis, let us consider a longitudinally uniform electron beam, which amounts to assuming

$$f(Z)=1 . \tag{2.16}$$

The matrix elements $f_{n,n'}(\tau,\tau')$ are then [13]

$$f_{nn'}(\tau)=\phi_n^{n'-n}\left[\left[\frac{\mu_E\tau}{\sqrt{2}}\right]^2\right] , \tag{2.17}$$

where $\phi_n^{n'-n}$ denotes the function

$$\phi_n^l(x^2)=\left[\frac{n!}{(n+l)!}\right]^{1/2}x^l e^{-x^2/2}L_n^{(l)}(x^2) , \tag{2.18}$$

thus providing the following explicit expression for the recursive relations (2.9):

$$\frac{d}{d\tau}a_n^{(k)}(\tau)=-i\pi\sum_{l=-n}^{\infty}\int_0^{\tau}d\tau' e^{i\nu\tau'}\phi_n^l\left[\left[\frac{\mu_E\tau'}{\sqrt{2}}\right]^2\right] \times a_{n+l}^{(k-1)}(\tau-\tau') , \tag{2.19}$$

$$a_n^{(-k)}(\tau)=0 . \tag{2.20}$$

At the zero order in g_0 we immediately recover the initial conditions

$$a_n^{(0)}(\tau)=a_n(0) \; . \tag{2.21}$$

As a consequence, assuming the initial field to be given by the basis function of zero order $u_0(Z)$, i.e.,

$$a(Z,0)=\frac{1}{\pi^{1/4}}\exp(-Z^2/2) \; , \tag{2.22}$$

the equation for the first-order coefficient $a_n^{(1)}(\tau)$ takes the simple form

$$\frac{d}{d\tau}a_n^{(1)}(\tau)=-\frac{i\pi}{\sqrt{n!}}\int_0^\tau d\tau'\tau' e^{i\nu\tau'}\left[-\frac{\mu_E\tau'}{\sqrt{2}}\right]^n \times\exp(-\mu_E^2\tau'^2/4) \; , \tag{2.23}$$

where the property of the Laguerre function

$$\phi_n^{-n}(x^2)=\frac{(-x)^n}{\sqrt{n!}}\exp(-x^2/2) \tag{2.24}$$

has been exploited.

Equation (2.23) can be formally integrated, thus giving

$$a_n^{(1)}(\tau)=-\frac{i\pi}{\sqrt{n!}}\left[-\frac{\mu_E}{\sqrt{2}}\right]^n\int_0^\tau d\tau'(\tau-\tau')\tau'^{n+1} \times e^{i\nu\tau'}e^{-\mu_E^2\tau'^2/4} \; . \tag{2.25}$$

Inserting the above expression into (2.20) relevant to $k=2$, after some algebra, we can specialize the second-order coefficient $a_n^{(2)}(\tau)$ as

$$a_n^{(2)}(\tau)=\frac{(-i\pi)^2}{12\sqrt{n!}}\left[-\frac{\mu_E}{\sqrt{2}}\right]^n\int_0^\tau d\tau'(\tau-\tau')\tau'^{n-3} \times e^{i\nu\tau'}e^{-\mu_E^2\tau'^2/4} \; . \tag{2.26}$$

It is easy to verify that kth-order function $a_n^{(k)}(\tau)$ can be given the explicit expression

$$a_n^{(k)}(\tau)=\frac{(-i\pi)^k}{k!}\frac{1}{(2k-1)!\sqrt{n!}}\left[-\frac{\mu_e}{\sqrt{2}}\right]^n \times\int_0^\tau d\tau'\tau'^{2k+n-1}(\tau-\tau')^k e^{i\nu\tau'}e^{-\mu_E^2\tau'^2/4} \; . \tag{2.27}$$

The above discussion clearly reveals that the slippage mechanism, accounted for in the evolution equation (2.1), by the parameter μ_E, provides the CLM coupling. In fact, starting, e.g., with the fundamental CLM, the FEL interaction turns into an excitation of all the other modes, whose contribution in modeling the field is expressed at the first order in g_0 by the coefficients $a_n^{(1)}(\tau)$. In particular, according to (2.14) at the lowest order in g_0 the gain is provided by the real part of $a_n^{(1)}(\tau)$ evaluated at $\tau=1$, i.e.,

$$G(\nu)\sim 2g_0\mathrm{Re}\,a_n^{(1)}(1) \; , \tag{2.28}$$

which therefore can be understood as a first-order gain function relevant to an initial optical field given by the nth-order basis function (2.3). Consequently, it is of some interest to consider in detail the first-order coefficients $a_n^{(1)}(\tau)$ as given by (2.25), which can be rewritten as

$$a_n^{(1)}(\tau)=\left[\frac{i\mu_E}{\sqrt{2}}\right]^n\frac{1}{\sqrt{n!}}\frac{\partial^n}{\partial\nu^n}G_1(\nu,\mu_E,\tau) \; , \tag{2.29}$$

where the function $G_1(\nu,\mu_E,\tau)$, explicitly given by

$$G_1(\nu,\mu_E,\tau)=-i\pi\int_0^\tau d\tau'\int_0^{\tau'}d\tau'' e^{i\nu\tau''}e^{-\mu_E\tau''^2/4} \; , \tag{2.30}$$

can be regarded as a first-order gain function relevant to a Gaussian-shaped input field.

Just to share the language of Ref. [14], let us rewrite the above equation as

$$a_n^{(1)}(\tau)=\left[-\frac{\mu_E}{\sqrt{2}}\right]^n\frac{1}{\sqrt{n!}}G_{n+1}(\nu,\mu_E,\tau) \; , \tag{2.31}$$

with $G_{n+1}(\nu,\mu_E,\tau)$ being defined in terms of the nth derivative of G_1 with respect to ν, i.e.,

$$G_{n+1}(\nu,\mu_E,\tau)=(-i)^n\frac{\partial^n}{\partial\nu^n}G_1(\nu,\mu_E,\tau) \; . \tag{2.32}$$

For $\mu_E=0$, the expression (2.30) turns into the "one-dimensional" gain function, the parameter μ_E playing within the present context a role similar to that of the inhomogeneous broadening parameter μ_ϵ (linked to the e-beam energy spread) [15].

In Fig. 1 we show the gain curve for $g_0=0.1$ and different values of μ_E. The plots are relevant to Eq. (2.28). The validity of such results has been checked with a full numerical integration of the pulse-propagation equation and for this range of parameters both procedures yield the same answer. The parameter μ_E produces a broadening and a reduction of the gain curve. A more detailed discussion of this topic is carried out in Ref. [16] within the context of an analysis of the violations to Madey's theorem, induced by the transverse and longitudinal dynamics.

The value of the resonance parameter $\nu_{\max}(\mu_E)$, where the gain is maximum and the corresponding values of the gain $G_{\max}(\mu_E)$ as functions of μ_E are reported for $g_0=0.1$ in Figs. 2 and 3, respectively. The behavior of $\nu_{\max}(\mu_E)$ and $G_{\max}(\mu_E)$ in the range of μ_E shown in the figures can be accounted for by the following functional relations, obtained from a numerical fit:

$$\nu_{\max}(\mu_E)\cong 2.6+1.2\mu_E^{1.7} \; , \qquad G_{\max}(\mu_E)\cong\frac{0.085g_0}{1+0.086\mu_E^2} \; . \tag{2.33}$$

According to the above-noticed correspondence between μ_E and μ_ϵ, the expressions (2.33) are in good agreement with the numerical fit, provided in Refs. [14] and [17] for $G_{\max}$ and $\nu_{\max}$ in terms of the parameters μ_ϵ. Just to visualize the effect of the higher-order terms in the expression (2.15), we have reported in Fig. 4 the gain curves for $g_0=5.0$ and μ_E ranging from 0 to 1. The correspondence of μ_E to μ_ϵ is further confirmed by the general trend of the gain curves, which by increasing μ_E recovers the antisymmetric shape, thus indicating that the param-

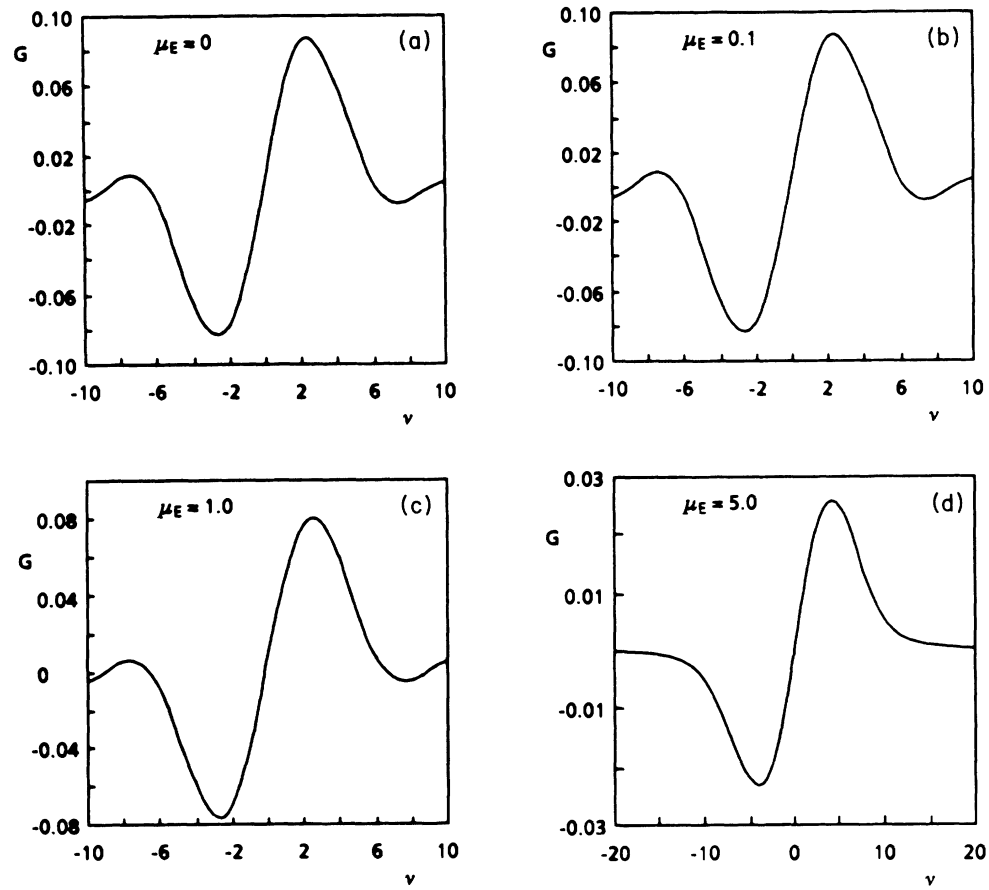

FIG. 1. Gain function vs ν for different values of μ_E.

eter μ_E reduces the contribution of higher-order terms in (2.15), which destroy the antisymmetric behavior of the gain curves.

Let us consider now the effect of the FEL interaction on the "optical field." We assume as indicative parameters, the "center of gravity" of the optical packet defined as

$$\bar{Z}(\tau) \equiv \frac{\int_{-\infty}^{+\infty} Z|a(Z,\tau)|^2 dZ}{\int_{-\infty}^{+\infty} dZ|a(Z,\tau)|^2} \tag{2.34}$$

and the standard deviation given by

$$\sigma(\tau) \equiv \frac{\int_{-\infty}^{+\infty} dZ(Z-\bar{Z})|^2 a(Z,\tau)|^2}{\int_{-\infty}^{+\infty} dZ|a(Z,\tau)|^2} . \tag{2.35}$$

Assuming as input field the lowest-order H function $u_0(Z)$, according to (2.34) and (2.11) we can write $\bar{Z}$ and σ to the first order in g_0 as

$$\bar{Z}(\tau) \cong \sqrt{2} g_0 \mathrm{Re} a_1^{(1)}(\tau) = -g_0 \mu_E \mathrm{Re} G_2(\nu, \mu_E, \tau) \tag{2.36}$$

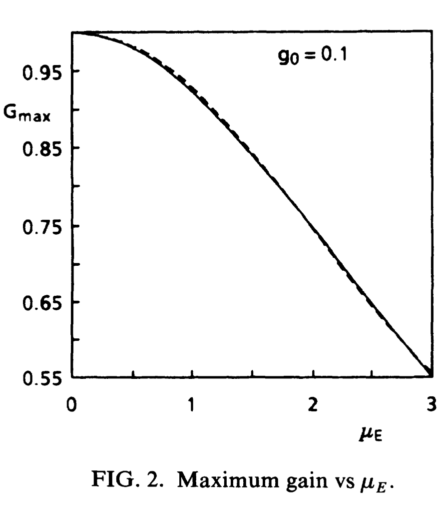

FIG. 2. Maximum gain vs μ_E.

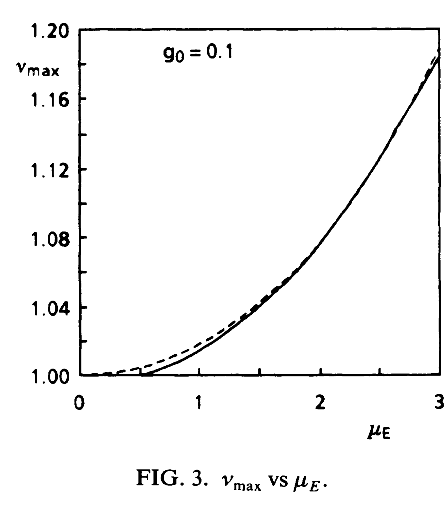

FIG. 3. ν_{max} vs μ_E.

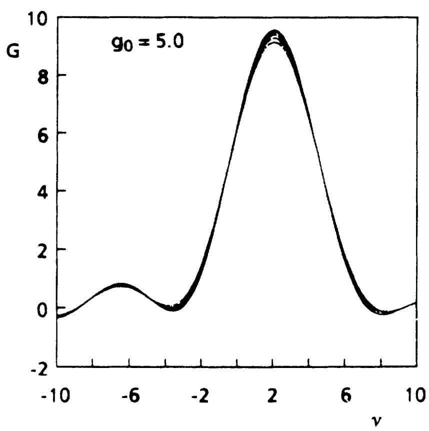

FIG. 4. Gain vs μ_E with values ranging from 0 to 1.

and

$$\sigma(\tau) \cong \sigma_0[1+g_0\sqrt{2}\mathrm{Re}a_2^{(1)}(\tau)]$$
$$=\sigma_0\left[1+g_0\frac{\mu_E^2}{2}\mathrm{Re}G_3(\nu,\mu_E,\tau)\right] \tag{2.37}$$

with σ_0 denoting $\sigma(O)$, which for the case we are considering is $\sigma(O)=1/\sqrt{2}$.

An accurate analysis of the trend of the functions G_2 and G_3 vs ν at $\tau=1$ allows us to state that, at the maximum gain, $\overline{Z}(1)$ is negative, thus indicating that the gain acts as the propagation through a medium with refractive index greater than unity. It is easy indeed to deduce from (2.36) the phase velocity as

$$\beta(\tau)=1+\frac{\sigma_E}{L_u}\overline{Z}(\tau)=1-g_0\frac{\Delta}{L_u}\mathrm{Re}G_2(\nu,\mu_E,\tau) , \tag{2.38}$$

which at the maximum gain ($\mathrm{Re}G_2>0$) is less than 1 (the pulse is delayed with respect to an ideal packet, traveling at the light velocity). Therefore, the FEL refractive index can be given by the following expression at the lowest order in g_0:

$$n(\tau)=\frac{1}{\beta(\tau)}\sim 1+\frac{g_0\Delta}{L_u}\mathrm{Re}G_2(\nu,\mu_E,\tau) , \tag{2.39}$$

which contains the dependence on ν and μ_E.

Correspondingly, $\sigma(1)$ is greater than σ_0; as a consequence, the "propagation" occurs "through a dispersive medium."

In Figs. 5(a) and 5(b), $\overline{Z}(1)$ and $\sigma(1)$ are, respectively, reported as functions of ν for $g_0=0.1,0.2,0.5$ and μ_E ranging from 0 to 1. Figure 6 shows the behavior of $\overline{Z}(1)$

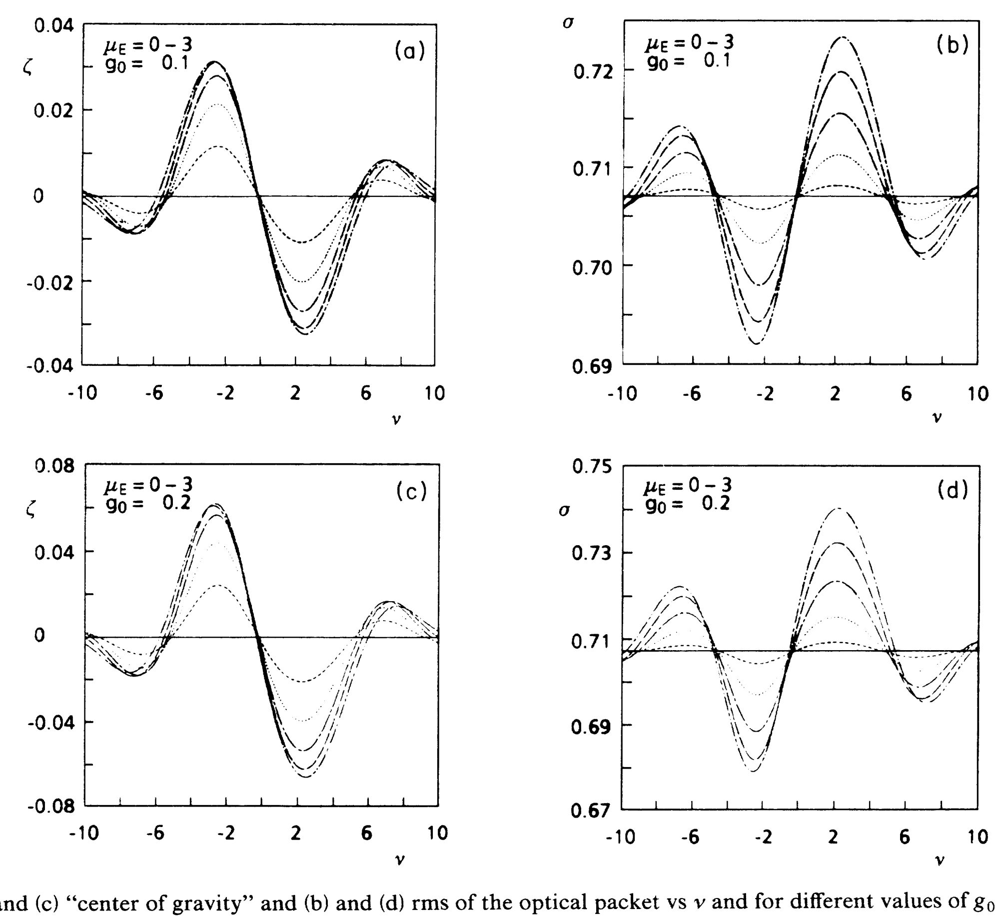

FIG. 5. (a) and (c) "center of gravity" and (b) and (d) rms of the optical packet vs ν and for different values of g_0. The values of μ_E are 0, 0.5, 1, 2, and 3. The curves with largest amplitudes have lower values of μ_E.

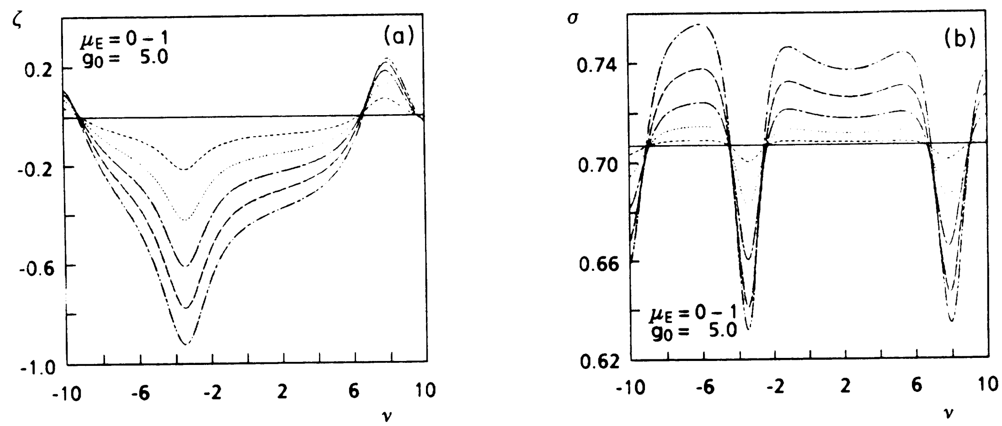

FIG. 6. Same as in Fig. 5 for different values of μ_E and for $g_0=5$. The values of μ_E are 0, 0.4, 0.6, 0.8, and 1. The curves with largest amplitudes have lower values of μ_E.

and $\sigma(1)$ as functions of ν for $g_0=5$ and $\mu_E=0\div 1$. The different shapes between the curves in Figs. 5 and 6 are due to the different values of the small-signal gain parameter.

The above analysis allows one to dispel a "long-standing mystery," namely, that relevant to the dependence of the supermode gain on the coupling parameter

$$\mu_c=\frac{\Delta}{\sigma_z}, \tag{2.40}$$

where σ_Z is the rms length of the electron bunch. According to Ref. [18] the spatial width of the fundamental supermode is just

$$\sigma_E \sim 0.5\sqrt{\Delta\sigma_Z}\ . \tag{2.41}$$

The corresponding σ_E is thus given by

$$\mu_E \sim 2\sqrt{\mu_c}\ . \tag{2.42}$$

Inserting therefore the above expression in the second of Eqs. (2.33) we find

$$G_{\max}(\mu_c)\cong 0.85\frac{g_0}{1+\frac{1}{3}\mu_c}, \tag{2.43}$$

which is the gain expression containing the supermode correction and quoted in Ref. [18]. The above considerations offer the possibility of including the effect of gain depression due to the energy spread in a rather simple way. Since, as already stressed, the effects of energy spread and longitudinal mode width combine quadratically, we get the following simple relation including both energy spread and coupling parameter:

$$G_{\max}(\mu_c,\mu_\epsilon)\cong\frac{0.85g_0}{1+1.7(\frac{1}{5}\mu_c+\mu_\epsilon^2)}\ . \tag{2.44}$$

As a further example of the flexibility of the so-far-developed analysis, we quote the possibility of getting from Eq. (2.44) a gain optimization criterion, for FEL's operating, e.g., with rf linac. In this case the longitudinal phase-space area (σ_ϵ is the e-beam rms relative energy spread)

$$A_l=\sigma_Z\sigma_\epsilon \tag{2.45}$$

is a conserved quantity. The gain coefficient g_0 is proportional to the peak current, which is in turn linked to the average current $\bar{I}$ and to the duty cycle δ by the relation

$$\hat{I}=\frac{\bar{I}}{\delta}\ . \tag{2.46}$$

The duty cycle is the ratio between the electron-bunch length σ_Z and the bunch-bunch distance L_l fixed by the rf period ($L_l=2\pi c/\omega_{\rm rf}$); for this reason we have

$$\hat{I}=\hat{I}_\Delta\mu_c,\quad \hat{I}_\Delta=\frac{L_l}{\Delta}\bar{I}, \tag{2.47}$$

where $(L_l/\Delta)/\bar{I}$ is the peak current corresponding to a bunch, whose length is equal to the slippage. According to Eqs. (2.47) and (2.45), the gain dependence can be parametrized in terms of the coupling parameter only as follows:

$$G_{\max}\simeq 0.85g_{0,\Delta}\frac{\mu_c}{1+1.7[\frac{1}{5}\mu_c+16(A_l^2/\lambda^2)\mu_c^2]}, \tag{2.48}$$

where $g_{0,\Delta}$ is the gain coefficient corresponding to $\hat{I}_\Delta$.

The last equation holds true, within the framework of low-gain approximation. The value of μ_c which maximizes (2.48) can be found immediately, thus getting

$$\mu_c^*=0.19\frac{\lambda}{A_l}, \tag{2.49}$$

which corresponds to a bunch length and to an energy spread, respectively, given by

$$\sigma_Z^*\sim 5(NA_l),\quad \sigma_\epsilon^*\sim\frac{1}{5N}, \tag{2.50}$$

which are within the values attainable with a conventional rf linac [19]. Equation (2.48) contains further important information and in fact it states that with increasing μ_c (and thus with decreasing σ_Z) the gain does not reach a value which is just proportional to the charge in the bunch, but starts to decrease when the effect due to the energy spread becomes dominant.

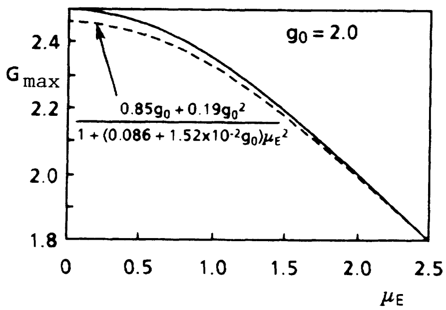

FIG. 7. Maximum gain vs μ_E (continuous line) and fit curve (dotted line).

We have so far discussed simple gain formulas which are capable of including contributions up to the lowest order in σ_0. We want to emphasize that high-gain effects can be accounted for, still within the framework of easily manageable expressions. Inspecting, e.g., Fig. 7 we can state that the gain dependence on μ_E (valid for $0.5 < g_0 < 4$) can be cast in the form

$$G_{\max}(\mu_E, g_0) = \frac{0.85 g_0 + 0.19 g_0^2}{1 + (0.086 + 1.52\times 10^{-2} g_0)\mu_E^2} . \quad (2.51)$$

The above relation also allows one to get the high-gain correction for the supermode operations. According to Eqs. (2.51) and (2.42), maximum gain scales vs μ_c and g_0 as ($\mu_c \leq 1$),

$$G_{\max}(\mu_c, g_0) \cong g_0 \frac{0.85}{(1+\mu_c/3)} + g_0^2 \frac{0.19}{(1+\mu_c/3)^2} . \quad (2.52)$$

The above relations must be taken as rough indications of the FEL gain scaling versus the relevant parameters; they provide, however, clear examples of the possibility of combining numerical and analytical results to get simple, insightful, and useful expressions to be exploited to design FEL devices.

III. LONGITUDINALLY SHAPED ELECTRON BUNCHES AND COLLECTIVE LONGITUDINAL MODE DYNAMICS

In the preceding section we have discussed the FEL optical-field evolution, using an expansion on a harmonic-oscillator basis. The analysis developed so far has been specialized to the case of an electron beam whose longitudinal shape can be considered as almost continuous. This approximation holds true when the slippage distance is negligibly small with respect to the electron-bunch length (typically short-wave storage-ring FEL's). Furthermore, the analysis of the preceding section has elucidated what is the gain of a continuous-electron-beam FEL amplifying a mode-locked input optical beam and what is the interplay with the inhomogeneous broadening gain reduction induced by the electron energy spread. On the other hand, the role played by a noncontinuous bunch distribution has been touched on, discussing the gain of the fundamental supermode. In some previous works [3], it has been shown that the electron-beam distribution acts like a potential, specifying the form and the interaction of the CLM which can also be viewed as a kind of quasiparticle. The above point of view can be further supported by the following rather simple examples. Going back to Eq. (2.5) and making the low-gain approximation, namely, neglecting the τ' dependence of a_n in its right-hand side and then expanding the integral in Eq. (2.7) up to the first order in μ_E, assuming that $f(Z)=1$, we get [20]

$$i\frac{d}{d\tau}a_n = \pi g_0 F(\nu,\tau) a_n - i\frac{\pi g_0}{\sqrt{2}}\mu_E \frac{\partial}{\partial\nu}F(\nu,\tau) \times(\sqrt{n+1}a_{n+1} - \sqrt{n}\,a_{n-1}) , \quad (3.1)$$

where

$$F(\nu,\tau) = \int_0^\tau d\tau' \tau' e^{i\nu\tau'} . \quad (3.2)$$

Equation (3.1) is a kind of Schrödinger equation that can be derived from a Hamiltonian, which in the second quantization formalism is written as

$$\hat{H} = \pi g_0 F(\nu,\tau)\hat{1} - i\frac{\pi g_0}{\sqrt{2}}\mu_E \frac{\partial}{\partial\nu}F(\nu,\tau)(\hat{A} - \hat{A}^+) , \quad (3.3)$$

where $\hat{A}, \hat{A}^+$ are annihilation and creation operators. [It is to be stressed that we are just borrowing the formalism from quantum mechanics but we are not claiming real quantum effects (depending on the Planck constant) in FEL theory.] The above Hamiltonian has the typical structure of that accounting for the coupling of a quantized electromagnetic field with an external classical dipole, which in this case is provided by

$$d = \frac{\pi g_0}{\sqrt{2}}\mu_E \frac{\partial}{\partial\nu}F(\nu,\tau) . \quad (3.4)$$

The above result, albeit of limited practical interest, yields at least two important conclusions: (a) It justifies the interpretation of the CLM as quasiparticles; and (b) it allows an immediate link with other problems in quantum optics such as, e.g., the generation of coherent states. This last point can be better understood noticing that the solution of Eq. (3.1) can be written in the form [21] $[a_n(O)=\delta_{n,0}]$

$$a_n(\tau) = \exp\left[-i\pi g_0 \int_0^\tau d\tau' F(\nu,\tau')\right] \times\exp\left[-\frac{1}{2}\left[\frac{\pi}{\sqrt{2}}g_0\mu_E \frac{\partial}{\partial\nu}\int_0^\tau d\tau' F(\nu,\tau')\right]^2\right] \times\frac{1}{\sqrt{n!}}\left[\frac{\pi}{\sqrt{2}}g_0\mu_E\frac{\partial}{\partial\nu}\int_0^\tau d\tau' F(\nu,\tau')\right]^n \quad (3.5)$$

and recognizing that the "probability amplitudes" $|a_n|^2$ yield a Poisson distribution. Accordingly, the average number of excited CLM's can be calculated as follows [22]:

$$\bar{n} = \sum_{n=0}^{\infty} n|a_n|^2 \cong \exp[g_0 \mathrm{Re}G_1(\nu)] . \quad (3.6)$$

The statistical behavior of CLM's induced by Eq. (3.1) is similar to that of coherent Glauber states. We can indeed easily prove that, within the framework of the approximations leading to Eq. (3.1), the "state" $a(Z,\tau)$ is an eigenvalue of the annihilation operator (see the concluding section for further remarks). It is clear that, keeping further terms in the expansion (2.7), one gets equivalent Hamiltonians of the type (3.2) involving nonlinear terms in the creation and annihilation operators and thus justifying the possibility of non-Poissonian CLM statistics. Let us now consider the case of nonconstant electron-beam distribution $f(Z)$ and assume that it has a parabolic shape, namely.

$$f(Z)=1-\frac{Z^2}{2\sigma^2}\ , \tag{3.7}$$

where $\sigma\equiv\sigma_z/\sigma_\epsilon$. The equation of motion for the time-dependent coefficients $a_n(\tau)$ now becomes

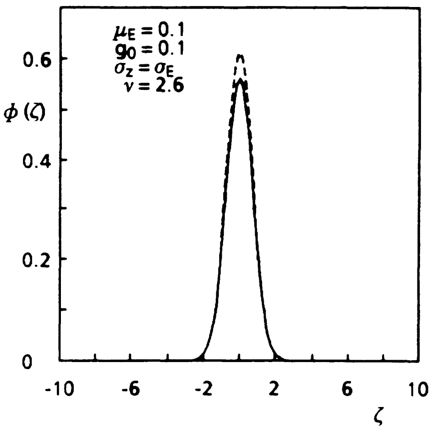

FIG. 8. Output (dotted line) and input (solid line) optical field distribution vs z for $\nu=\nu_{\max}$.

$$i\frac{d}{d\tau}a_n=\pi g_0\left[1-\frac{1}{4\sigma^2}\right]F(\nu,\tau)a_n-\frac{\pi g_0}{2\sigma^2}F(\nu,\tau)na_n-i\frac{\pi g_0}{\sqrt{2}}\left[\frac{\partial}{\partial\nu}F(\nu,\tau)\right]\mu_E(\sqrt{n+1}a_{n+1}-\sqrt{n}\,a_{n-1})$$
$$-\frac{\pi g_0}{4\sigma^2}F(\nu,\tau)[\sqrt{(n+1)(n+2)}a_{n+2}+\sqrt{n(n-1)}a_{n-2}]\ , \tag{3.8}$$

where it has been assumed that $\mu_c\ll\sigma$, which amounts to saying $\Delta\ll\sigma_Z$ and $\sigma_E\ll\sigma_Z$ (the so-called long-bunch approximation).

It is clear that the parabolic shape induces coupling between modes which are not the nearest neighbors only. It is therefore clear that, owing to the further CLM coupling induced by the longitudinal current shape, the FEL gain will experience a further reduction, which is in turn linked to the electron-bunch length. It is furthermore evident that the number of CLM's involved in the interaction becomes larger and larger with decreasing electron-bunch length σ_Z (or better when $\sigma\ll1$). The above point can be understood analytically. It can be shown (see the Appendix) that in the limit $\sigma_E\ll\sigma_Z$ the current form factor $f_{n,m}(\tau)$ can be cast, to lowest order in σ, in the form

$$f_{n,m}(\tau)\cong(-)^{n+m}\sqrt{2\pi}\sigma u_n(\mu_E\tau)u_m(\mu_E(\tau-\tau'))\ . \tag{3.9}$$

Accordingly, it can be shown that at the lowest order in g_0 and σ, the gain for an input Gaussian field, scales as

$$G(\nu,\sigma,\mu_E)\cong\frac{0.85g_0\sqrt{2\sigma}}{(1+0.086\mu_E^2)}\ , \tag{3.10}$$

thus indicating that, for very short electron bunches, the gain is just proportional to the charge in the bunch. (Here we have assumed a very short input bunch and we did not include the effect of the energy spread.)

After the above discussion aimed at clarifying the role played by the various physical quantities we comment on some numerical results.

In Fig. 8 we show the low-gain evolution for an input Gaussian field undergoing a FEL interaction with $g_0=0.1$ (the other relevant parameters are quoted in the

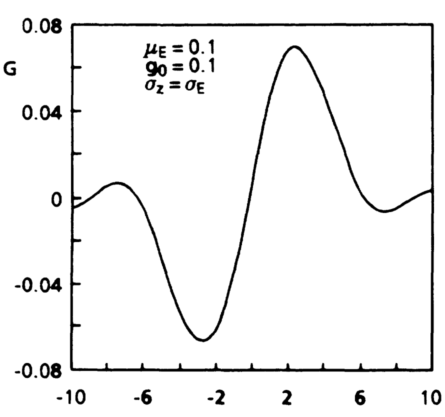

FIG. 9. Gain vs ν for a z-shaped e beam.

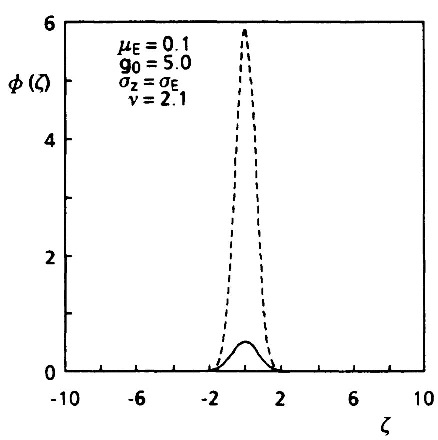

FIG. 10. Same as in Fig. 8 for larger g_0.

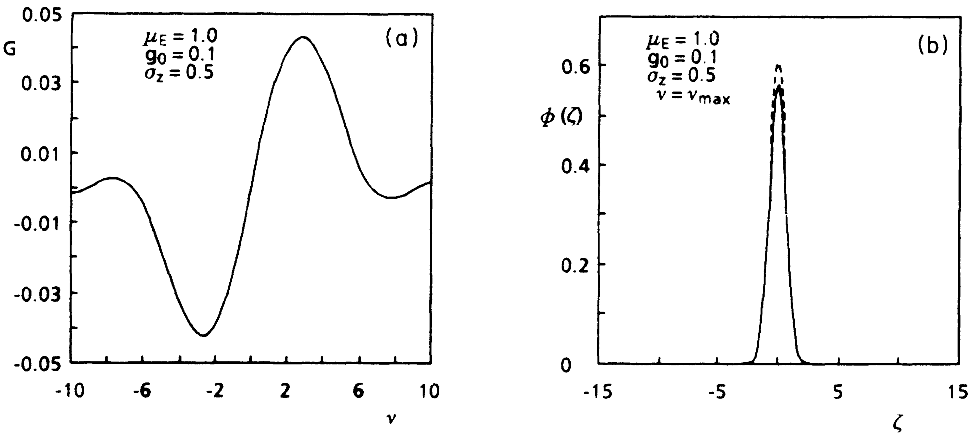

FIG. 11. Gain curve vs ν for $\mu_E=0.1$ and $\sigma_z=0.5$ (a) output (dotted line) and input (solid line) optical field distribution vs z.

figure caption). It is evident that the packet undergoes an amplification and a shift back of the centroid (although very tiny) due to the lethargy. In Fig. 9 the relevant gain curve is reported and a significant reduction of the maximum is shown, due to the combined effect of the finiteness of the electron and input field bunches. An example of the laser operation in the intermediate-gain region is provided in Fig. 10 relevant to $g_0=5$. Owing to the larger gain, the packet shifting back due to the lethargy is more evident. Figures 11 and 12 are relevant to a physical situation in which the optical bunch is much shorter than the input optical field. It is easy to realize that the maximum gain of Fig. 11 scales according to Eq. (3.10), while the maximum gain of Fig. 12 can be calculated from the simple relation

$$G_{max} \cong 0.85\tilde{g}_0 + 0.19\tilde{g}_0^2 , \tag{3.11}$$

where the effective gain coefficient is defined as

$$\tilde{g}_0 = \sqrt{2} g_0 \sigma . \tag{3.12}$$

In this section we have developed some considerations which may be helpful to understanding the FEL evolution in the small-signal pulsed regime. Use has been made of rather simple analytical and numerical tools. The physical effects underlying the combined dynamics of pulsed electron beams and input optical fields have been partially elucidated. Further comments will be presented in the next section.

IV. CONCLUDING REMARKS

In this paper we have discussed the role played by the CLM in the FEL longitudinal dynamics. We have seen that this type of mode behaves as a quasiparticle and that typical concepts of quantum optics can be used to understand their relevant phenomenology. Collective longitudinal modes can, in fact, undergo processes like squeezing, antibunching, and so on. The above language has not been used just to establish a formal, albeit useful, analogy with another field in physics, but to stress physically pregnant points. Although the above ideas are

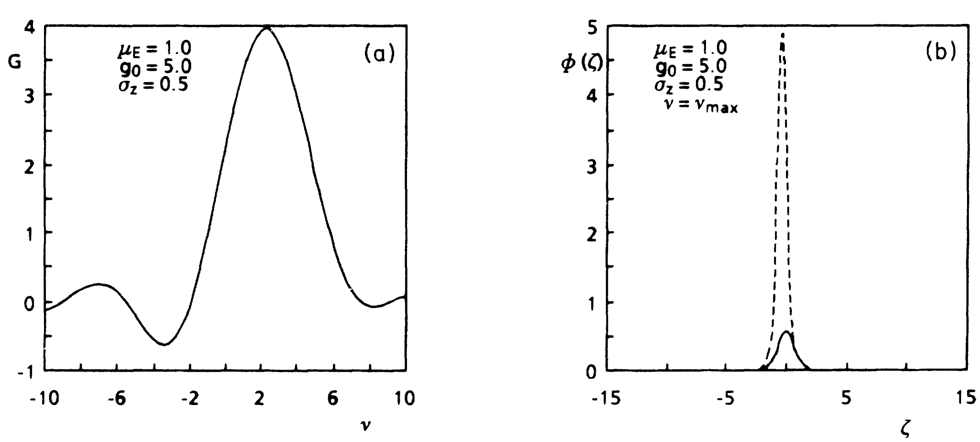

FIG. 12. Same as in Fig. 11 in the case of larger gain.

planned to be developed further, here we notice that quantities like the average number of excited CLM's may give an idea of the structure of the amplified input field. Furthermore, the rms values of the operators [see Eq. (3.2)]

$$\hat{A}_c=\frac{\hat{A}+\hat{A}^+}{2}\,,\qquad \hat{A}_s=\frac{\hat{A}-\hat{A}^+}{2i}\,, \tag{4.1}$$

may give an idea of how the optical-bunch dimensions are modified by the FEL interaction and whether they can undergo positive or negative squeezing. Within the above framework, it is worth stressing that the FEL interaction may be exploited in many flexible ways, to modify, e.g., the structure of an input optical bunch. It is indeed shown in Fig. 13 that, working at a value of ν corresponding to minimum gain, at $g_0=5$ and $\mu_E=1$ FEL amplifiers may transform a Gaussian input field in an output configuration, exhibiting two narrow peaks.

In the preceding section we discussed the gain depression induced on the fundamental CLM by the μ_E parameter. It is therefore worth discussing how higher-order CLM's are affected by the slippage. Assuming that the input field is an mth-order mode, we get for the first-order coefficient $a_n^{(1)}(\tau)$ the following equation of motion:

$$\frac{d}{d\tau}a_n^{(1)}(\tau)=-i\pi\int_\pi^\tau d\tau'\tau' e^{i\nu\tau'}\phi_n^{m-n}\left[\left[\frac{\mu_E\tau'}{\sqrt{2}}\right]^2\right]. \tag{4.2}$$

The equation relevant to the mth mode can, therefore, be cast in the form

$$\frac{d}{d\tau}a_m^{(1)}(\tau)=-i\pi\int_0^\tau d\tau'\tau' e^{i\nu\tau'}\exp\left[-\frac{\mu_E^2\tau'^2}{4}\right]\times L_m\left[\left[\frac{\mu_E\tau'}{\sqrt{2}}\right]^2\right] \tag{4.3}$$

and finally the gain function is written as

$$G_{1,m}(\nu,\mu_E)=2\pi i\int_0^1 d\tau\int_0^\tau d\tau'\tau'\exp\left[i\nu\tau'-\frac{\mu_E\tau'^2}{4}\right]\times L_m\left[\left[\frac{\mu_E\tau'}{2}\right]^2\right]. \tag{4.4}$$

The gain depression is not transparent from the above equation; we can, however, gain some insight. For $\sqrt{m}\,\mu_E \ll 1$ we have

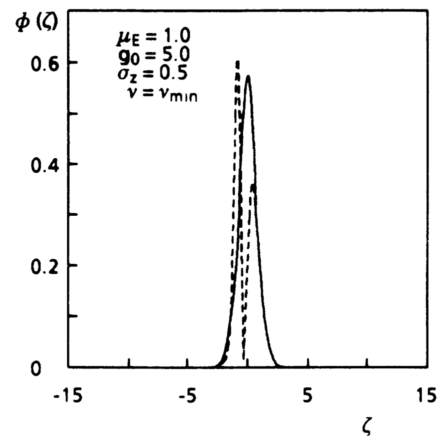

FIG. 13. Output (dotted line) input (solid line) beam distribution vs z input (solid line) beam distribution for $\nu=\nu_{\max}$.

$$L_m\left[\left[\frac{\mu_E\tau}{\sqrt{2}}\right]^2\right]\sim 1-m\frac{\mu_E^2\tau^2}{2}\cong\exp\left[-2m\frac{\mu_E^2\tau^2}{4}\right], \tag{4.5}$$

which inserted into (4.4) yields

$$G_{1,m}(\nu,\mu_E)\cong -2\pi i\times\int_0^1 d\tau\int_0^\tau d\tau'\tau' e^{i\nu\tau'}\times\exp\left[-(2m+1)\frac{\mu_E^2\tau'^2}{4}\right]. \tag{4.6}$$

The effect of μ_E is therefore $(2m+1)$ times more important for the mth CLM. We can, therefore, expect that

$$G_{1,m}(\mu_E)\cong\frac{0.85g_0}{1+0.086(2m+1)\mu_E^2}\,. \tag{4.7}$$

(For a more general expression, see the Appendix.)

We have so far discussed the role played on the gain by the finite electron- and optical-bunch lengths. We have studied limiting cases where the effect of one or the other is dominant. We did not specify, however, a unique parameter which may account for both effects.

The pulse FEL evolution equation can be solved in the low-gain regime, and in the hypothesis of a quasicontinuous electron beam (i.e., $\sigma_Z \gg \Delta$) we get

$$a(Z,1)=\left[1+g_0/2e^{-Z^2/2\sigma_Z^2}G_1\left[\nu-i\Delta\frac{\partial}{\partial Z}\right]\right]a(Z,0)\,. \tag{4.8}$$

Assuming that the input field is Gaussian, we end up with a rather simple expression,

$$a(Z,1)=\frac{1}{(\pi\sigma_E^2)^{1/4}}\left[1+\frac{g_0}{2}e^{-Z^2/2\sigma_Z^2}\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\Delta^n\frac{\partial^n}{\partial\nu^n}G_1(\nu)H_n\left[\frac{Z}{\sqrt{2}\sigma_E}\right]\right]e^{-Z^2/2\sigma_E^2}\,. \tag{4.9}$$

We can, therefore, rearrange the above equation as follows:

$$a(Z,1)=\frac{1}{(\pi\sigma_E^2)^{1/4}}\left\{e^{-Z^2/2\sigma_E^2}+\frac{g_0}{2}\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\Delta^n\frac{\partial^n}{\partial\nu^n}G_1(\nu)e^{-\bar{Z}^2/2}H_n\left[\frac{\bar{Z}}{\sqrt{2}}\left[1+\left[\frac{\sigma_E}{\sigma_Z}\right]^2\right]^{1/2}\right]\right\}, \tag{4.10}$$

where $\bar{Z}=Z/\tilde{\sigma}$ and

$$\tilde{\sigma}=\frac{\sigma_E}{[1+(\sigma_E/\sigma_Z)^2]^{1/2}}. \tag{4.11}$$

Albeit it appears rather difficult to get the gain in a closed form (see the Appendix), Eq. (4.10) yields the strong indication that the parameter which can play the above-quoted role is

$$\tilde{\mu}=\mu_E\left[1+\left[\frac{\mu_c}{\mu_E}\right]^2\right]^{1/2}. \tag{4.12}$$

Another important point touched on in the paper is that relevant to the lethargy, and we provided an explicit expression for the packet-velocity slowing down. It is perhaps worth giving a more transparent feeling of the lethargy dynamics within the framework of a simple example. Equation (2.1) can be written in the low-gain approximation, at first order in μ_E and for constant $f(Z)$, as follows:

$$\frac{\partial}{\partial\tau}a(Z,\tau)=-\pi g_0\left[iF(\nu,\tau)+\mu_E\frac{\partial}{\partial\nu}F(\nu,\tau)\frac{\partial}{\partial Z}\right]a(Z,\tau), \tag{4.13}$$

whose solution is rather straightforwardly written as $[a(Z,0)=a(Z)]$

$$a(Z,\tau)=\exp\left[-\pi g_0 i\int_0^\tau F(\nu,\tau')d\tau'\right]\times a\left[Z-\pi g_0\mu_E\int_0^\tau\frac{\partial}{\partial\nu}F(\nu,\tau')d\tau'\right]. \tag{4.14}$$

The exponent in (4.14) accounts just for the amplification process, while the second term provides, in a rather insightful way, the role of the lethargy centroid shift back. One may argue that, when higher-order terms in μ_E come into play, the situation will not be so simple. This is certainly true, but one can easily separate the roles. Second-order contributions in μ_E, and thus second-order derivatives on Z, will behave like a diffusive term which will be responsible for the packet spread given by Eq. (2.37). The sizeable effect of third-order terms may be of relevance if $\mu_E>2$.

Before concluding this section, we will address a final comment on the possibility of getting a perturbative solution of Eq. (2.1) including $f(Z)$ and passing through an expansion of the type (2.3). We write, therefore,

$$\frac{\partial}{\partial\tau}a(Z,\tau)=-i\pi g_0 f(Z+\mu_E\tau)\times\int_0^\tau d\tau'\tau' e^{i\hat{\nu}\tau'}a(Z,\tau-\tau'),\quad \hat{\nu}=\nu-i\mu_E\frac{\partial}{\partial Z}. \tag{4.15}$$

Furthermore, we assume that

$$a(Z,\tau)=\sum_{n=0}^{\infty}g_0^n a_n(Z,\tau),\quad a_n(Z,0)=a_0(Z)\delta_{n,0}. \tag{4.16}$$

We get, therefore,

$$a_1(Z,\tau)=-i\pi f(Z+\mu_E\tau)\int_0^\tau d\tau'\int_0^{\tau'}d\tau''\tau''e^{i\hat{\nu}\tau''}a_0(Z),$$
$$\frac{d}{d\tau}a_2(Z,\tau)=-\pi^2 f(Z+\mu_E\tau)\int_0^\tau d\tau'\tau'\int_0^{\tau-\tau'}d\tau''e^{i\hat{\nu}\tau''}f(Z+\mu_E\tau'')e^{i\hat{\nu}\tau''}\int_0^{\tau''}d\tau'''\tau'''e^{i\hat{\nu}(\tau'+\tau''')}a_0(Z), \tag{4.17}$$

and so on.

Noticing that

$$e^{i\hat{\nu}\tau'}f(Z)e^{-i\hat{\nu}\tau'}=f(Z)-i\tau'[f(Z),\hat{\nu}]-\frac{\tau'^2}{2}[[f(Z),\hat{\nu}],\hat{\nu}]+\cdots \tag{4.18}$$

and retaining the first commutator only in the expansion, we end up with

$$\begin{aligned}\frac{d}{d\tau}a_2(Z,\tau)=&-\pi^2 f(Z+\mu_E\tau)\int_0^\tau d\tau'\tau'\int_0^{\tau-\tau'}d\tau''f(Z+\mu_E\tau'')\int_0^{\tau''}d\tau'''\tau'''e^{i\hat{\nu}(\tau'+\tau''')}a_0(Z)\\&-\pi^2\mu_E f(Z+\mu_E\tau)\int_0^\tau d\tau'\tau'^2\int_0^{\tau-\tau'}\frac{d}{dZ}f(Z+\mu_E\tau'')\int_0^{\tau''}d\tau'''\tau'''e^{i\hat{\nu}(\tau'+\tau''')}a_0(Z)\\&+\pi^2\mu_E f(Z+\mu_E\tau)\int_0^\tau d\tau'\tau'^2\int_0^{\tau-\tau'}d\tau''f(Z+\mu_E\tau'')\int_0^{\tau''}d\tau'''\tau'''e^{i\hat{\nu}(\tau'+\tau''')}\frac{d}{dZ}a_0(Z),\end{aligned} \tag{4.19}$$

where the effect of $f(Z)$, its derivative, and the derivative of the input field are clearly displayed. Finally, let us point out that the paper, albeit preliminary, has been aimed at analyzing a large body of the FEL pulse-propagation dynamics. However, many problems, like the pulse behavior in the very-high-gain regime, have not been discussed. They will, however, be a matter of future investigation.

APPENDIX

This appendix is devoted to some computational details not explicitly carried out in the paper.

1. Derivation of the matrix elements $f_{n,m}(\tau)$ for a Gaussian-shaped longitudinal current

Equation (2.7) yields

$$f_{n,m}(\tau'\tau)=\int_{-\infty}^{+\infty}dZ\,u_n(Z)\exp[-(Z+\mu_E\tau)^2/2\sigma^2]\times u_m(Z+\mu_E\tau')\,, \tag{A1}$$

which, using the addition theorem for Hermite polynomials,

$$H_m(Z+\mu_E\tau')=\sum_{k=0}^{m}\begin{bmatrix} m\\ k\end{bmatrix}(2\mu_E\tau')^{m-k}H_k(Z) \tag{A2}$$

becomes

$$f_{n,m}(\tau)=\frac{1}{(\pi 2^{n+m}n!m!)^{1/2}}\sum_{k=0}^{m}\begin{bmatrix} m\\ k\end{bmatrix}(2\mu_E\tau')^{m-k}\exp[-\tfrac{1}{2}\mu_E^2(\tau'^2+\tau^2/\sigma^2)]\times\int_{-\infty}^{+\infty}dZ\,H_n(Z)H_k(Z)\exp\left[-Z^2\left[1+\frac{1}{2\sigma^2}\right]-Z\mu_E(\tau'+\tau/\sigma^2)\right]. \tag{A3}$$

The integrals in (A3) can be worked out explicitly, thus getting

$$f_{n,m}(\tau,\tau')=\left[\frac{2}{1+2\sigma^2}\right]^{1/2}\sigma\exp\left\{-\frac{\mu_E^2}{4(1+1/2\sigma^2)}\left[\tau'^2\left[1+\frac{1}{\sigma^2}\right]+\frac{2}{\sigma^2}(\tau^2-\tau\tau')\right]\right\}\frac{1}{\sqrt{2^{n+m}n!m!}}\times\sum_{k=0}^{m}\begin{bmatrix} m\\ k\end{bmatrix}(2\mu_E\tau')^{m-n}\sum_{j=0}^{\min(n,k)}2^j j!\begin{bmatrix} m\\ j\end{bmatrix}\begin{bmatrix} k\\ j\end{bmatrix}\left[\frac{1}{1+2\sigma^2}\right]^{(n+k)/2-j}H_{n+k-2j}\left[-\frac{\mu_E\tau+\sigma^2\tau'}{\sqrt{1+2\sigma^2}}\right], \tag{A4}$$

which in the limit $\sigma \ll 1$ yields Eq. (3.8).

2. Gain function for higher-order CLM's

Recalling that

$$L_n(x)=\sum_{k=0}^{n}\frac{1}{k!}\begin{bmatrix} n\\ k\end{bmatrix}(-x)^k\,, \tag{A5}$$

we can cast the gain function (4.4) in the form

$$G_{1,n}(\nu,\mu_E)=-2\pi i\int_0^1 d\tau\int_0^\tau d\tau'\tau' e^{i\nu\tau'-(1/4)\mu_E^2\tau'^2}\sum_{k=0}^{n}\frac{1}{k!}\begin{bmatrix} n\\ k\end{bmatrix}\left[-\left[\frac{\mu_E\tau}{\sqrt{2}}\right]^2\right]^k = \sum_{k=0}^{n}\frac{(-1)^k}{k!}\begin{bmatrix} n\\ k\end{bmatrix}\left[\frac{\mu_E}{\sqrt{2}}\right]^{2k}\frac{\partial^{2k}}{\partial\nu^{2k}}G_1(\nu,\mu_E)\,. \tag{A6}$$

Furthermore, since

$$G_1(\nu,\mu_E)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\left[\frac{\mu_E}{\sqrt{2}}\right]^{2s}\frac{\partial^{2s}}{\partial\nu^{2s}}G_1(\nu)\,, \tag{A7}$$

we end up with

$$G_{1,n}(\nu,\mu_E)=\sum_{s=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{k+s}}{k!s!}\begin{bmatrix} n\\ k\end{bmatrix}\left[\frac{\mu_E}{\sqrt{2}}\right]^{2(k+s)}\times\left[\frac{\partial}{\partial\nu}\right]^{2(k+s)}G_1(\nu)\,, \tag{A8}$$

and since

$$G_{n+1}(\nu)=(-i)^n\frac{\partial^n}{\partial\nu^n}G_1(\nu)\,, \tag{A9}$$

we finally get

$$G_{1,n}(\nu,\mu_E)=\sum_{s=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}\left[\frac{\mu_E}{\sqrt{2}}\right]^{2(k+s)}G_{2(k+s)+1}(\nu)\,, \tag{A10}$$

which can be easily exploited to get the explicit values of the gain function.

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$$f_{n,n'}(\tau') = \delta_{nn'} + \frac{\mu_E \tau'}{\sqrt{2}}(\sqrt{n'}\delta_{n,n'-1} - \sqrt{n'+1}\delta_{n,n'+1}) \ .$$

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