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Self-consistent interaction between the plasma wake field and the driving relativistic electron beam

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It is shown that the self-consistent interaction between wake fields and the driving electron bunch in a collisionless, unmagnetized, overdense ($n_p \gg n_b$) plasma is governed by three coupled equations. In the long-beam limit, they reduce to a pair consisting of an appropriate nonlinear Schrödinger equation for the *beam wave function* Ψ, and an equation for the wake-field potential that is driven by the transverse profile of the beam density, which is proportional to $|\Psi|^2$. The pair of equations are suitable for studying the beam self-focusing (self-pinching equilibrium) for the case in which the beam-spot size is larger (smaller) than the wavelength of the wake fields. It is demonstrated that our self-consistent theory, which is based on the recently proposed *thermal wave model for relativistic charged-particle beam propagation,* is capable of reproducing the main results for the beam-filamentation threshold and the self-pinching equilibrium condition that are already known in the conventional theory of the beam self-interaction in collisionless plasmas.

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I. INTRODUCTION

It is widely believed that plasma-based particle accelerators would be capable of accelerating charged particles to extremely high energies. In particular, for particle energies beyond 10 TeV one must invent schemes that can produce accelerating fields of 10 MeV cm^{-1}. Such an intense longitudinal electric field can be theoretically produced by space charge waves driven by (i) the resonant beating of two laser beams in a high-density plasma [1] (known as the plasma beat-wave accelerator or PBWA), (ii) bunched relativistic electron beams [2,3] (termed the plasma wake-field accelerator or PWFA), and (iii) short intense single laser pulses [1,4] (referred to as the laser wake-field accelerator or LWFA).

Of these three schemes, the PWFA concept has received a great deal of attention [5–8] because of the possibility of achieving ultrahigh accelerating gradients for high-energy particles through this scheme. Furthermore, the transverse wake field in plasmas can have a possible application as a final focusing scheme for linear colliders [5,6] in order to produce very high luminosity at the interaction point. The experimental test of the physical principles of the PWFA scheme has been successfully performed at the Argonne National Laboratory [7] and in Japan [8].

On the other hand, the interaction of a relativistic charged-particle beam with plasmas is also of interest in connection with the generation of short-wavelength radiation [9,10] through the free-electron-laser process involving plasma waves as wigglers (undulators). Thus, the study of the intense charged-particle beam-plasma interaction is of much significance.

Beam-plasma interactions can produce several plasma instabilities [11–15], as well as have relevance to adiabatic focusing [16]. Specifically, a high-intensity electron beam of finite extent can be subjected to longitudinal instabilities such as the two-stream instability [12], whereas transverse effects can cause self-focusing or self-defocusing, beam filamentation, and self-pinching. The longitudinal instabilities could be avoided, but the transverse instabilities can cause distortion of the beam profile. Thus, the efficiency of energy transfer from the driving beam to the plasma wakefield can be drastically reduced.

The transverse beam dynamics is related to the transverse density profile of the beam, where the focusing or defocusing effect occurs at the same time as the beam-emittance spreading. Clearly, one of the problems still open is related to the proper determination of the real transverse density profile of the beam, taking into account both the emittance spreading and the transverse self-force. This calculation would be quite valuable for estimating the luminosity at the interaction point of linear colliders (allowing for spherical aberrations), and it would also provide a better understanding of changes that have occurred in the transverse profile while a relativistic charged-particle beam propagates through the plasma.

Recently, a *wave model for relativistic charged-particle propagation* has been proposed [17,18]. This model successfully recovers all the usual results of the relativistic charged-particle-beam optics and appears to be useful for describing and understanding several important problems in particle accelerators, viz. luminosity estimates in linear colliders by taking into account the spherical aberration corrections, and the interaction of wake fields with the driving bunch in both the conventional and the new accelerator schemes.

In this paper, we employ the recently proposed *thermal wave model* [17] in order to study the self-consistent interaction between the plasma wake fields and the driving relativistic electron (positron) bunch in a collisionless, unmagnetized, overdense plasma. The paper is organized as follows. In Sec. II we briefly summarize the salient features of the thermal wave model. In Sec. III we derive the dynamical equations for the plasma wake field in the presence of the transverse profile of the beam density. In the *long-beam limit* (or *coasting beam),* we find a pair of equations that describe the coupling between the beam wave function (BWF) and the wake-field potential. The beam self-focusing (self-pinching) is considered in Sec. IV, by assuming that the beam spot size R is much larger (smaller) than the wavelength k_p^{-1} of the wake field. Finally, Sec. V contains a summary of our results.

II. MAIN FEATURES OF THE THERMAL WAVE MODEL

In this section, we briefly summarize the salient features of the thermal wave model [17]. First, the stationary configuration of a relativistic charged-particle beam of transverse emittance ϵ, traveling along the z axis with velocity βc ($\beta\approx 1$), under the action of a potential $u(r,\phi,z)$, is described by the Schrödinger-like equation for the so-called *beam wave function* $\Psi(r,\phi,z)$:

$$i\epsilon\frac{\partial}{\partial z}\Psi=-\frac{\epsilon^2}{2}\nabla_\perp^2\Psi+U(r,\phi,z)\Psi\ , \tag{1}$$

where $\nabla_\perp$ is the gradient, and

$$U(r,z,\phi)=\frac{u(r,z,\phi)}{m_0\gamma\beta^2c^2}\ . \tag{2}$$

Here m_0 is the particle rest mass, c is the speed of light, and $\gamma=(1-\beta^2)^{-1/2}$ is the relativistic factor. Equation (1) must be coupled with an equation for the total field force ($\mathbf{F}_\perp=-m_0\gamma\beta^2c^2\nabla_\perp U$) acting on the system. In analogy with nonrelativistic quantum mechanics, z and ϵ play the role of time and Planck's constant, respectively; while in analogy with electromagnetic beam optics in the geometrical approximation, ϵ plays the role of the inverse of the wave number $k^{-1}=\lambda/2\pi$, and U that of the refractive index of the nonlinear medium, so that Eq. (1) corresponds to the well-known Fock-Leontovich equation [19].

Second, the norm $\mathcal{N}$ of Ψ, viz.

$$\mathcal{N}\equiv\|\Psi\|=\left[\int_0^{2\pi}d\phi\int_0^{\infty}r\,dr|\Psi|^2\right]^{1/2}, \tag{3}$$

is conserved ($d\mathcal{N}/dz=0$) and the BWF has the following meaning: If N is the total number of the particles of the beam, then the transverse beam-number density $\sigma(r,\phi,z)$ (number of particles per unit transverse section) is given by

$$\sigma(r,z,\phi)=\frac{N}{\mathcal{N}^2}|\Psi(r,z,\phi)|^2\ . \tag{4}$$

Thus, $|\Psi|^2$ gives the transverse beam-density profile. Note that here Ψ is dimensionless. Let us denote with $g(\xi)$ and σ_z the dimensionless longitudinal density profile and the longitudinal beam length, respectively; where $\xi=z-\beta ct$ (t being the time). Then, it is easy to prove that the beam volumic number density $\rho_b(r,\phi,z)$ assumes the form

$$\rho_b(r,z,\phi)=n_bg(\xi)|\Psi(r,z,\phi)|^2\ , \tag{5}$$

where $n_b=N/(\mathcal{N}^2\sigma_z)$.

Third, by defining the averaged total energy $\mathcal{E}$ and the effective beam radius (spot size) R as (for more details see Ref. [17]):

$$\mathcal{E}(z)=\frac{1}{\mathcal{N}^2}\int_0^{2\pi}d\phi\int_0^{\infty}r\,dr\left[\frac{\epsilon^2}{2}|\nabla_\perp\Psi|^2+U|\Psi|^2\right], \tag{6}$$

$$R(z)=\left[\frac{1}{\mathcal{N}}\int_0^{2\pi}d\phi\int_0^{\infty}r\,dr\,r^2|\Psi|^2\right]^{1/2}, \tag{7}$$

we can derive the following relations assuming cylindrical symmetry:

$$\frac{d\mathcal{E}}{dz}=\left\langle\frac{\partial}{\partial z}U\right\rangle=\frac{2\pi}{\mathcal{N}^2}\int_0^{\infty}\left[\frac{\partial}{\partial z}U\right]|\Psi|^2r\,dr\ , \tag{8}$$

and

$$\frac{d^2}{dz^2}R^2(z)=4\mathcal{E}-4\mathcal{V}-4\pi\int_0^{\infty}\left[r\frac{\partial}{\partial r}U\right]|\Psi|^2r\,dr\ , \tag{9}$$

which are the energy-variation equation and the virial theorem, respectively. Here, the averaged potential energy is denoted by

$$\mathcal{V}(z)=\frac{1}{\mathcal{N}^2}\int_0^{2\pi}d\phi\int_0^{\infty}r\,dr\,U|\Psi|^2\ . \tag{10}$$

Following Ref. [17], it is instructive to note that Eqs. (8) and (9) are derived by differentiating (6) once and the square of (7) twice, respectively, and making use of (1) and its complex conjugate. We observe that the energy is conserved only if the potential does not explicitly depend on z.

Fourth, an uncertainty principle, similar to the one in quantum mechanics, holds [17]. That means

$$RP\geq\epsilon\ , \tag{11}$$

where the total averaged transverse linear momentum is denoted by

$$P(z)=\epsilon\left[\frac{1}{\mathcal{N}^2}\int_0^{2\pi}d\phi\int_0^{\infty}r\,dr|\nabla_\perp\Psi|^2\right]^{1/2}. \tag{12}$$

By denoting with P_r and P_ϕ the radial and the azimuthal components of P, respectively,

$$P_r=\frac{d}{dz}R\ , \tag{13}$$

and

$$P_\phi=(P^2-P_r^2)^{1/2}\ , \tag{14}$$

we have

$$PR=\left[\left[\frac{dR}{dz}\right]^2+P_\phi^2\right]^{1/2}R\geq P_\phi R\ . \tag{15}$$

In general $P_\phi R \geq \epsilon$. However, if the beam wave function is Gaussian, then the following equalities hold:

$$P_\phi R = P_0 R_0 = \epsilon \ , \tag{16}$$

where $R_0 = R(z=0)$ and $P_0 = P(z=0)$. According to a well-known quantum theorem, this circumstance occurs for both the free motion ($U=0$) and the harmoniclike potential $[U=\frac{1}{2}K(z)r^2]$ only. For other beam profiles, the minimum value of PR is larger than ϵ (for more details see Ref. [17]).

III. WAKE-FIELD DYNAMICS IN THE PRESENCE OF THE BEAM

Let us now consider, in cylindrical symmetry, a relativistic electron (positron) beam traveling along the z axis with velocity βc in a collisionless, unmagnetized plasma of density n_p. As theoretically investigated [3,5,6] and experimentally confirmed [7,8], the self-focusing of the beam can occur while it is propagating through an overdense plasma ($n_p \gg n_b$). This effect is an aspect of the plasma wake field that is excited by the beam itself. On using the fluid theory, it can be shown that within the linear approximation the wake field driven by the charged-particle beam obeys the following equations [3]:

$$\left[\frac{\partial^2}{\partial \xi^2} + k_p^2\right] n_1 = \mp k_p^2 \rho_b(r,z,\xi) \ , \tag{17}$$

and

$$(\nabla_\perp^2 - k_p^2)\Omega = -4\pi e n_1 \ , \tag{18}$$

where $n_1 = n_1(r,z,\xi)$ is the plasma number-density perturbation, $\rho_b(r,z,\xi)$ is the beam-number density, $\Omega = \Omega(r,z,\xi)$ is the plasma wake-field potential, and $k_p = \omega_p/\beta c = (4\pi n_p e^2/m_0)^{1/2}/\beta c$ is the wave number of the plasma wave. On the right-hand side of Eq. (17) the $-$ $(+)$ sign is for the electron (positron) beam. All the previous papers [3,5] have considered $\rho_b(r,z,\xi)$ as a given source and have computed the wake fields. However, this procedure is not self-consistent because it does not take into account the reaction of the wake field on the evolution of the transverse beam profile. In a realistic situation, the expression $\rho_b(r,z,\xi)$ in terms of $|\Psi|^2$ given by Eqs. (1) and (5) with $U = u/m_0\gamma\beta^2c^2 = \pm e\Omega/m_0\gamma\beta^2c^2$ must be supplemented with Eqs. (17) and (18). In the following, we present a self-consistent description of the wake-field-beam interaction in the long-beam limit (viz. $k_p\sigma_z \gg 1$ or $k_p^2 \gg |\partial^2/\partial\xi^2|$). Here, we get

$$n_1 = \mp n_b g |\Psi|^2 \ . \tag{19}$$

Substituting for n_1 from (19) into (18), we have

$$(\nabla_\perp^2 - k_p^2)\Omega = \pm 4\pi e n_b g |\Psi|^2 \ . \tag{20}$$

On the other hand, Eq. (1) takes the form

$$i\epsilon \frac{\partial}{\partial z}\Psi = -\frac{\epsilon^2}{2}\nabla_\perp^2 \Psi \pm \frac{e}{m_0\gamma\beta^2c^2}\Omega\Psi \ . \tag{21}$$

We observe that, in the long-beam limit, $g=1$, and the screening of the beam space charge is provided by Eq. (19). This is the so-called adiabatic screening. Equations (20) and (21) describe the self-interaction of a relativistic electron (positron) beam traveling in a collisionless unmagnetized plasma. If we solve (20) for Ω, we should find that it is a function of $|\Psi|^2$, i.e., $\Omega = \Omega(|\Psi|^2)$. Consequently, by putting the latter in (21) we get a nonlinear Schrödinger equation, which can be analyzed.

IV. SELF-FOCUSING AND SELF-PINCHING OF THE DRIVING BEAM

In order to illustrate how the system of equations (20) and (21) self-consistently describes the beam self-focusing, we consider two limiting cases. First, we assume that the beam-spot size is larger than the plasma wavelength (viz. $k_p R \gg 1$ or $k_p^2 \gg |\nabla_\perp|^2$). Here, Eq. (20) becomes

$$\Omega = \pm \frac{4\pi e n_b}{k_p^2} |\Psi(r,z)|^2 \ . \tag{22}$$

Substituting for Ω from (22) into (21), we obtain the cubic nonlinear Schrödinger equation

$$i\epsilon \frac{\partial}{\partial z}\Psi = -\frac{\epsilon^2}{2}\nabla_\perp^2\Psi - \frac{n_b}{n_p\gamma}|\Psi|^2\Psi \ . \tag{23}$$

Note that in (23), we have $U = -(n_b/n_p\gamma)|\Psi|^2$. Equation (23) is similar in structure to the equation that describes the self-focusing of a coherent electromagnetic beam in a nonlinear medium [19].

By using (6), we obtain the average energy associated with the relativistic charged-particle beam

$$\mathcal{E} = \frac{2\pi}{\mathcal{N}^2}\int_0^\infty \left[\frac{\epsilon^2}{2}|\nabla_\perp\Psi|^2 - \frac{n_b}{n_p\gamma}|\Psi|^4\right] r\,dr \ , \tag{24}$$

and by using (8) it is readily seen that the beam energy is not conserved. In fact, the charged-particle beam interacting with the plasma is an open system. However, by using (8) it can be proved that the quantity

$$\begin{aligned}\mathcal{A} &\equiv \mathcal{E} + \frac{1}{2}\left[\frac{2\pi}{\mathcal{N}^2}\frac{n_b}{n_p\gamma}\int_0^\infty |\Psi|^4 r\,dr\right] \\ &= \frac{2\pi}{\mathcal{N}^2}\int_0^\infty \left[\frac{\epsilon^2}{2}|\nabla_\perp\Psi|^2 - \frac{1}{2}\frac{n_b}{n_p\gamma}|\Psi|^4\right] r\,dr \ ,\end{aligned} \tag{25}$$

is conserved; namely

$$\frac{d}{dz}\mathcal{A} = 0 \ . \tag{26}$$

However, this is not the averaged beam energy. Furthermore, by substituting (25) into the virial equation (9), we get

$$\frac{d^2}{dz^2}R^2 = 4\mathcal{A} = \text{const} \ , \tag{27}$$

which can be immediately solved with the initial conditions $R_0 = R(z=0)$ and $(dR/dz)_{z=0} = 0$. We find

$$R^2(z) = R_0^2 + 2\mathcal{A}z^2 \ . \tag{28}$$

Equation (25) shows that $\mathcal{A}$ is negative (positive) if the thermal energy,

$$(2\pi/\mathcal{N}^2)(\epsilon^2/2)\int_0^\infty |\nabla_\perp\Psi|^2 r\,dr\ ,$$

is smaller (larger) than half of the self-energy,

$$(2\pi/\mathcal{N}^2)(n_b/n_p\gamma)\int_0^\infty |\Psi|^4 r\,dr\ .$$

Consequently, the charged-particle beam would focus (defocus), as is evident from Eq. (28) for the beam caustic. A stationary solution also occurs when the thermal energy is exactly balanced by half of the self-energy ($\mathcal{A}=0$). It corresponds to a stabilization of the beam at the initial radius, which is analogous to the Bennett pinch equilibrium [20]. A simple criterion for finding the threshold of self-focusing can now be established. First of all, we observe that

$$\mathcal{A}=\frac{P^2(z)}{2}-\tfrac{1}{2}\mathcal{V}_b(z)=\frac{P_0^2}{2}-\tfrac{1}{2}\mathcal{V}_0=\text{const}\ , \tag{29}$$

where

$$\mathcal{V}_b(z)\equiv\frac{n_b}{n_p\gamma}\frac{2\pi}{\mathcal{N}^2}\int_0^\infty|\Psi|^4 r\,dr \tag{30}$$

and

$$\mathcal{V}_0\equiv\mathcal{V}_b(z=0),\quad P_0=P(z=0)\ . \tag{31}$$

If we assume a Gaussian profile as an initial condition for the BWF, we find then

$$P_0^2\approx\frac{\epsilon^2}{R_0^2} \tag{32}$$

and

$$\mathcal{V}_0\approx\frac{n_b}{2\gamma n_p}\ . \tag{33}$$

Accordingly, we have

$$\mathcal{A}\approx\frac{1}{2}\left[\frac{\epsilon^2}{R_0^2}-\frac{1}{2}\frac{n_b}{n_p\gamma}\right]\ . \tag{34}$$

By imposing $\mathcal{A}\approx 0$, we immediately obtain the threshold condition

$$\frac{\epsilon^2}{R_0^2}\approx\frac{1}{2}\frac{n_b}{n_p\gamma}\ . \tag{35}$$

Since

$$\frac{\epsilon^2}{R_0^2}=\frac{T}{m_0\gamma\beta^2c^2}\approx\frac{T}{m_0\gamma c^2}\ , \tag{36}$$

where T is the transverse beam temperature expressed in energy units, we can write (35) in terms of the thermal velocity $v_{\text{th}}\equiv(T/m_0\gamma)^{1/2}$

$$\beta_\perp\equiv\frac{v_{\text{th}}}{c}\approx 0.7\left[\frac{n_b}{n_p\gamma}\right]^{1/2}\ . \tag{37}$$

Equation (37) is precisely the threshold for the Weibel (or the filamentation) instability [14]. As an illustration, we mention that if the plasma number density is $n_p=10^{17}$ cm^{-3}, the beam-number density is $n_b=10^{16}$ cm^{-3}, and the beam energy is $\gamma=100$, then (37) gives $v_{\text{th}}/c\approx 0.022$.

Next, we consider the limit in which the beam-spot size is smaller than the plasma wavelength [viz. $k_pR\ll 1$ or $k_p^2\ll|\nabla_\perp|^2$)]. For this case, (20) gives

$$r\frac{\partial}{\partial r}U=-\left[\frac{2\pi}{\mathcal{N}^2}\right]2K_0\int_0^r|\Psi(r',z)|^2r'dr'\ , \tag{38}$$

where we have made use of the relation between U and Ω, and have defined

$$K_0=e^2\frac{N/\sigma_z}{m_0\gamma\beta^2c^2}\ . \tag{39}$$

Consequently, the virial equation (9) becomes

$$\frac{d^2}{dz^2}R^2=4\mathcal{E}-4\mathcal{V}-\frac{16\pi^2}{N^4}K_0\int_0^\infty\left[\int_0^r\Psi(r',z)|^2r'dr'\right]\times|\Psi(r,z)|^2r\,dr\ . \tag{40}$$

By observing that

$$d^2R^2/dz^2=2[Rd^2R/dz^2+(dR/dz)^2]\ ,$$

Eq. (40) with the initial conditions $R_0=R(z=0)$ and $(dR/dz)_{z=0}=0$ allows us to establish the equilibrium condition ($d^2R/dz^2=0$):

$$\frac{2\pi}{\mathcal{N}^2}\frac{\epsilon^2}{2}\int_0^\infty|\nabla_\perp\Psi(r,0)|^2r\,dr=\frac{4\pi^2}{\mathcal{N}^4}K_0\int_0^\infty\left[\int_0^r|\Psi(r',0)|^2r'dr'\right]\times|\Psi(r,0)|^2r\,dr\ , \tag{41}$$

where (6) and (10) have been used. Note that $\Psi(r,0)$ represents the initial condition for (1) when the potential U is given by (38). Since it is arbitrary, we choose an initial Gaussian profile in order to find the pinch equilibrium. In this case, we have

$$\frac{2\pi}{\mathcal{N}^2}\frac{\epsilon^2}{2}\int_0^\infty|\nabla_\perp\Psi(r,0)|^2r\,dr=\frac{\epsilon^2}{2R_0^2}\ , \tag{42}$$

and

$$\frac{4\pi^2}{\mathcal{N}^4}K_0\int_0^\infty\left[\int_0^r|\Psi(r',0)|^2r'dr'\right]|\Psi(r,0)|^2r\,dr=\tfrac{1}{2}K_0\ . \tag{43}$$

Thus, the equilibrium condition (41) becomes

$$\frac{\epsilon^2}{R_0^2}=K_0\ , \tag{44}$$

which is the well-known Bennett self-pinch equilibrium condition [20]. It can be equivalently obtained by starting from the well-known envelope equation including the self-interaction, and imposing a perfect beam neutralization [21]. By introducing the definition of the beam current

$$I=\frac{eN\beta c}{\sigma_z}\approx\frac{eNc}{\sigma_z}\ , \tag{45}$$

and using the transverse-beam-emittance definition in terms of the transverse temperature T [see, Eq. (36)], Eq. (45) gives the well-known Bennett pinch relation [20,21] (cgs units):

$$\frac{I^2}{c^2}=N_z T\ , \tag{46}$$

where $N_z=N/\sigma_z$ estimates the number of particles per unit longitudinal beam length.

V. SUMMARY

In this paper, we have presented a theory for a self-consistent interaction between the wake field and the driving relativistic electron (positron) beam in an unmagnetized, overdense, collisionless plasma. For our purposes, we have employed the recently proposed *thermal wave model for relativistic charged-particle-beam propagation* [17], and in addition, the fluid equations are used to study the wake-field dynamics in the presence of the transverse profile of the beam density. We thus have a consistent coupling between the driving beam and the wake fields, and our treatment represents an improvement of previous approaches [3,5], which have completely ignored the reaction of the wake field on the driver and did not consider the spatial evolution of the beam.

When the wavelength of the wake field is smaller than the longitudinal beam length, then the self-consistent interaction of the plasma wake field and the driving bunch is governed by the pair of equations (20) and (21). The latter are analyzed in two limiting cases in order to investigate the beam self-focusing or self-defocusing and the self-pinch equilibrium. It is found that the present model is capable of reproducing the main results found in the conventional theory of the beam self-interaction.

It is of interest to note that in two space dimensions our equations (20) and (21) are similar in structure to Eqs. (1) and (2) of Mironov *et al.* [22], who considered localized nonlinear wave structures in the nonlinear photon accelerator. Accordingly, for the two-dimensional problem, we expect that the numerical solutions of (20) and (21) could also yield a specific class of nonlinear wave structures consisting of spatially localized BWF's trapped by a locally deformed plasma-wave lattice, which are similar to the self-localized color centers in ionic crystals.

Finally, we stress that the thermal wave model has the advantage of describing the evolution of the relativistic beam profile due to its interaction with the plasma waves which are excited by the beam itself. Our results have demonstrated that the self-consistent beam-plasma interaction in multidimensional space is governed by Eqs. (1), (17), and (18). Numerical studies of the latter using realistic boundary conditions are planned. It is anticipated that numerical investigations involving the boundary-value problems of our general set of equations could be quite involved. However, the results deduced from this study should certainly throw significant light on the dynamics of the relativistic beam, which is employed for producing intense wake fields for the purpose of accelerating electrons to very high energies.

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