# Fréedericksz transitions in zero-field distorted nematic liquid crystals

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(Received 30 July 1991; revised manuscript received 22 October 1991)

The twist, splay, and bend Fréedericksz transitions induced by electric or magnetic fields in zero-field undistorted nematic cells are known to be second order, except the bend transition induced by an electric field, which can be first order. By a theoretical analysis, we show that similar transitions, with threshold and symmetry breaking, can be obtained in zero-field distorted cells—the distortion being imposed by a $+\varphi_b$ and a $-\varphi_b$ rotation of the director at the two boundary planes—with pure twist, predominantly splay and predominantly bend distortion. The transitions that are second order for $\varphi_b \rightarrow 0$ preserve their order for any value of $\varphi_b$, and the critical field is an increasing function of such angle, whereas the first-order bend transition induced by an electric field becomes second order above a critical value of $\varphi_b$. We discuss the stability of the possible director configurations and the role of the critical fluctuations in driving the transitions.

PACS number(s): 61.30.Gd, 64.70.Md

## I. INTRODUCTION

A nematic liquid crystal uniformly aligned between parallel planes can undergo a transition to a deformed state under the action of magnetic, electric, or optical fields. Under suitable symmetry conditions the transition resembles a first- or second-order phase transition [1]. In fact the deformed state occurs above a threshold value of the field, and the deformation breaks some symmetry element of the system. The maximum distortion angle plays the role of an order parameter. The critical constants are generally found by considering the limit of small distortions, where the free energy of the sample is well approximated by a series expansion up to terms of the fourth or sixth power in the order parameter. Such transitions have been intensively studied in recent decades and are now well known. In particular the twist, splay, and bend Fréedericksz transitions induced by static or low-frequency electric and magnetic fields are known to be second order, except the bend transition induced by an electric field, which can be first order [2,3].

In this paper we discuss the more general cases in which the boundary conditions impose a zero-field symmetric distortion, with predominantly splay, predominantly bend, and pure twist deformations, as shown in Fig. 1. The splay-type cell has received particular attention in the past decade [4–9], in view of the potential display applications. It has been shown that above a threshold value, a new type of distortion appears that is asymmetric with respect to the midplane of the cell. The behavior of such cells near the threshold field is not so clear as in the case of zero-field undistorted cells. In fact, simple analytical expressions of the free energy are not available, and so the standard perturbative approach cannot be applied. Here we tackle this problem by considering the behavior of the extremal curves of the free energy and the well-known amplitude-period relation for the curves that are periodic functions of $z$. In this way we can analyze at the same time the order of the transitions and the stability of the solutions.

The method of analysis is described in Sec. II by considering the simple case of twist-type distortion in a magnetic field; a discussion of the possible twist distortions and of their stability is given in Sec. III. In Secs. IV and

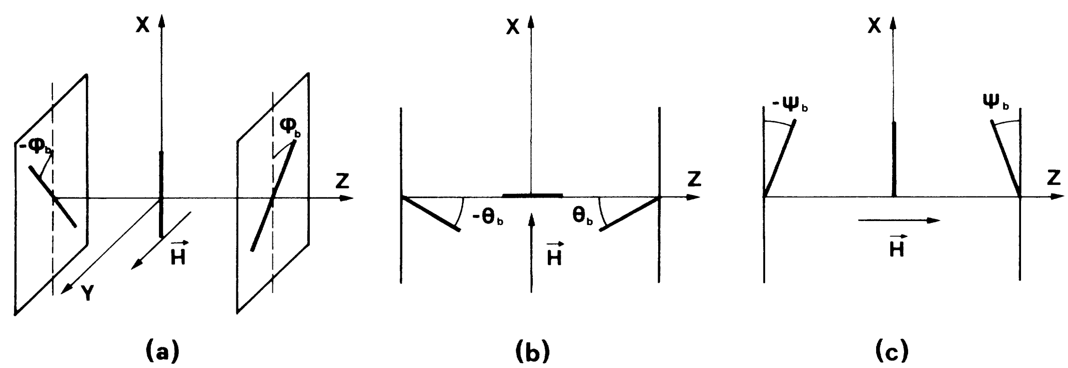


FIG. 1. Twist-type (a), bend-type (b), and splay-type (c) geometries. The director is orthogonal to the $z$ axis in (a), while it lies in the $(x,z)$ plane in (b) and (c). The figure points out the symmetries of the director configurations at zero field.

 

V the other types of transitions are considered and discussed. Finally in Sec. VI we analyze the role of the thermal fluctuations in driving the transitions. This clarifies the mechanism of the symmetric-asymmetric transitions and gives a deeper insight into the physics of these phenomena.

## II. THEORY: FIELD OF EXTREMALS AND STABILITY CRITERIA

For the sake of definiteness let us consider the magnetic-field-induced twist Fréedericksz transition. This transition is generally obtained by placing a cell with a given thickness $d$ in a uniform magnetic field **H** suitably oriented, and by increasing the amplitude $H$ of the field. In this situation all the relevant parameters only depend on the product $Hd$. From the analytical point of view it is therefore completely equivalent to consider as control parameter the thickness $d$ instead of the field intensity $H$. The main results are first obtained by considering this point of view, which allows a simpler use of the concept of field of extremals, and are then reformulated by considering $H$ as the control parameter.

Following this procedure, we consider a cell with a pure twist deformation in a given magnetic field, with the following assumptions [see Fig. 1(a)]: (i) positive diamagnetic anisotropy $\chi_a=\chi_\parallel-\chi_\perp$, where $\chi_\parallel$ and $\chi_\perp$ are the magnetic permeabilities parallel and perpendicular to the director, respectively; (ii) magnetic field **H** parallel to the $y$ axis; (iii) director **n** lying in the $xy$ plane and forming with respect to the $x$ axis a twist angle $\varphi(z)$ depending only on the space coordinate $z$; and (iv) strong anchoring conditions $\varphi(z_1)=\varphi_1$, $\varphi(z_2)=\varphi_2$. Under these hypotheses the bulk free-energy density $F(\varphi,\varphi_z)$ is given by $\frac{1}{2}k_2\varphi_z^2-\frac{1}{2}\mu_0\chi_a H^2\sin^2\varphi$, where $\varphi_z$ indicates the derivative of $\varphi$ with respect to $z$, $k_2$ is the twist elastic constant, and $\mu_0$ the free-space permeability. The stable and metastable director configurations correspond to local minima of the free energy of the system and are solutions of the Euler equation derived from the free-energy density.

Let us now consider a central family of extremals, obtained by taking all the solutions of the Euler equation having a common point in the $(z,\varphi)$ plane. In Fig. 2 such a family of extremals is plotted, with center in a point $C$ of the $z$ axis. Any arc $P_1P_2$ of any one of such curves gives a possible director configuration—for a given field $H$—for a cell having the boundaries coincident with the planes $z=z_1$ and $z=z_2$ with the strong anchoring conditions $\varphi(z_1)=\varphi_1$, $\varphi(z_2)=\chi_2$. The configuration is stable against small fluctuations (locally stable) if it gives a local minimum of the free energy. This requires that the Legendre and the Jacobi conditions be satisfied [10]. The Legendre condition reads

$$\frac{\partial^2F(\varphi,\varphi_z)}{\partial\varphi_z^2}>0\ , \tag{2.1}$$

and is always satisfied by all the free-energy expressions considered in this paper. The Jacoby condition requires that the arc $P_1P_2$ may be included in a field of extremals. This means that there exists a region $D$ of the $(z,\varphi)$ plane,

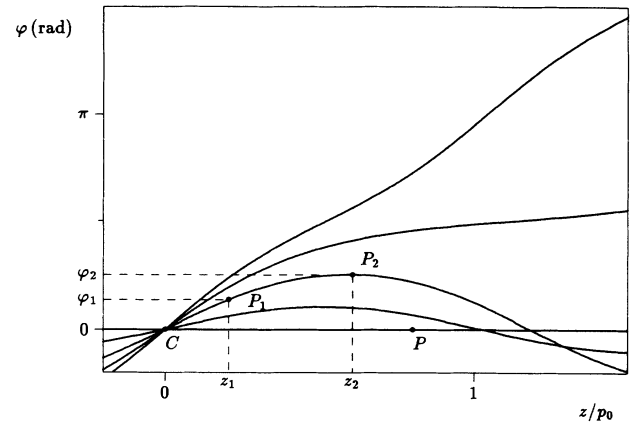


FIG. 2. Family of solutions $\varphi$ of the Euler equation for the twist-type geometry with center in the origin $C$ of the $z$ axis as a function of the normalized coordinate $z/p_0$ [see Eq. (2.3)].

which includes the arc $P_1P_2$ and which is such that one and only one curve of a family of extremals passes through every point of $D$. As a limiting case, one of the points $P_1$, $P_2$ can be coincident with the center of the family. As an example, the solution corresponding to the arc $P_1P_2$ of Fig. 2 is obviously included in a field of extremals, and therefore corresponds to a minimum of the free energy.

In the following, a main role is played by the curves of Fig. 2, which are periodic functions of $z$ with half-period $p$, and by the function $A(p)$, which gives the maximum value (amplitude) of $\varphi$ for the curve with half period $p$. The inverse function $p(A)$ is immediately obtained from the first integral of the Euler equation, and is given by

$$p(A)=\frac{2}{H}\left[\frac{k_2}{\mu_0\chi_a}\right]^{1/2}\int_0^{\pi/2}(1-\sin^2A\,\sin^2\beta)^{-1/2}d\beta\ . \tag{2.2}$$

The half-period $p$ is an increasing function of $A$ with a minimum value $p_0$ given by

$$p_0=\lim_{A\to 0}p(A)=\frac{\pi}{H}\left[\frac{k_2}{\mu_0\chi_a}\right]^{1/2}. \tag{2.3}$$

## III. FRÉEDERICKSZ TRANSITIONS FOR TWIST DEFORMATION IN A MAGNETIC FIELD

Let us consider a configuration with a zero-field distortion imposed by the strong anchoring conditions

$$\varphi(z_1)=-\varphi_b,\quad \varphi(z_2)=+\varphi_b\ . \tag{3.1}$$

We will show that a critical value $d_c$ of $d$ exists, which separates two different types of stable solutions, and which is equal to the half-period $p$ of the extremal with amplitude $A$ equal to $\varphi_b$. This means that $d_c=p(A=\varphi_b)$. The maximum distortion angle in the sample $\varphi_m(d)$ for the stable solutions is plotted in Fig. 3 (solid line). A simple inspection of Eq. (2.2) shows that

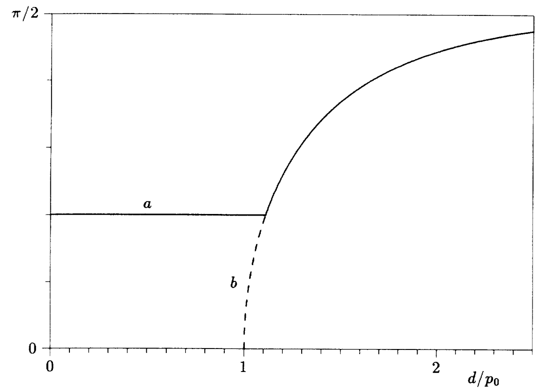


FIG. 3. Maximum distortion angle $\varphi_m$ in the sample as a function of $d/p_0$ for a symmetrically twisted cell with $\varphi_b=0.2\pi$ rad (solid line). The horizontal part (curve $a$) corresponds to symmetric solutions whose maximum distortion angle $\varphi_m$ coincides with the boundary angle $\varphi_b$. These solutions become unstable for $d>d_c=1.109p_0$. For $d>d_c$ the stable solutions are two arcs of the periodic curve whose half-period $p$ coincides with the cell thickness $d$, and therefore the maximum distortion angle $\varphi_m$ coincides with the amplitude $A$ of the extremal curve with $p=d$ (curve $b$).

the amplitude magnetic-field intensity plot at fixed $d$ is given by a curve having exactly the same shape. For $\varphi_b=0$ it reduces to the well-known plot giving the maximum distortion angle as a function of $H$ for zero-field undistorted cells.

In order to show the above results, let us first consider the case $\varphi(z_1)=\varphi(z_2)=0$. Such conditions are met by any arc $CP$ of the extremal curve $\varphi(z)=0$ of Fig. 2, corresponding to the undistorted configuration for a cell whose thickness $d$ is equal to the length of the arc. For $d<p_0$ this arc is included in a field of extremals, as is evident. By increasing $d$, the point $P$ moves to the right, and for $d>p_0$ two other solutions with the same boundary conditions are available, corresponding to the periodic curves with $p=d$ and opposite distortions. For these curves the maximum distortion angle $\varphi_m$ is equal to the amplitude A of the periodic solution, and therefore the function $\varphi_m(d)$ coincides with the function $A(p)$ with $p=d$. Furthermore it is evident that each one of these latter curves is included in the field constituted by a pencil of extremals starting from $C$, whereas the arc $CP$ of the curve $\varphi(z)=0$ can no longer be included in a field of extremals, since it is intersected by the periodic curves with small enough amplitude. The above considerations are straightforwardly extended to the case $\varphi_b\neq 0$. The role previously played by the arc $CP$ of the curve $\varphi(z)=0$ is now replaced by an arc having $C$ as midpoint and values of $\varphi$ at the ending points equal to $-\varphi_b$ and $+\varphi_b$. For any given value of $\varphi_b$, this arc lies on an aperiodic curve for small enough values of $d$, on a periodic curve for higher values.

The critical value $d_c$ of $d$ is reached when the arc $(-\varphi_b,\varphi_b)$ corresponds to a half-period of the extremal curve having $A=\varphi_b$. Beyond this limit, the arc can no longer be included in an extremal field, as is easily understood, and therefore the solution becomes unstable. However two other equivalent asymmetric solutions are now allowed, which are given by the arcs $AA'$ and $BB'$ of Fig. 4. They correspond exactly to one half-period and therefore their $\varphi_m(d)$ relation is coincident with the $A(p)$ relation, as in the previously discussed case of zero-field undistorted cells. The corresponding solutions are stable (see Sec. V), and are asymmetric with respect to the midplane of the cell. A symmetric-asymmetric transition is therefore obtained by increasing $d$, and the critical value $d_c$ of $d$ is given by the half-period of the periodic curve whose amplitude is equal to $\varphi_b$.

A suitable order parameter for this transition is the angle $\varphi_0$ at the midplane of the cell. It is easily shown that immediately above the transition point the order parameter increases as $(d-d_c)^{1/2}$. The transition is therefore second order. We first derive an approximate solution for such a relation, valid in the limit of small distortions. The asymmetric extremal curve is approximated by the harmonic function

$$\varphi(z)=A\sin\left[\frac{\pi}{d}z+\delta\right]$$
$$=A\left[\cos\delta\sin\left[\frac{\pi}{d}z\right]+\sin\delta\cos\left[\frac{\pi}{d}z\right]\right]. \tag{3.2}$$

The boundary conditions $\varphi(\pm d/2)=\pm\varphi_b$ give

$$A\cos\delta=\varphi_b\ . \tag{3.3}$$

The order parameter is given by

$$\varphi_0=A\sin\delta=(A^2-\varphi_b^2)^{1/2}\ . \tag{3.4}$$

By using the $A(p)$ relation, and recalling that for the asymmetric solution the thickness $d$ is equal to the half-period $p$, we finally obtain

$$\varphi_0=[A^2(d)-A^2(d_c)]^{1/2}\simeq 2\left[\frac{d-d_c}{p_0}\right]^{1/2}, \tag{3.5}$$

where the well-known small-amplitude approximation

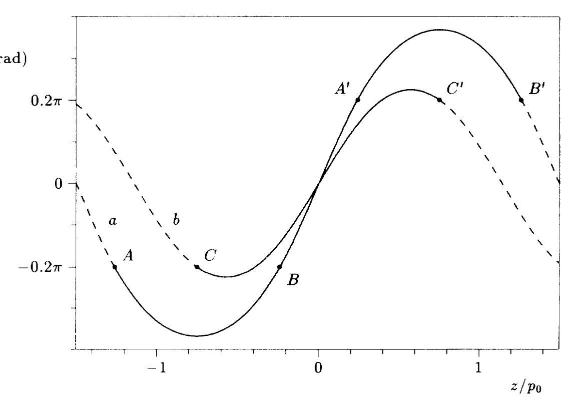


FIG. 4. Extremal curves for a symmetrically distorted cell with $\varphi_b=0.2\pi$ rad, $d=1.5p_0>d_c=1.109p_0$. Three solutions are allowed, corresponding to the arcs $AA'$ and $BB'$ of the periodic curve $a$ whose half-period $p$ is equal to $d$, and to the arc $CC'$ of a periodic curve $b$ whose half-period $p$ is greater than $d$. The latter solution is unstable since $p>d$ and therefore it cannot be included in a field of extremals.

$A \simeq 2(p/p_0 - 1)^{1/2}$ of Eq. (2.2) has been used.

From the plots of Fig. 3, and from Eq. (3.5), it appears that the Fréedericksz transition in a zero-field undistorted cell can be considered the particular case, for $\varphi_b \to 0$, of the transitions considered here. It must be further noticed that for small $\varphi_b$ values the critical thickness $d_c$ and the relation between the order parameter and $(d - d_c)$ are practically independent of $\varphi_b$. More precisely $d_c(\varphi_b) = p_0 + O(\varphi_b^2)$.

For higher values of $\varphi_b$ an approximate $(\varphi_0, d)$ relation can be found by the following procedure. Let us consider the plot of Fig. 5, which represents an asymmetric extremal curve with the origin of the $z$ axis in the point where $\varphi = 0$ and the midplane of the cell in $z_0$. From the symmetries of the curve we obtain

$$z_1 = -z_3, \quad d \equiv z_2 - z_1 = z_2 + z_3 ,$$
$$z_0 \equiv (z_2 + z_1)/2 = (z_2 - z_3)/2, \quad z_A = (z_2 + z_3)/2 . \tag{3.6}$$

Let us now define

$$a = \frac{1}{2}\left[\frac{d^2\varphi}{dz^2}\right]_{z=z_A}, \quad b = \left[\frac{d\varphi}{dz}\right]_{z=0}, \quad c = \left[\frac{dA}{dp}\right]_{p=d_c} \tag{3.7}$$

and use the approximations

$$\varphi(z) = A + a(z - z_A)^2 \quad \text{for the arc } P_3P_2 , \tag{3.8a}$$

$$A \equiv A(d) = A(d_c) + c(d - d_c) = \varphi_b + c(d - d_c) , \tag{3.8b}$$

$$\varphi_0 = bz_0 . \tag{3.8c}$$

By equating the expressions of $A - \varphi_b$ obtained from Eq. (3.8a) (for $z = z_2$) and from Eq. (3.8b) we get with the help of the last two expressions of Eqs. (3.6), $z_0^2 = c(d_c - d)/a$. Inserting this expression in Eq. (3.8c) we finally have

$$\varphi_0 = b[c(d_c - d)/a]^{1/2} . \tag{3.9}$$

If necessary, the quantities $a$, $b$, and $c$ are very easily expressed as a function of $\varphi_b$ by making use of Eq. (2.2) and of the Euler equation. Here we are only interested in pointing out the dependence of the order parameter $\varphi_0$ on $(d - d_c)$.

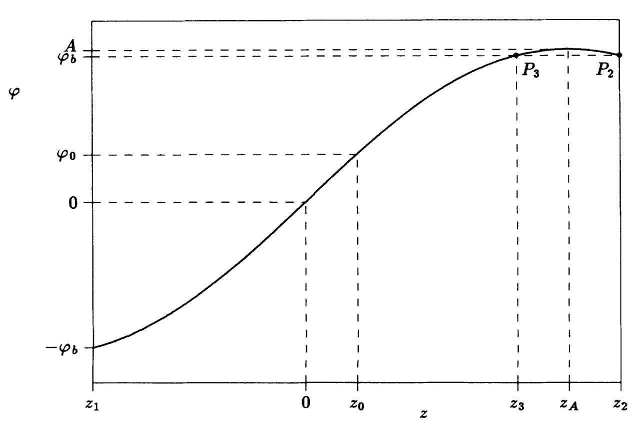


FIG. 5. Asymmetric solution for $\varphi_b = 0.2\pi$ rad and cell thickness $d = z_2 - z_1 = 1.12 p_0$ slightly above the critical value $d_c = 1.109 p_0$. The midplane of the cell is at $z = z_0$; $\varphi_0 = \varphi(z_0)$ is the order parameter.

Let us finally observe that other locally stable solutions are obtained by considering the fact that the angle $\varphi$ is defined modulus $\pi$, since $\varphi$, $\varphi \pm \pi$, $\varphi \pm 2\pi, \ldots,$ correspond to the same arrangement of molecules. This fact poses the problem of finding the absolute minimum of the free energy. It is particularly interesting to compare the above solutions, where $\varphi$ ranges from $-\varphi_b$ to $+\varphi_b$, with the solutions where $\varphi$ ranges from $\pi - \varphi_b$ to $\varphi_b$ and is equal to $\pi/2$ at the midplane. For $\varphi_b = \pi/4$ and zero field they are energetically equivalent, as evident, the second one becoming more and more stable by increasing the field, since the magnetic energy density reaches its minimum value for $\varphi = \pi/2$. By a numerical computation of the free energy we have found that the previously discussed asymmetric solutions that are locally stable above the critical field correspond to the absolute minimum of the free energy only for $\varphi_b < 0.289$ rad and a not too high field intensity.

## IV. SPLAY- AND BEND-TYPE TRANSITIONS IN A MAGNETIC FIELD

In order to discuss the other types of transitions, it is convenient to summarize the essential points of our previous analysis. A second-order transition is expected in a nematic cell with boundary conditions $\varphi(z_2) = -\varphi(z_1) = \varphi_b$ if $d = z_2 - z_1$ is the control parameter and if the Euler equation admits periodic solutions whose amplitude $A$ is an increasing function of the half-period $p$. The mechanism of the transition is the following. For any value of $\varphi_b$ and for any thickness $d$, there exists a symmetric solution with $\varphi(d/2) = -\varphi(-d/2) = \varphi_b$; such a solution is stable if and only if $A \geq \varphi_b$, or equivalently, $d \leq d_c = p(A = \varphi_b)$. For $d > d_c$ other types of solutions are allowed, with $p = d$ and $\varphi(z_1) = \pm\varphi_b$. In fact, the condition $d = p$ implies that $\varphi(z_1 + d) = -\varphi(z_1)$, and so the other boundary condition is automatically satisfied.

All the above conditions are satisfied in the splay- and bend-type Fréedericksz transitions induced by a magnetic field. The bend-type transition can be obtained with a nematic cell with $\chi_a > 0$, a magnetic field **H** parallel to the $x$ axis, and director lying in the plane $(x,z)$ [see Fig. 1(b)]. Here the dependent variable is the polar angle $\vartheta$ between the director and the $z$ axis, the boundary conditions are $\vartheta(z_2) = -\vartheta(z_1) = \vartheta_b$, and the elastic free-energy density is equal to $\frac{1}{2}(k_1 \sin^2\vartheta + k_3 \cos^2\vartheta)\vartheta_z^2$, where $k_1$ $(k_3)$ is the splay (bend) elastic constant. The half-period $p$ as a function of the amplitude $A$ is given by

$$p = \frac{2}{H}\left[\frac{k_3}{\mu_0 \chi_a}\right]^{1/2} \times \int_0^{\pi/2} \left[\frac{1 + (k_1/k_3 - 1)\sin^2 A \sin^2\beta}{1 - \sin^2 A \sin^2\beta}\right]^{1/2} d\beta . \tag{4.1}$$

It is easily found that the Legendre condition (2.1) is always satisfied and that $p$ is an increasing function of $A$.

For the splay-type transition we can use the same equations with $k_1$ and $k_3$ interchanged, $\vartheta$ replaced by $\psi=\pi/2-\vartheta$, and **H** parallel to the $z$ axis [Fig. 1(c)].

## V. ELECTRIC-FIELD-INDUCED TRANSITIONS

Let us now consider the transitions induced by an electric field. They are generally very similar to the transitions induced by a magnetic field, except for the case of the bend transition in a cell with positive dielectric anisotropy $\epsilon_a=\epsilon_\parallel-\epsilon_\perp$. Here the new fact is due to the $A(p)$ relation, which for high enough elastic and dielectric anisotropies is no more a monotonically increasing function. In the following we will concentrate our attention on this transition. It requires an unusual experimental geometry, since the electric field must be applied parallel to the boundary planes, along the $x$ axis. According to Refs. [3] and [11], the $p(A)$ relation is given by

$$p=\frac{2}{\pi}p_0\int_0^{\pi/2}\left[\frac{(1-k\sin^2A\sin^2\beta)(1-u\sin^2A\sin^2\beta)(1-u\sin^2A)}{1-\sin^2A\sin^2\beta}\right]^{1/2}d\beta\,, \tag{5.1}$$

where

$$p_0=\frac{\pi D}{V}\left[\frac{k_3}{\epsilon_0\epsilon_\perp u}\right]^{1/2},\quad k=1-k_1/k_3\,,\quad u=1-\epsilon_\perp/\epsilon_\parallel\,; \tag{5.2}$$

$D$ being the length of the cell along $x$ (with $D\gg d$), $V$ the applied voltage, and $\epsilon_0$ the free-space permittivity. The $A(p)$ plot is given in Fig. 6 with the parameters for 5CB ($k_1/k_3=0.86$, $\epsilon_\perp=8.2$, $\epsilon_\parallel=18.8$). It shows the existence of an interval where the Euler equation admits five periodic solutions with the same half-period. In a cell of thickness in this range of half-periods the solutions corresponding to the part of the curve where $A$ is a decreasing function of $p$ do not satisfy the Jacobi condition and are therefore unstable, as shown in Fig. 7.

According to the previous discussion, the transitions are expected to be first order for $\vartheta_b<A_1$, second order for $\vartheta_b\geq A_1$, where $A_1$ corresponds to the turning point of the $A(p)$ curve (see Fig. 6). In the particularly important case where $\vartheta_b=0$ the transition is therefore first order, as has been proved experimentally [3].

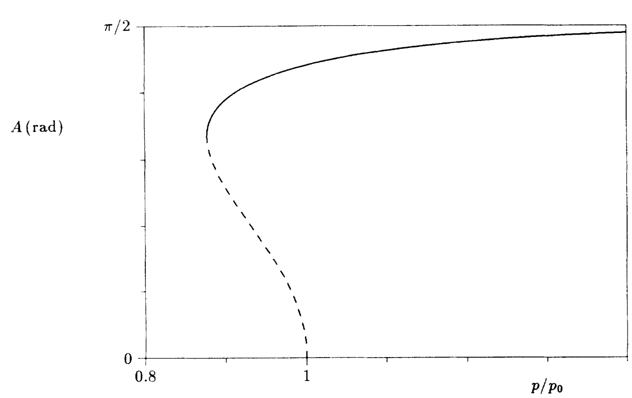


FIG. 6. Amplitude $A$ as a function of the normalized half-period $p/p_0$ for the periodic solutions in a bend geometry with an electric field for $k_1/k_3=0.86$, $\epsilon_\parallel=18.8$, $\epsilon_\perp=8.2$. The negative slope portion of the curve (dashed line), between $p=p_0$, $A=0$, and $p=0.877p_0$, $A=A_1=0.334\pi$ rad, corresponds to unstable solutions.

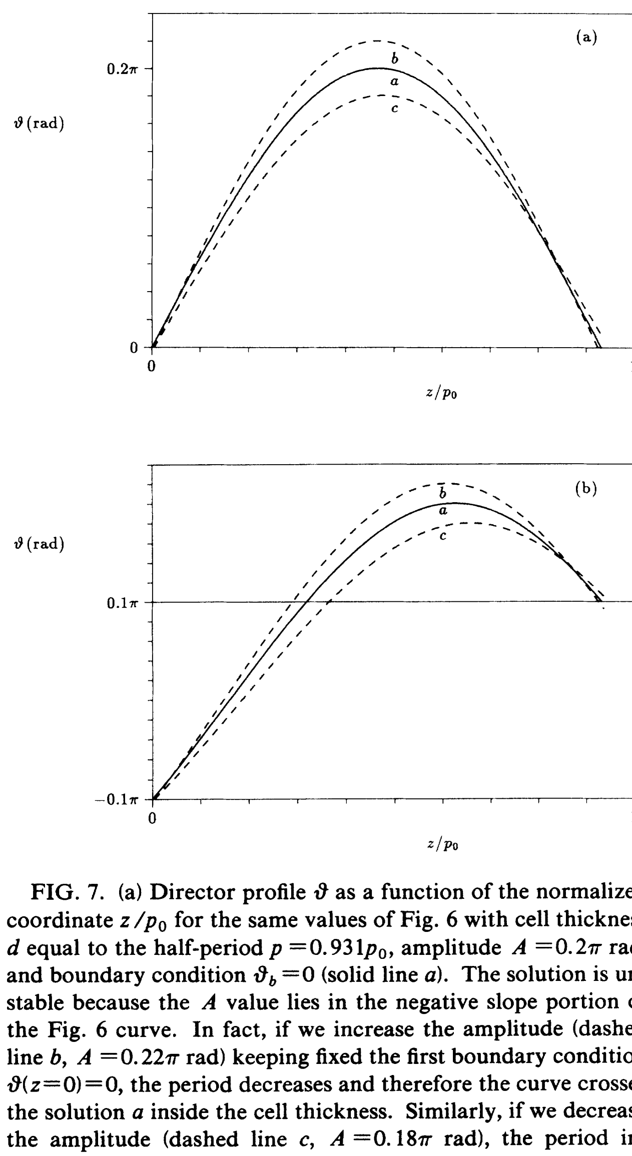


FIG. 7. (a) Director profile $\vartheta$ as a function of the normalized coordinate $z/p_0$ for the same values of Fig. 6 with cell thickness $d$ equal to the half-period $p=0.931p_0$, amplitude $A=0.2\pi$ rad, and boundary condition $\vartheta_b=0$ (solid line $a$). The solution is unstable because the $A$ value lies in the negative slope portion of the Fig. 6 curve. In fact, if we increase the amplitude (dashed line $b$, $A=0.22\pi$ rad) keeping fixed the first boundary condition $\vartheta(z=0)=0$, the period decreases and therefore the curve crosses the solution $a$ inside the cell thickness. Similarly, if we decrease the amplitude (dashed line $c$, $A=0.18\pi$ rad), the period increases and therefore the curve again crosses the solution. Hence the solution $a$ cannot be included in a field of extremals and is unstable. (b) Same as (a), but with $\vartheta_b=0.1\pi$ rad. Again the solution $a$, which corresponds to one half-period, is unstable.

It must be noted that in the actual experiments the control parameter is the voltage $V$ along $D$. However, it is straightforward to restate the theory in terms of $V$ instead of $d$, since $p_0$ is inversely proportional to $V$ [see Eq. (5.2)].

## VI. THERMAL FLUCTUATIONS AT THE CRITICAL POINT

In the preceding sections we have considered the static of the transitions. A full account of the dynamics is beyond the aim of this paper. However, it is interesting to analyze the effects of the thermal fluctuations on the director configuration at the symmetric-asymmetric second-order transitions.

As already stated, the transitions that we have considered resemble a second-order phase transition. In the Landau model for such transitions, the free energy is expanded in a power series of the order parameter, i.e.,

$$\delta\mathcal{F}=a(T)\eta^2+b(T)\eta^4+O(\eta^6) , \quad (6.1)$$

where $T$ is the temperature, which plays the role of a control parameter. At the critical point $T_c$ of a second-order phase transition, the free-energy dependence on $\eta$ becomes

$$\delta\mathcal{F}=O(\eta^4)>0 , \quad (6.2)$$

since $a(T_c)=0$ and $b(T_c)>0$. This explains the huge fluctuations of $\eta$ and their critical slowing down.

Let us now consider the effects of the thermal fluctuations in a nematic cell with strong anchoring conditions at $z=\pm d/2$. If we consider only the critical mode of the fluctuations, the director field at the critical point can be written as

$$\varphi(z,t)=\varphi_s(z)+\eta(t)f(z) , \quad (6.3)$$

where $\varphi_s(z)$ is the stationary director configuration, $\eta(t)$ is the order parameter, which can be identified with the distortion angle at $z=0$, and $f(z)$ is an even function of $z$, which is equal to zero at $z=\pm d/2$. In a zero-field undistorted cell the function $\varphi_s(z)$ is identically zero, $f(z)$ can be approximated by $\cos(\pi z/d)$, and $\eta$ coincides with the amplitude $A$ of the distorted field. Therefore huge thermal fluctuations of the amplitude $A$ are expected. Instead, in a zero-field distorted cell $\varphi_s(z)$ is a periodic function of $z$ with half-period $p=d$ and amplitude $A=\varphi_b$. Since this function reaches its minimum and maximum values at the boundary walls, where $f(z)=0$, it is easily understood that for $\varphi_b \gg \eta$ the amplitude of the function $\varphi(z,t)$ is changed very little by the fluctuations. The main effect appears as a simple shift along $z$ of the function $\varphi_s(z)$. Practically, the thermally fluctuating field $\varphi(z,t)$ can be visualized as an oscillation of the distortion in the $z$ direction. This is evidently at the origin of the symmetric-asymmetric transition.

From the Landau theory, the change $\delta\mathcal{F}$ of the free energy is expected to be of the order $\eta^4$. This can be directly proved as follows. The displacement $z_0$ of the director field is proportional to the order parameter $\eta$, as shown by Fig. 5 and Eq. (3.8c). Now for a simple rigid displacement of the function $\varphi_s(z)$ the free energy would be unchanged, since it is in any case integrated over a half-period. However, in such a translation the boundary conditions are no longer satisfied. The compensation comes from the function $\eta(t)f(z)$, and is of order $z_0^2$, since at the boundaries $d\varphi_s/dz=0$. Now Eq. (6.3) can be rewritten as

$$\varphi(z,t)=\varphi_s(z-z_0)+\delta\varphi(z,t) , \quad (6.4)$$

where the compensating function $\delta\varphi(z,t)$ is of order $z_0^2$. Since the function $\varphi_s(z-z_0)$ is in any case an extremal of the free energy, $\delta\mathcal{F}$ depends quadratically on the integral of $\delta\varphi(z,t)$. This gives $\delta\mathcal{F}=O(z_0^4)=O(\eta^4)$.

This result can be summarized as follows: a thermal fluctuation of the type (6.3) gives in general a change $\delta\mathcal{F}=O(\eta^2)$ in the free energy of the system. At the critical point, instead, $\delta\mathcal{F}=O(\eta^4)$: this is related to the fact that at the boundary walls we have both $f(z)=0$ and $d\varphi_s/dz=0$.

Such analysis offers a very simple picture of the second-order symmetric-asymmetric transition, which can be of great help in more complicated situations. For example, let us consider the splay-type geometry of Fig. 1(c). At zero field an angle $\psi_c$ exists, such that for $\psi_b>\psi_c$ a twisted director configuration with an overall twist angle $\delta\varphi=\pi$ is an extremal curve for the free energy [12,13]. The critical angle $\psi_c$ depends on the actual values of the elastic constants. The midplane of the cell is a symmetry element of the system, with the director lying in the $(x,z)$ plane. The effect of a field along the $z$ axis on the equilibrium director configuration is not easily found, since here both angles $\psi$ and $\varphi$ depend on $z$. The above considerations suggest that a symmetric-asymmetric second-order phase transition can occur at the critical point where $d\vartheta/dz=0$ at the boundaries.

## VII. CONCLUDING REMARKS

The bistable behavior of zero-field distorted nematic liquid-crystal cells under the action of an electric field has been intensively studied in recent years, in view of its potential display application. Such behavior is associated with the existence of two equivalent asymmetric states, which appear above a critical value of the field. Below the critical field, the director configuration is symmetric with respect to the midplane of the cell. In the literature the main emphasis has been focused toward the possibility of switching between different configurations and to the high-field limit.

Here we have analyzed the behavior of the cells near the symmetric-asymmetric transition. We have shown that this transition is generally second order, and that the main effect of the critical fluctuations is to give an oscillation along $z$ to the symmetric stationary solution. The actual director configuration at the critical point appears therefore as an oscillation between the two stationary asymmetric solutions. This oscillation is very slow and its amplitude $z_0$ extremely high, since the free-energy change during the oscillation is zero up to second-order terms in $z_0$. This fact allows for a very useful visualization of the transition and gives a deeper insight in the transition mechanism.

The formalism developed here together with the employed symbols are such that they point out the analogy of these transitions with the well-known Fréedericksz transitions in undistorted cells, which appear as particular cases of a more general picture. However, it must be stressed that these particular cases are singular points in the framework of the general theory. In fact, the zero values of the parameters that measure the initial distortion correspond to a change in the symmetry of the problem, since new elements of symmetry appear. Some important differences between zero-field undistorted and distorted cells are therefore expected, and deserve a few comments. A most important difference concerns the thermal fluctuations. In the ideal case of a zero-field undistorted cell a huge increase in the amplitude of the critical or nearly critical fluctuations at the transition point occurs, which can give rise to an intensity peak in the scattered light. For a zero-field distorted cell with $\varphi > 10^{-3}$ rad the amplitude of the fluctuations is practically unchanged, and some other experiment must be devised in order to detected the critical noise.

Another interesting difference concerns the case of weak anchoring conditions. A second-order transition can be obtained in a zero-field distorted cell only if the surface anchoring energies at the two boundaries are exactly the same, whereas in a zero-field undistorted cell such a condition is no longer required. The transition may be second order even in the case of strong anchoring at one of the boundary planes and weak anchoring at the other one, as can be easily understood.

Finally, let us recall that we have only considered the simplest case of distortion configurations with only one dependent and one independent variable. Other more complicated situations of first- and second-order phase transitions are found in the literature. Most of these transitions have been studied only in zero-field undistorted cells as, for instance, the transition giving rise to a periodic structure along one transverse coordinate [14,15], and the [14,15], and the transition obtained in the presence of both electric and magnetic fields [11,16]. An extension of our theory to the case of zero-field distorted cells could be of great interest both from the theoretical and from the practical point of view. In fact, the actual experiments are generally performed on slightly distorted cells, since perfectly planar or homeotropic anchoring cannot be easily obtained.


[1] See, e.g., P. G. de Gennes, *The Physics of Liquid Crystals* (Clarendon, Oxford, 1974). A transition of this type was first observed by Fréedericksz in 1927, hence the name given to these transitions.

[2] S. M. Arakelyan, A. S. Karayan, and Yu. S. Chilingaryan, Dokl. Akad. Nauk SSSR **275**, 52 (1984) [Sov. Phys.—Dokl. **29**, 202 (1984)].

[3] B. J. Frisken and P. Palffy-Muhoray, Phys. Rev. A **39**, 1513 (1989).

[4] G. D. Boyd, J. Cheng, and P. D. T. Ngo, Appl. Phys. Lett. **36**, 556 (1980).

[5] J. Cheng, R. N. Thurston, and D. W. Berreman, J. Appl. Phys. **52**, 2756 (1981).

[6] R. N. Thurston, J. Appl. Phys. **54**, 4966 (1983).

[7] R. N. Thurston, J. Cheng, and G. D. Boyd, J. Appl. Phys. **53**, 4463 (1982); R. B. Meyer and R. N. Thurston, Appl. Phys. Lett. **43**, 342 (1983).

[8] R. N. Thurston, J. Cheng, R. B. Meyer, and G. D. Boyd, J. Appl. Phys. **56**, 263 (1984).

[9] L. Komitov, G. Hauck, and H. D. Koswig, Phys. Status Solidi A **97**, 645 (1986).

[10] See, e.g., V. Smirnov, *Cours de Mathématique Supérieures,* (Mir, Moscow, 1975), Vol. IV.

[11] B. J. Frisken and P. Palffy-Muhoray, Phys. Rev. A **40**, 6099 (1989).

[12] F. M. Leslie, Pramana Suppl. No. 1, 41 (1975); G. Porte and J. P. Jadot, J. Phys. (Paris) **39**, 213 (1978).

[13] R. N. Thurston, J. Phys. (Paris) **42**, 413 (1981).

[14] F. Lonberg and R. B. Meyer, Phys. Rev. Lett. **55**, 718 (1985); G. Srajer, F. Lonberg, and R. B. Meyer, *ibid.* **67**, 1102 (1991).

[15] E. Miraldi, C. Oldano, and A. Strigazzi, Phys. Rev. A **34**, 4348 (1986).

[16] G. Barbero, E. Miraldi, and C. Oldano, Phys. Rev. A **38**, 3027 (1988).