Continuous-time description of jump clustering for fractal random processes: An alternative approach to stochastic renormalization

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A continuous-time approach for stochastic renormalization of fractal random processes is suggested. The jump clustering is described in terms of two waiting time densities: the probability density $\eta_c(\tau)$ for the time interval τ within which the clustering takes place and the density $\eta_r(\tau)$ for the relaxation time attached to the probability that a clustering event occurs. The condition of scale invariance requires that $\eta_c(\tau)$ and $\eta_r(\tau)$ are exponential functions of time. The statistical properties of the number n of jumps from a cluster are determined by a single parameter $H=\bar{\tau}_r/\bar{\tau}_c$, where $\bar{\tau}_r$ and $\bar{\tau}_c$ are the mean times attached to the distributions $\eta_r(\tau)$ and $\eta_c(\tau)$. For $0<H\leq 1$ all moments of n are infinite. In this case the renormalization leads to a Lévy flight whose characteristic exponent is equal to $2H$. The occurrence of lacunary series is not a necessary condition for the nonanalytic behavior of renormalized structure functions. The fractal behavior of the renormalized random walk is due only to the fractal behavior of the number n of jumps from a cluster.

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Ten years ago Shlesinger and Hughes introduced a general stochastic renormalization method for fractal random processes [1]. Their approach is based on an analogy with the real-space renormalization-group transformations of free energy and with the way in which the points of a Cantor set are selected. For instance, starting from a D-dimensional random walk on a hypercubic lattice for which the jump probability has the following form:

$$p(\mathbf{r})=(2D)^{-1}\sum_{l=1}^{D}[\delta(\mathbf{r}-\mathbf{u}_l)+\delta(\mathbf{r}+\mathbf{u}_l)]\ , \tag{1}$$

we get the following renormalized expression for the same quantity:

$$\tilde{p}(\mathbf{r})=(2D)^{-1}(N-1)\times\sum_{l=1}^{D}\sum_{q=0}^{\infty}N^{-(q+1)}[\delta(\mathbf{r}-\mathbf{u}_l b^q)+\delta(\mathbf{r}+\mathbf{u}_l b^q)]\ . \tag{2}$$

Here $\mathbf{r}$ is the position vector, $\mathbf{u}_l$, $l=1,2,\ldots$ are the lattice constants and $b>1$, $N>1$ are two scaling parameters.

The Shlesinger-Hughes renormalization approach has been successfully applied to a broad class of problems from condensed-matter physics, hydrodynamics, biophysics, nonlinear optics, or even economics or scientometry (see [2–5] and references therein). Although very useful, the general physical significance of the renormalization transformations like Eq. (2) is not very clear. It is obvious that Eq. (2) is based on the assumption that a kind of clustering takes place. However, no explicit mechanism for the way in which the clustering occurs has been suggested.

The purpose of this paper is to present a stochastic renormalization method for which a more detailed description of the jump clustering dynamics is possible. Unlike the case of the Shlesinger-Hughes method, the starting point of our approach is not the analogy with other renormalization procedures, but the way in which the time evolution of jump clustering can be described.

We shall consider a symmetric Markovian random walk in a d-dimensional continuous space and denote by $p(\mathbf{r})d\mathbf{r}$ the probability that the displacement of a jump is between $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$. The elementary jumps are grouped into clusters of variable size. A renormalized jump is in fact a succession of elementary jumps forming a cluster. Denoting by $\xi(n)$ the probability that a cluster consists of n jumps, it follows that the renormalized jump probability density $\tilde{p}(\mathbf{r})$ is given by

$$\tilde{p}(\mathbf{r})=\sum_{n=1}^{\infty}\xi(n)[p(\mathbf{r})\otimes]^n \tag{3}$$

where $[p(\mathbf{r})\otimes]^n$ is the n-fold convolution product of $p(\mathbf{r})$.

Our approach is based on the following assumptions.

(a) The jump clustering is a collection of independent events. We outline that the way in which the jumps are clustering should not be confounded with the random walk itself which is Markovian.

(b) The temporal development of the jump clustering is described in terms of two different times: the time in which the clustering as a whole occurs and the time required for a clustering event. We shall assume that the clustering process is rate-determining; that is, the random walk itself is very fast. Therefore, we can consider that all jumps from a cluster take place essentially at the same time.

(c) The time evolution of the clustering process is scale invariant, i.e., the distribution functions of the two times introduced above have the same form for any initial time.

By making use of the hypotheses (a) and (b) we can express $\xi(n)$ as an average over time

$$\xi(n)=\int_0^\infty \eta_c(\tau)\lambda(\tau)^{n-1}[1-\lambda(\tau)]d\tau \ , \tag{4}$$

where $\eta_c(\tau)$ is the probability density of the clustering time and $\lambda(\tau)$ is the probability that a clustering event takes place in the time interval $(0,\tau)$. $\lambda(\tau)$ can be expressed as

$$\lambda(\tau)=\int_0^\tau \eta_r(\tau)d\tau \ , \tag{5}$$

where $\eta_r(\tau)$ is the probability density of the time required for the occurrence of a clustering event. $\lambda(\tau)$ depends strongly on the shape of $\eta_r(\tau)$. In fact $\eta_r(\tau)$ can be considered as the density function of the relaxation time of λ towards its asymptotical value $\lambda(\infty)=1$.

The form of the functions $\eta_c(\tau)$ and $\eta_r(\tau)$ may be determined by making use of the condition of scale invariance. Usually, the initial time corresponding to $\eta_{c,r}(\tau)$ is $\tau=0$. Assuming that the initial time is $\tau'\geq 0$ it is necessary that the densities $\eta_{c,r}(\tau-\tau')$ have the same form as $\eta_{c,r}(\tau)$. But $\eta_{c,r}(\tau-\tau')$ can be expressed by dividing $\eta_{c,r}(\tau)$ by the probabilities

$$\gamma_{c,r}(\tau')=1-\int_0^{\tau'}\eta_{c,r}(\tau'')d\tau''=\int_{\tau'}^\infty \eta_{c,r}(\tau'')d\tau''$$

that in the time interval $0,\tau'$ no events took place. We have

$$\eta_{c,r}(\tau-\tau')=\eta_{c,r}(\tau)\Big/\int_{\tau'}^\infty \eta_{c,r}(\tau'')d\tau'' \ . \tag{6}$$

Equation (6) is in fact a set of two functional equations for η_c and η_r. They can be solved by integrating over τ from τ to ∞. This yields

$$\gamma_{c,r}(\tau-\tau')\gamma_{c,r}(\tau')=\gamma_{c,r}(\tau) \ ,$$

i.e., $\gamma_{c,r}(\tau)$ are exponential functions of τ. Coming back from $\gamma_{c,r}(\tau)$ to $\eta_{c,r}(\tau)$ and making use of the normalization conditions $\int \eta_{c,r}(\tau)d\tau=1$, we get

$$\eta_{c,r}(\tau)=\bar{\tau}_{c,r}^{-1}\exp(-\tau/\bar{\tau}_{c,r}) \ , \tag{7}$$

where $\bar{\tau}_{c,r}=\int_0^\infty \tau\eta_{c,r}(\tau)d\tau$ are average times.

Now $\xi(n)$ can be easily evaluated from Eq. (4). A straightforward calculation gives

$$\xi(n)=H(n-1)!/[(H+1)\cdots(H+n)] \ , \tag{8}$$

where H is the ratio between the relaxation time $\bar{\tau}_r$ and the clustering time $\bar{\tau}_c$

$$H=\bar{\tau}_r/\bar{\tau}_c \ . \tag{8'}$$

The factorial moments corresponding to the probability $\xi(n)$ given by Eq. (8) are equal to

$$\langle n(n-1)\cdots(n-m+1)\rangle=[(m-1)!]^2 mH\sum_{k=0}^{m-1}(-1)^k[k!(m-k-1)!(H-m-k)]^{-1} \quad \text{for } m<H \ , \tag{9a}$$

$$=\infty \quad \text{for } m\geq H \ . \tag{9b}$$

Thus the behavior of the number n of jumps from a cluster is determined by the ratio between the two mean times. If $(m+1)\bar{\tau}_c\geq\bar{\tau}_r>m\bar{\tau}_c$ only the first m moments are finite. In particular, for $\bar{\tau}_c\geq\bar{\tau}_r>0$; i.e., for $1\geq H>0$ all moments are infinite. The physical significance of this fact is clear. From Eq. (4) it turns out that the factorial moments of n can be expressed as

$$\langle n(n-1)\cdots(n-m+1)\rangle=m!\int_0^\infty \eta_c(\tau)\lambda(\tau)^{m-1}[1-\lambda(\tau)]^{-m}d\tau \ . \tag{10}$$

We see that as $\lambda\to 1$ the cumulants are infinite. When $\bar{\tau}_r>\bar{\tau}_c$, $\eta_c(\tau)$ falls off faster as $\lambda(\tau)$ increases towards its asymptotic value 1 and the very large clusters have only a small contribution to the moments; in this case only the superior moments are infinite. On the contrary, when $\bar{\tau}_c\geq\bar{\tau}_r$, $\lambda(\tau)$ increases faster as $\eta_c(\tau)$ decreases and the contribution of very large clusters is significant; in this case all moments of n are infinite and we expect that $\xi(n)$ has the properties of a statistical fractal. Indeed, by evaluating the asymptotic behavior of $\xi(n)$ we come to

$$\xi(n)\cong H\Gamma(1+H)/n^{1+H} \quad \text{as} \quad n\to\infty \ , \tag{11}$$

where $\Gamma(1+H)=\int_0^\infty x^H\exp(-x)dx$ is the usual Euler's Γ function.

The next step is to discuss the renormalized random walk. We introduce the structure functions

$$\bar{p}(\mathbf{k})=\int e^{i\mathbf{k}\cdot\mathbf{r}}p(\mathbf{r})d\mathbf{r} \ ,$$
$$\bar{\bar{p}}(\mathbf{k})=\int e^{i\mathbf{k}\cdot\mathbf{r}}\bar{p}(\mathbf{r})d\mathbf{r} \ . \tag{12}$$

Combining Eqs. (3), (8), and (12), after lengthy manipulations we get the following expression for $\bar{\bar{p}}(\mathbf{k})$:

$$\bar{\bar{p}}(\mathbf{k})=H\int_0^1 y^H[\rho(\bar{p}(\mathbf{k}))+y]^{-1}dy \ , \tag{13}$$

where

$$\rho(\bar{p}(\mathbf{k}))=[1-\bar{p}(\mathbf{k})]/\bar{p}(\mathbf{k}) \ . \tag{14}$$

By replacing $(\rho+y)^{-1}$ by its inverse Mellin transform and performing the integration over y we can express $\bar{\bar{p}}(\mathbf{k})$ as a complex integral; by taking account of the corresponding poles and evaluating the complex integral we obtain

$$\bar{\bar{p}}(\mathbf{k})=1-\left[\frac{1-\bar{p}(\mathbf{k})}{\bar{p}(\mathbf{k})}\right]^H\frac{\pi H}{\sin(\pi H)}+\mathcal{M}(\rho(\mathbf{k})) \ , \tag{15}$$

where $\mathcal{M}(\rho)$ is an analytic function in ρ

$$\mathcal{M}(\rho)=\sum_{n=1}^\infty(-1)^{n+1}H(n-H)^{-1}\rho^n \ . \tag{16}$$

We assume that all moments of the nonrenormalized jump probability density exist and are finite. Thus, the initial structure function of a symmetric random walk can be represented as

$$\bar{p}(\mathbf{k})=1-\langle r_0^2\rangle|\mathbf{k}|^2/2D+o(|\mathbf{k}|^4)\ , \qquad (17)$$

where $\langle r_0^2\rangle$ is the mean square displacement corresponding to a nonrenormalized step. From Eqs. (14) and (17) it follows that

$$\rho(\mathbf{k})\approx\langle r_0^2\rangle|\mathbf{k}|^2/2D+o(|\mathbf{k}|^4)\ . \qquad (18)$$

For $1>H>0$ the nonanalytic term of Eq. (15) has a dominant contribution to $\tilde{\bar{p}}(\mathbf{k})$. By using Eq. (18) we obtain

$$\tilde{\bar{p}}(\mathbf{k})\cong 1-|\mathbf{k}|^{2H}\frac{\langle r_0^2\rangle^H}{2D}\frac{\pi H}{\sin(\pi H)}\ , \quad |\mathbf{k}|\to 0\ . \qquad (19)$$

Considering now a succession of N renormalized steps and coming back to $\mathbf{r}$ variable we can compute the probability $\tilde{P}_N(\mathbf{r})d\mathbf{r}$ that a walker is at a position between $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$ after N renormalized steps provided that the initial position was $\mathbf{r}=\mathbf{0}$. In the limit $N\to\infty$ we get the following asymptotic expression:

$$\tilde{P}_N(\mathbf{r})=(2\pi i)^{-D}\int e^{-i\mathbf{k}\cdot\mathbf{r}}\left[1-|\mathbf{k}|^{2H}\frac{\langle r_0^2\rangle^H}{2D}\frac{\pi H}{\sin(\pi H)}+\cdots\right]^N d\mathbf{k}$$
$$\cong(2\pi i)^{-D}\int\exp(-i\mathbf{k}\cdot\mathbf{r}-b|\mathbf{k}|^{2H})d\mathbf{k}\quad\text{as } N\to\infty\ , \qquad (20)$$

where

$$b=\frac{N\langle r_0^2\rangle^H}{2D}\frac{\pi H}{\sin(\pi H)}\ . \qquad (21)$$

Therefore for large N, $\tilde{P}_N(\mathbf{r})$ is described by a Lévy flight with parameters $2H$ and b given by Eqs. (8′) and (21).

To outline the differences between our approach and the Shlesinger-Hughes method we shall compute the renormalized jump probability density in the particular case of the lattice random walk described by Eq. (1). By applying the above theory we get

$$\tilde{p}(\mathbf{r})=\sum_{n=1}^{\infty}\{H[(n-1)!]^2 n/[(2D)^n(H+1)\cdots(H+n)]\}$$
$$\times\sum_{\substack{n_1,\ldots,n_D\\ \sum_l n_l=n}}\ \sum_{m_1=0}^{n_1}\cdots\sum_{m_D=0}^{n_D}\prod_{l=1}^{D}\{\delta(\mathbf{r}-\mathbf{u}_l(2m_l-n_l))/[m_l!(n_l-m_l)!]\}\ . \qquad (22)$$

The usual explanation of the nonanalytic behavior within the case of the Shlesinger-Hughes approach is related to the lacunarity in the series expansion of the renormalized jump probability [6]. For example, in Eq. (2) as $b>1$ the intermediate points $\mathbf{r}=m\mathbf{u}_l$ where $m\neq\pm b^q$ have no contributions to $\tilde{p}(\mathbf{r})$. The expansions similar to Eq. (2) are simply related to certain lacunary Taylor series which can generate the nonanalytic behavior of $\tilde{\bar{p}}(\mathbf{k})$.

However, this explanation is not valid within the framework of our approach. By examining Eq. (22) we see that we have no lacunarity; for a given n all intermediate points $-n,\ -(n-1),\ldots,\ 0,1,\ldots,n-1,n$ have contributions to $\tilde{\bar{p}}(\mathbf{k})$. The physical significance of our approach is clear: the nonanalytic behavior is simply due to the fact that the number of jumps from a cluster has the properties of a statistical fractal.

Another difference between the two approaches is related to the expansion (19) of $\tilde{\bar{p}}(\mathbf{k})$ for small wave vectors. The Shlesinger-Hughes approach leads to an expansion similar to Eq. (19) with the difference that H has a different significance and the coefficient of $|\mathbf{k}|^{2H}$ is a periodic function of $\ln|\mathbf{k}|$. In our opinion this periodicity is due to the fact that the Shlesinger-Hughes transformation has a discrete nature, whereas in our case the renormalization transformation is defined in terms of a continuous variable, the time of clustering. Although more simple, our equations still preserve the basic features of the renormalization process.

We have considered here only the space scaling. However, the method could be extended to other random variables as well. The time scaling seems to be more difficult while in this case we should consider the time interval required for the occurrence of different jumps.

The main idea of this paper is that the renormalization could be described in terms of clusters of jumps of variable size n obeying a certain probability density $\xi(n)$. The evaluation of $\xi(n)$ is not necessarily related to a continuum-time description. Unfortunately, in the case of a discrete representation the computation of $\xi(n)$ is more complicated. However, we can distinguish at least two different types of discrete models for which the function $\xi(n)$ can be evaluated analytically: (a) the clustering of jumps may be described by a hierarchical process; (b) the clustering can be considered as a branching process.

Although more complicated, the discrete approach has the advantage that it can be applied directly both to space and time scaling. This method is planned to be presented elsewhere.

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