Temporal profile and linewidth of coherent harmonics generated by fundamental mode bunching

Wu Ding
Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100 088, People's Republic of China

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In this paper, we have obtained the power expression of the coherent harmonic emissions generated by fundamental mode bunching and explained their temporal profile and linewidth.

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I. INTRODUCTION

The measurements for the temporal profile, linewidth, and absolute energy of the first seven coherent harmonics emitted in the Mark III free-electron laser (FEL) at Stanford University have been published [1]. In this paper we try to give the physical explanation for the main measurement results.

II. POWER OF THE COHERENT HARMONICS GENERATED BY FUNDAMENTAL MODE BUNCHING

A. Equation of mode evolution

According to Ref. [2], we can obtain a general equation that describes the evolution of longitudinal modes in a free-electron laser:

$$\frac{\partial \mathbf{a}_n(z)}{\partial z} = -\frac{4\pi e}{2\alpha_p mc^2\omega_n}\mathbf{J}_n(z) , \tag{1}$$

$$\mathbf{a}_n(z) = \frac{e\mathbf{E}_n}{\alpha_p mc^2 K_n} , \tag{2}$$

$$\omega_n = n\omega_1 = K_n , \tag{3}$$

$$\alpha_p = \begin{cases} 1 & \text{for circular polarization} \\ \sqrt{2} & \text{for linear polarization ,} \end{cases} \tag{4}$$

$$\mathbf{J}_n(z) = \frac{1}{T}\int_{-T/2}^{T/2}\mathbf{J}(z,t)e^{-iK_nz+i\omega_nt}dt , \tag{5}$$

$$T = \frac{2\pi}{\omega_1} , \tag{6}$$

where ω_1 is the frequency of the fundamental mode and

$$t = t_0 + \int_0^z \frac{1}{c\beta_\parallel}dz . \tag{7}$$

If the thermal effects of the electron beam are not taken into account, the current is

$$\mathbf{J}(z,t) = -ecn_0\int \beta_\parallel^0 c\,dt_0\boldsymbol{\beta}_\perp\delta(z-z_e(t,t_0)) . \tag{8}$$

If we only consider the fundamental mode bunching and neglect the self-bunching of the modes considered, sideband instability and the effective energy spread caused by the applied focusing magnetic field, we have [2]

$$\beta_\parallel = \beta_\parallel^{(0)} + \Delta\beta_\parallel^{(0)} + \Delta\beta_\parallel^{(1)} . \tag{9}$$

Substituting (7), (8), and (9) into (5) and (1) yields

$$\frac{\partial \mathbf{a}_n}{\partial z} = \frac{\omega_b^2}{2\alpha_p c\omega_n}\frac{1}{T}\int_{-T/2}^{T/2}dt_0\boldsymbol{\beta}_\perp \exp\left[-iK_nz + i\omega_nt_0 + i\frac{\omega_nz}{c\beta_\parallel^{(0)}} - i\frac{\omega_n}{c}\int\Delta\beta_\parallel^{(0)}dz - i\frac{\omega_n}{c}\int\Delta\beta_\parallel^{(1)}dz\right] , \tag{10}$$

$$\omega_b^2 = \frac{4\pi e^2 n_0}{m} . \tag{11}$$

We first calculate the phase factor as follows:

$$\exp\left[-i\frac{\omega_n}{c}\int_0^z \Delta\beta_\parallel^{(0)}dz'\right] . \tag{12}$$

Among the coherent harmonics generated by the fundamental mode bunching in the free-electron laser oscillator, the odd harmonics are excited by the linear polarization wiggler, and the even harmonics are excited by the misalignment [3] between the optical beam and the electron beam.

Thus, for a linear polarization wiggler, we have

$$\boldsymbol{\beta}_\perp = \frac{\sqrt{2}a_w}{\gamma_0}\tfrac{1}{2}(e^{iK_wz} + e^{-iK_wz})\hat{\mathbf{e}}_x , \tag{13}$$

$$\mathbf{a}_n = a_n\hat{\mathbf{e}}_x , \tag{14}$$

$$\Delta\beta_\parallel^{(0)} = -\frac{a_w^2}{2\gamma_0^2}\cos 2K_wz + \frac{a_w\theta}{\gamma_0}\cos K_wz , \tag{15}$$

where θ is the misalignment angle between the optical beam and the electron beam,

$$-i\frac{\omega_n}{c}\int_0^z dz\left[-\frac{a_w^2}{2\gamma_0^2}\cos 2K_w z+\frac{a_w\theta}{\gamma_0}\cos K_w z\right]$$
$$=in\xi_L^2\sin 2K_w z-in\xi_\theta\sin K_w z\ , \quad (16)$$

$$\xi_L=\frac{a_w^2}{2(1+a_w^2+\gamma_0^2\theta^2)},\quad \xi_\theta=\frac{2a_w\gamma_0\theta}{(1+a_w^2+\gamma_0^2\theta^2)}\ . \quad (17)$$

Substituting (13) and (16) into (10), we note the exponential terms

$$(e^{iK_w z}+e^{-iK_w z})\exp\left[-inK_1 z+\frac{in\omega_1 z}{c\beta_\parallel^{(0)}}+in\xi_L\sin 2K_w z-in\xi_0\sin K_w z\right]$$
$$=\exp\left[-in(K_1+K_w)z+\frac{in\omega_1 z}{c\beta_\parallel^{(0)}}+i(n+1)K_w z+in\xi_L\sin 2K_w z-in\xi_0\sin K_w z\right]$$
$$+\exp\left[-in(K_1+K_w)z+\frac{in\omega_1 z}{c\beta_\parallel^{(0)}}+i(n-1)K_w z+in\xi_L\sin 2K_w z-in\xi_0\sin K_w z\right]\ . \quad (18)$$

Because

$$\exp(in\xi_L\sin 2K_w z)=\sum_{l=-\infty}^{\infty}J_l(n\xi_L)e^{il2K_w z}\ , \quad (19)$$

$$\exp(-in\xi_\theta\sin K_w z)=\sum_{m=-\infty}^{\infty}(-1)^m J_m(n\xi_\theta)e^{imK_w z}\ , \quad (20)$$

therefore (18) becomes

$$\exp\left[-in(K_w+K_1)z+\frac{in\omega_1 z}{c\beta_\parallel^0}\right]\sum_m(-1)^{(n-1+3m)/2}J_m(n\xi_\theta)[J_{(n-1+m)/2}(n\xi_L)-J_{(n+1+m)/2}(n\xi_L)]\ . \quad (21)$$

Substituting (14) and (21) into (10), we obtain

$$\frac{\partial a_n(z)}{\partial z}=\frac{\omega_b^2 a_w F_{L\theta n}}{4c\gamma_0\omega_n}e^{-i\Delta K_n z}\frac{1}{T}\times\int_{-T/2}^{T/2}dt_0 e^{i\omega_n t_0}F_n(z,t_0)\ , \quad (22)$$

$$F_{L\theta n}=\sum_{m=-\infty}^{\infty}(-1)^{(n-1+3m)/2}J_m(n\xi_\theta)\times[J_{(n-1+m)/2}(n\xi_L)-J_{(n+1+m)/2}(n\xi_L)]\ , \quad (23)$$

$$\Delta K_n=n\Delta K_1\ , \quad (24)$$

$$\Delta K_1=K_w-K_1\left[\frac{1}{\beta_\parallel^{(0)}}-1\right]\ , \quad (25)$$

$$F_n(z,t_0)=\exp\left[-i\frac{\omega_n}{c}\int_0^z\Delta\beta_\parallel^{(1)}dz\right]\ , \quad (26)$$

$$\Delta\beta_\parallel^{(1)}=\frac{\mu^2}{\gamma_0^3}\Delta\gamma^{(1)}\ , \quad (27)$$

$$\Delta\gamma^{(1)}=-\int\frac{\omega_1 a_w F_{L\theta 1}|a_{1B}|}{c\gamma_0}\sin\psi_{1B}dz\ , \quad (28)$$

$$\psi_{1B}=\left[K_w-K_1\left[\frac{1}{\beta_\parallel^{(0)}}-1\right]\right]z+\frac{\omega_n}{c}\int\Delta\beta_\parallel^{(1)}dz-\omega_1 t_0+\varphi_{1B}\ , \quad (29)$$

where $|a_{1B}|$ and φ_{1B} are the real amplitude and phase of the optical field of the bunching fundamental mode, respectively, and ψ_{1B} is the phase of the electrons in the ponderomotive potential well formed by the optical field of the bunching fundamental mode and the wiggler field.

In (29), $\omega_1/c\int\Delta\beta_\parallel^{(1)}dz$ is the modulated term caused by the bunching fundamental mode.

When

$$K_{sy}L_w<1\ ,$$
$$K_{sy}=\frac{\sqrt{a_w|a_{1B}|F_{L\theta 1}}2K_w}{\mu}\ , \quad (30)$$

we have

$$\frac{\omega_1}{c}\int\Delta\beta_\parallel^{(1)}dz\ll\omega_1 t_0\ , \quad (31)$$

i.e., this term can be neglected.

Assuming that the fundamental mode nearly resonates with the electron beam

$$K_w-K_1\left[\frac{1}{\beta_\parallel^{(0)}}-1\right]\approx 0 \quad (32)$$

and

$$\varphi_{1B}\ll\omega_1 t_0\ , \quad (33)$$

then

$$\psi_{1B}=-\omega_1 t_0\ . \quad (34)$$

Substituting (34) into (28), (27), and (26), we obtain

$$F_n=\exp(iA_{n1}\sin\omega_1 t_0)=\sum_{l=-\infty}^{\infty} J_l(A_{nl})e^{il\omega_1 t_0} , \quad (35)$$

$$A_{nl}=-\frac{\omega_n}{c}\frac{\mu^2}{\gamma_0^3}\int_0^z \Delta\gamma_{\rm mod}dz' , \quad (36)$$

$$\Delta\gamma_{\rm mod}=\int_0^{z'}\frac{\omega_1 a_w F_{L\theta 1}|a_{1B}|}{c\gamma_0}dz'' , \quad (37)$$

where $\Delta\gamma_{\rm mod}$ is the modulation width.

Inserting (35) in (22) and integrating over t_0, we have

$$\frac{\partial a_n}{\partial z}=(-1)^n\frac{\omega_b^2 a_w F_{L\theta n}}{4c\gamma_0\omega_n}e^{-i\Delta K_n z}J_n(|A_{n1}|) . \quad (38)$$

Equation (38) describes the coherent harmonics generated by the fundamental mode bunching.

B. Expression of power

In the Mark III FEL oscillator, when the fundamental mode grows large enough but does not reach saturation, coherent harmonics will appear in the system due to the slippage effects between the optical beam and the electron beam. The coherent-harmonic generation includes three physical processes, i.e., modulation, dispersion, and emissions, which are performed in the same wiggler. As the fundamental mode passes through the wiggler each time (in each pass), a series of coherent-harmonic micropulses are emitted. Each coherent-harmonic macropulse is a superposition of its micropulses. The macropulses reach saturation when their growth rate becomes equal to their decay rate. The saturation legnth is a characteristic quantity of coherent harmonics.

Integrating both sides of Eq. (38) over z, under the condition of

$$a_n(0)=0 , \quad (39)$$

we obtain

$$a_n(z)=(-1)^n\frac{\omega_b^2 a_w F_{L\theta n}}{4\gamma_0 c\omega_n}\int_0^z J_n(|A_{n1}(z')|)e^{-i\Delta K_n z'}dz' . \quad (40)$$

Then, the power of the coherent harmonic is

$$P_n(z)=\frac{c}{4\pi}\left[\frac{\sqrt{2}mc^2K_n}{e}\right]^2|a_n(z)|^2$$
$$=\frac{\pi}{2}n_0^2e^2c\frac{a_\omega^2}{\gamma_0^2}F_{L\theta n}^2\left|\int_0^z J_n(|A_{n1}(z')|)e^{-i\Delta K_n z'}dz'\right|^2 . \quad (41)$$

Formula (41) shows that the coherent harmonic is a superradiance; its power is proportional to the square of the electron-beam density

$$P_n\propto n_0^2 . \quad (42)$$

C. Temporal profile of the coherent harmonic

1. Saturation length

At the saturation point, we have

$$\frac{d}{dz}|a_n(z)|=0 . \quad (43)$$

From (38), we obtain

$$J_n(|A_{n1}(z_c^{(n)})|)=0 , \quad (44)$$

where $z_c^{(n)}$ is the saturation length of the nth-order coherent harmonic.

There are a series of zero-point positions R_{mn} of Bessel function. Let us use an approximate formula to get the values of R_{mn} [4]:

$$R_{mn}\approx\left[m+\frac{n}{2}-\frac{1}{4}\right]\pi , \quad (45)$$

where m and n, respectively, represent the orders of the zero points and the coherent harmonics.

We take the first zero point of Bessel function as the saturation point, after which the power of the coherent harmonics will appear as oscillation.

From (44), we have

$$R_{1n}=|A_{n1}(z_c^{(n)})|=\frac{\omega_n}{c}\frac{\mu^2}{\gamma_0^3}z_c^{(n)}\int_0^1\Delta\gamma_{\rm mod}dx^{(n)}$$
$$=n\frac{\omega_1}{c}\frac{\mu^2}{\gamma_0^3}z_c^{(n)}\int_0^1\Delta\gamma_{\rm mod}dx^{(n)} . \quad (46)$$

From (46), we have

$$z_c^{(n)}\propto\frac{R_{1n}}{n} . \quad (47)$$

Using formula (45), we calculate out the values of R_{1n} listed in Table I. From (47) and Table I, we calculate out the values of $z_c^{(n+1)}/z_c^{(n)}$ listed in Table II. It is shown in Table II that the higher the order n of the coherent harmonic, the shorter its saturated length.

2. Oscillatory flat top

All intervals between two zero points of Bessel function are equal, as follows from (45):

$$R_{m+1n}-R_{mn}=\pi . \quad (48)$$

Because

$$\frac{Z_{m+1}^{(n)}}{Z_m^{(n)}}=\frac{R_{m+1n}}{R_{mn}} ,$$

$$\frac{Z_{m+1}^{(n)}-Z_m^{(n)}}{Z_m^{(n)}}=\frac{R_{m+1n}-R_{mn}}{R_{mn}}=\frac{\pi}{R_{mn}} ,$$

TABLE I. Values of R_{1n} calculated from Eq. (45).

n	1	2	3	4	5	6	7
R_{1n}	3.92	5.50	7.07	8.64	10.2	11.78	13.35

TABLE II. Values of $z_c^{(N+1)}/z_c^{(n)}$ calculated from Eq. (47) and Table I.

n	1	2	3	4	5	6	7
$z_c^{(n+1)}/z_c^{(n)}$	0.702	0.857	0.917	0.944	0.962	0.971	

therefore,

$$\Delta Z_{m+1n}^{(n)} = Z_m^{(n)} \frac{\pi}{R_{mn}} . \tag{49}$$

As n increases, R_{mn} increases and $Z_m^{(n)}$ decreases, which leads $\Delta Z_{m+1n}^{(n)}$ to decrease, i.e., the larger n, the higher the oscillatory frequency. But the oscillatory amplitude decreases because of

$$|J_{n+1}^{\text{peak}}| < |J_n^{\text{peak}}| .$$

In other words, there is a flat top for the higher-order harmonic.

D. Linewidth of the coherent harmonic

From (40), we can derive a complex electrical field with dimension

$$E_n(z) = \frac{\sqrt{2}mc^2K_n}{e} a_n(z)$$

$$= (-1)^n \frac{\sqrt{2}\pi e n_0 a_w F_{L\theta n}}{\gamma_0} \times \int_0^z J_n(|A_{n1}(z')|) e^{-i\Delta K_n z'} dz' , \tag{50}$$

$$|A_{n1}(z)| = \frac{\omega_n}{c} \frac{\mu^2}{\gamma_0^3} \int_0^z \Delta\gamma_{\text{mod}} dz' = \alpha(z)\omega_n . \tag{51}$$

Since the power of the coherent harmonic reaches a maximum at the saturation point, we come to evaluate the inverse Fourier transform of the complex electrical field with dimension at the saturation point:

$$E(z_c^{(n)},t) = \sum_n E_n(z_c^{(n)}) e^{i(K_n z_c^{(n)} - \omega_n t)} = \frac{T}{2\pi} \int E(z_c^{(n)},\omega) e^{-i\omega(t - z_c^{(n)}/c)} d\omega$$

$$= (-1)^n \frac{T}{2\pi} \frac{\sqrt{2}\pi e n_0 z_c^{(n)} F_{L\theta n}}{\gamma_0} \int_0^1 \frac{e^{-inK_w z_c^{(n)} x} g(t, z_c^{(n)}, x)}{\tau_R(x)} dx , \tag{52}$$

$$g(t, z_c^{(n)} x) = \begin{cases} \dfrac{2(-i)^n T_n \left[\dfrac{t - \tau_d(z_c^{(n)},x)}{\tau_R(x)} \right]}{\left[1 - \left[\dfrac{t - \tau_d(z_c^{(n)},x)}{\tau_R(x)} \right]^2 \right]^{1/2}}, & \left[\dfrac{t - \tau_d(z_c^{(n)},x)}{\tau_R(x)} \right]^2 < 1 \\ 0, & \left[\dfrac{t - \tau_d(z_c^{(n)},x)}{\tau_R(x)} \right]^2 > 1 , \end{cases} \tag{53}$$

$$x = \frac{z}{z_c^{(n)}} , \tag{54}$$

where T_n is the first kind of Chebyshev polynomial; its form and property can be seen in Ref. [5].

τ_d is the delay time of emission:

$$\tau_d(z_c^{(n)},x) = \frac{z_c^{(n)}}{c} + \frac{z_c^{(n)}}{c} \frac{\mu^2}{2\gamma_0^2} x \approx \frac{z_c^{(n)}}{c} . \tag{55}$$

$\tau_R(x)$ is the lifetime function of superradiance:

$$\tau_R = \frac{z_c^{(n)}}{c} \frac{\mu^2}{\gamma_0^3} \int_0^x \Delta\gamma_{\text{mod}} dx' . \tag{56}$$

Let

$$y = \frac{t - \tau_d^{(n)}}{\tau_R(x)} , \tag{57}$$

$$\tau_d^{(n)} = \frac{z_c^{(n)}}{c} . \tag{58}$$

From (57) and (56), the following can be derived:

$$\frac{dx}{\tau_R(x)} = -\frac{dy}{\tau_R^{(n)} \eta(y) y} , \tag{59}$$

$$\tau_R^{(n)} = \frac{z_c^{(n)}}{c} \frac{\mu^2}{\gamma_0^3} \int_0^1 \Delta\gamma_{\text{mod}} dx , \tag{60}$$

$$\eta(y) = \frac{\Delta\gamma_{\text{mod}}(y)}{\int_0^1 \Delta\gamma_{\text{mod}} dx} , \tag{61}$$

and from (57), we can derive

$$x=\varphi\left[\frac{t-\tau_d^{(n)}}{y}\right], \tag{62}$$

where φ is a function of $t-\tau_d^{(n)}$ and y.

Substituting (59) and (62) into (52) and noticing the change of the integral limits

$$x=0,\quad y=1\quad [y>1,\quad g(t,z_c^{(n)},x)=0]\ ,$$

$$x=1,\quad y=\frac{t-\tau_d^{(n)}}{\tau_R^{(n)}}\ , \tag{63}$$

we obtain

$$E(z_c^{(n)},t)=2(i)^n\frac{T}{2\pi}\frac{\sqrt{2}\pi e n_0 a_w F_{L\theta n} z_c^{(n)}}{\gamma_0}\times\int_{(t-\tau_d^{(n)})/\tau_R^{(n)}}^{1}\frac{f_n(y)dy}{\tau_R^{(n)}\eta(y)}\ , \tag{64}$$

$$f_n(y)=\frac{\exp[-inK_w z_c^{(n)}\varphi[(t-\tau_d^{(n)})/y]T_n(y)}{y\sqrt{1-y^2}}\ . \tag{65}$$

If we define an average life of superradiance,

$$\overline{\left[\frac{1}{\tau_R}\right]^{(n)}}=\frac{\displaystyle\int_{(t-\tau_d^{(n)})/\tau_R^{(n)}}^{1}\frac{f_n(y)dy}{\tau_R^{(n)}\eta(y)}}{\displaystyle\int_{(\hat{t}-\tau_d^{(n)})/\tau_R^{(n)}}^{1}f_n(y)dy}=\frac{1}{\tau_R^{(n)}}\overline{\left[\frac{1}{\eta}\right]^{(n)}} \tag{66}$$

then (64) becomes

$$E(z_c^{(n)},t)=2(i)^n\frac{T}{2\pi}\frac{\sqrt{2}\pi e n_0 a_w F_{L\theta n} z_c^{(n)}}{\gamma_0}\overline{\left[\frac{1}{\tau_R}\right]^{(n)}}\times\int_{(t-\tau_d^{(n)})/\tau_R^{(n)}}^{1}f_n(y)dy\ . \tag{67}$$

By the principle of indeterminacy, from (66), the average absolute linewidth of the coherent harmonic can be obtained:

$$\overline{\Delta\omega_R^{(n)}}=\overline{\left[\frac{1}{\tau_R}\right]^{(n)}}=\Delta\omega_R^{(n)}\overline{\left[\frac{1}{\eta}\right]^{(n)}}\ , \tag{68}$$

$$\Delta\omega_R^{(n)}=\frac{1}{\tau_R^{(n)}}\ . \tag{69}$$

Thus, the relationship among the relative linewidths of the coherence harmonics, using (47) and (68), becomes

$$\frac{\overline{\Delta\omega_R^{(n)}}/\omega_n}{\overline{\Delta\omega_R^{(1)}}/\omega_1}=\frac{R_{11}}{R_{1n}}\frac{\overline{\left[\frac{1}{\eta}\right]^{(n)}}}{\overline{\left[\frac{1}{\eta}\right]^{(1)}}}\ , \tag{70}$$

where R_{11} and R_{1n} are the positions of the first zero point of Bessel functions J_1 and J_n, respectively.

If in all intervals from the time the fundamental mode approaches saturation to the time the coherent harmonics reach saturation, we have

$$\Delta\gamma_{\rm mod}={\rm const}\ , \tag{71}$$

then

$$\overline{\left[\frac{1}{\eta}\right]^{(n)}}=\overline{\left[\frac{1}{\eta}\right]^{(1)}}\ , \tag{72}$$

$$\frac{1}{n}<\frac{\overline{\Delta\omega_R^{(n)}}/\omega_n}{\overline{\Delta\omega_R^{(1)}}/\omega_1}=\frac{R_{11}}{R_{1n}}<1\ . \tag{73}$$

But the measured results in Ref. [1] show

$$\frac{\overline{\Delta\omega_R^{(n)}}/\omega_n}{\overline{\Delta\omega_R^{(1)}}/\omega_1}\geq 1 \tag{74}$$

which shows that $\Delta\gamma_{\rm mod}$ varies in the interval. Therefore, we obtain

$$\overline{\left[\frac{1}{\eta}\right]^{(n)}}>\overline{\left[\frac{1}{\eta}\right]^{(1)}}\ . \tag{75}$$

In fact, since

$$\eta(y)=\frac{\Delta\gamma_{\rm mod}(y)}{\int_0^1\Delta\gamma_{\rm mod}dx}\ , \tag{76}$$

$$\frac{t-\tau_d^{(1)}}{\tau_d^{(1)}}<\frac{t-\tau_d^{(n)}}{\tau_d^{(n)}}\quad (z_c^{(n)}<z_c^{(1)})\ , \tag{77}$$

$$f_n(y)=\frac{\exp[-inK_w z_c^{(n)}\varphi[(t-\tau_d^{(n)})/y]T_n(y)}{y\sqrt{1-y^2}}\ . \tag{78}$$

Though both of $\overline{(1/\eta)^{(n)}}$ and $\overline{(1/\eta)^{(1)}}$ are to be weighed mean of the function $1/[\Delta\gamma_{\rm mod}(y)]$, their results are different because of the different mean regions and weights. In the case of the experiment in Ref. [1], there is

$$\frac{R_{11}}{R_{1n}}\frac{\overline{\left[\frac{1}{\eta}\right]^{(n)}}}{\overline{\left[\frac{1}{\eta}\right]^{(1)}}}\geq 1\ . \tag{79}$$

From (70), we could design a laser to make the following inequality hold:

$$\frac{\overline{\left[\frac{1}{\eta}\right]^{(n)}}}{\overline{\left[\frac{1}{\eta}\right]^{(1)}}}<1\ . \tag{80}$$

Then we can acquire the high coherent harmonics with narrower linewidth.

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