

Correlated fluctuations in multielement systems: The stochastic-branching-process model

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A problem of correlated fluctuations in multielement systems is considered in the framework of the theory of stochastic branching processes and is formulated in terms of a population model. Correlations have been taken into account by means of the heredity function. An explicit mathematical method is proposed to solve the problem without referring to the complicated theory of general branching processes. Using this method the global statistical properties of the system (e.g., the ensemble average and the probability of extinction) have been found. Possible applications of the present model are discussed.

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I. INTRODUCTION

Various multielement systems with a local interaction of the elements invoking each other are of great interest for chemistry, physics, and biology. In particular, such systems are typical objects for investigations in evolution problems, population dynamics, branching and avalanche physicochemical processes, heuristic and neural networks, in various problems of optimization, etc. [1].

A useful model to study such systems is a stochastic branching process (SBP). The convenience of the SBP model is caused by its clearness and by the simplicity of the mathematical formulation. It allows one to make conclusions on the global evolution of the system on the basis of assumptions about its local properties. In addition, in some cases the inverse problem can also be solved [2].

One of the features of the SBP model is the existence of local fluctuations in the course of an elementary act of the SBP. The importance of fluctuations in the analysis of the complex behavior of various multielement systems has been considered in a number of publications (see, e.g., [3] and references therein).

Here we would like to emphasize another aspect of this problem, namely, the possibility of correlations of such fluctuations in real multielement physical systems. As far as we know, Ref. [4] was the first work calling the attention to this essential aspect of the problem using as an example the counting of nuclear decays.

The correlations mentioned above may occur either due to environmental and other external causes or due to some intrinsic properties of the elements. This implies that the SBP model describing these systems should include the *correlations of fluctuations* on the microscopic level.

For further consideration it is convenient to describe the SBP model in terms of population dynamics. Then it becomes clear that such fluctuations can result from variability of individuals in populations while their correlations are due to heredity.

Some attempts have been made to consider such correlations in the framework of the SBP model by introducing spatial diffusion of the reproducing entities, their mu-

tual transformations, age dependencies, etc. [5]. Note that in these cases the local correlated fluctuations of the reproducing entities number do not result from innate properties of the entities.

In contrast to these works we present here a variant of the SBP model in which the correlated fluctuations exist due to one of such innate properties, namely, the inheritance of fertility. Formally our approach constitutes the introduction of hereditary modes into the standard Watson-Galton model [6].

The concrete formulation of the model is given in Sec. II. In Sec. III we present a mathematical approach to treat the problem. We conclude that the main trends of the evolution of the system can be evaluated in the framework of this approach, which is very convenient from the computational point of view. In addition, for a wide class of concrete problems our formalism allows one to evaluate the main asymptotic properties of the SBP with hereditary modes analytically. Section IV contains an important example which illustrates the application of the general formalism. Finally, in Sec. V we summarize the main conclusions following from our investigation and discuss possible generalizations of the approach developed.

II. MODEL

Consider a number of individuals propagating independently from each other. The number of nearest descendants (children) of an individual m is a random number with a certain distribution $\phi(m)$, $m = 1, 2, 3, \dots$. The individuals have fixed identical lifetimes, so the process is considered in discrete time, i.e., the time variable adopts integer values $t = 0, 1, 2, \dots$ constituting the generations of individuals.

Let $\xi_i(t)$ be a number of descendants (not only nearest ones) of an initially given i th individual which exist at time t . Then the total number of individuals at time t is

$$\xi(t) = \xi_1(t) + \dots + \xi_q(t) \quad (2.1)$$

given q individuals at $t = 0$.

If $\phi(m)$ is the same for all individuals, we are dealing with the simple Watson-Galton branching process which

is a Markovian chain in terms of $\xi(t)$ [7].

Now we introduce heredity in the form

$$\mu_{\text{ch}} = f(m_p), \quad (2.2)$$

where μ_{ch} is the first moment of ϕ of each child of the parent that has yielded m_p children. The function f provides the direct correspondence between the expected number of grandchildren and the actual number of children. We shall call f the *heredity function*.

Equation (2.2) implies that children inherit the high (or low) fertility of their parents and the productive branches obtain a certain preference if f is a nondecreasing function. Therefore in the following we shall call the process with the property given by (2.2) the process with heredity.

For biological and physical systems described by such a process it is important to evaluate the quantities characterizing their main evolutionary trends, namely, the extinction probabilities and the ensemble average of the number of individuals. In principle this may be done by means of the formalism on general branching processes [8], as the model presented above constitutes a particular case of such processes. However, the direct application of the general formalism to the present model encounters essential difficulties.

First of all, the number of possible types of individuals in our case is infinite. Moreover, the manifold containing the types is unbounded and the structure of this manifold is considerably affected by the heredity function. These circumstances strongly complicate the evaluation of generating functionals, transition functions, and iterations of moments in the form presented by Moyal [9], Harris [6(c)] and others.

In Sec. III we describe a mathematical procedure which allows one to calculate the main global characteristics of the process with heredity without referring to the general formalism mentioned above.

III. MATHEMATICAL CONSIDERATIONS

The aim of this section is to evaluate iterations of the first moment (the ensemble average) $\langle \xi(t) \rangle$ and of the extinction probability $p_0(t) \equiv P\{\xi(t)=0\}$ of the process with heredity.

Since individuals breed independently, we assume that at $t=0$ there is a single individual with the feature μ_0 , i.e., characterized by the probability distribution $\phi_{\mu_0}(m)$, where the subscript marks the average value of the distribution. The generalization for an arbitrary number of initial individuals is obvious.

For the first iteration one obtains

$$\begin{aligned} \langle \chi(1) \rangle &= \mu_0, \\ p_0(1) &= \phi_{\mu_0}(0). \end{aligned} \quad (3.1)$$

For further consideration let us introduce the functions

$$\begin{aligned} F(k, t+1) &= \sum_{i=0}^{\infty} \phi_k(i) i F(f(i), t), \\ F(k, 1) &= k, \end{aligned} \quad (3.2)$$

and

$$\begin{aligned} Z(k, t+1) &= \sum_{i=0}^{\infty} \phi_k(i) [Z(f(i), t)]^i, \\ Z(k, 1) &= \phi_k(0), \end{aligned} \quad (3.3)$$

where k is a real positive variable.

Noting the independence of the individuals breeding, expressions (2.2) and (3.1), and using the induction method for the iterative relations (3.2) and (3.3) one can directly show that

$$\begin{aligned} \langle \chi(t) \rangle &= F(\mu_0, t), \\ p_0(t) &= Z(\mu_0, t), \end{aligned} \quad (3.4)$$

given the single initial individual with the feature μ_0 .

In contrast to the general formalism [10], the relations (3.2) and (3.3) have appeared to be very convenient for numerical calculations as well as for the analytical evaluations of trend asymptotic behavior and ζ dependencies.

For further consideration we shall assume that f is positive and nondecreasing (except one case discussed separately), i.e., the productivity is inherited and that ϕ "conserves the dimensionality," i.e., that for positive integer ν it provides

$$\langle m^\nu \rangle_{\phi_k} \equiv \sum_m m \phi_k(m) = \mathcal{P}_\nu(k), \quad (3.5a)$$

where \mathcal{P}_ν is the polynomial of the ν th degree of its argument and

$$\langle m^\nu \rangle_{\phi_k} \sim k^\nu \quad (3.5b)$$

for noninteger positive ν . The "natural" Poisson distribution

$$\phi_k(m) = \frac{k^m}{m!} \exp(-k) \quad (3.6)$$

satisfies the constraints (3.5). Moreover, for integer $\nu=2, 3, \dots$ one obtains

$$\langle m^\nu \rangle_{\text{Poisson } \phi_k} = k^\nu + k^{\nu-1}. \quad (3.7)$$

In the next section we shall use the relations (3.5)–(3.7) together with (3.2)–(3.4) to obtain asymptotic properties of the ensemble average and the extinction probability and numerical results.

Here we would like to note that even when (3.5) is not valid, numerical results are available by the direct usage of the iterative relations (3.2) and (3.3) for arbitrary ϕ and f . Numerical calculations due to (3.2) and (3.3) are trivial by virtue of the simplicity of these relations. When the constraints (3.5) are satisfied, an analytical investigation of particular cases is also possible.

IV. EXAMPLE

To exploit our approach for obtaining numerical results, one needs explicit forms of f and ϕ . These functions are given by the concrete problem. However, in this paper we are interested only in some typical properties of the system under consideration. As an example we

present here the analysis of one specific application, namely, for a problem of a population growth.

Recalling the illustrative character of the further calculations, we restrict ourselves to the two following forms of the heredity function:

$$f(k, \zeta) = \zeta k, \tag{4.1}$$

$$f(k, \zeta) = k^\zeta, \tag{4.2}$$

where ζ is a real positive parameter.

Using the relations (3.2) and (3.4) together with (3.5) for the heredity in the form (4.1) with $\zeta > 0$ one obtains

$$\langle \xi(t) \rangle \sim \mu_0^t \zeta^{0.5t^2 - 0.5t} \tag{4.3}$$

for $t \gg 1$. Moreover, for the Poisson distribution (3.6) precise iterative relations for $\langle \xi(t) \rangle$ can be evaluated:

$$\langle \xi(t+1) \rangle = \langle \xi(t) \rangle (\zeta^{t-1} \mu_0 + 1) \zeta. \tag{4.4}$$

In general, in the long-time limit (actually, as valuations show, for $t > \zeta$) the ensemble average for the process with heredity in the form (4.1) is given by

$$\langle \xi(t) \rangle \sim \exp(\gamma t^2 + \beta t), \tag{4.5}$$

where $\gamma \sim \ln \zeta$ and $\beta \sim \ln \mu_0 - \ln \zeta$.

For $\zeta > 1$ the relation (4.5) provides the growth of the ensemble average in time with the rate greater than exponential. The value $\zeta = 1$ provides the simple exponential growth. Note that the simple Watson-Galton process with $P\{\xi(\infty) = 0\} \neq 1$ provides an exponential growth of the ensemble average with the growth exponent $\gamma \sim \ln \mu_0$ [see Fig. 1(a)]. For $0 < \zeta < 1$ the relation (4.5) provides the decrease of the ensemble average down to zero for $t \rightarrow \infty$ while for short times the increase of the ensemble average may occur [see Fig. 1(c)]. In this case the increase and decrease rates are greater than exponential due to (4.5).

Using the relations (3.2) and (3.4) together with (3.5) for the heredity in the form (4.2) with $\zeta > 1$ one obtains

$$\langle \xi(t) \rangle \sim \exp \left[\beta \sum_{q=0}^t \exp(\gamma q) \right], \tag{4.6}$$

where $\gamma \sim \ln \zeta$ and $\beta \sim \ln \mu_0$.

Using the Poisson distribution (3.6) for integer $\zeta > 1$ one can evaluate the precise expression for the ensemble average:

$$\langle \xi(t) \rangle = \mu_0^{t-1} (\mu_0 + 1) (\mu_0^\zeta + 1) \cdots (\mu_0^{\zeta-t-2} + 1). \tag{4.7}$$

For $0 < \zeta < 1$ as well as for negative ζ the evaluation of the $\langle \xi(t) \rangle$ is available using the expressions (3.2) and (3.4) while for $\zeta > 1$ the relation (4.6) is valid.

Thus for $\zeta > 1$ the ensemble average of the process with heredity in the form (4.2) demonstrates an "overdrive" (exponent versus exponent) growth [see Fig. 1(a)]. For $\zeta = 1$ Eq. (4.2) reduces to the case considered above [$f(k) = k$]. For $0 < \zeta < 1$ the calculation due to (3.2) and (3.4) shows that $\langle \xi(t) \rangle$ grows slower than an exponent but reaches exponential growth asymptotically [see Fig. 1(b)]. For $\zeta = 0$ one comes to the set of trivial Watson-Galton processes (with $\mu = \text{const} = 1$) starting from the second generation. In this case $\langle \xi(t) \rangle = \text{const} = \mu_0$ [see

Fig. 1(b)]. For $\zeta < 0$ one should redefine the heredity function $f(k, \zeta)$ in $k = 0$. Defining $f(0, \zeta) = 0$, one obtains the overdrive decrease of $\langle \xi(t) \rangle$ [see Fig. 1(c)].

The extinction probability $p_0(t) \equiv P\{\xi(t) = 0\}$, obviously, increases with time and reaches its asymptote $p_0^* \equiv P\{\xi(\infty) = 0\}$ as is shown in Fig. 2. Since the probability distribution $p_n(t) \equiv P\{\xi(t) = n\}$ is defined on positive integer numbers n , it can be shown that $p_0(t) = 1$

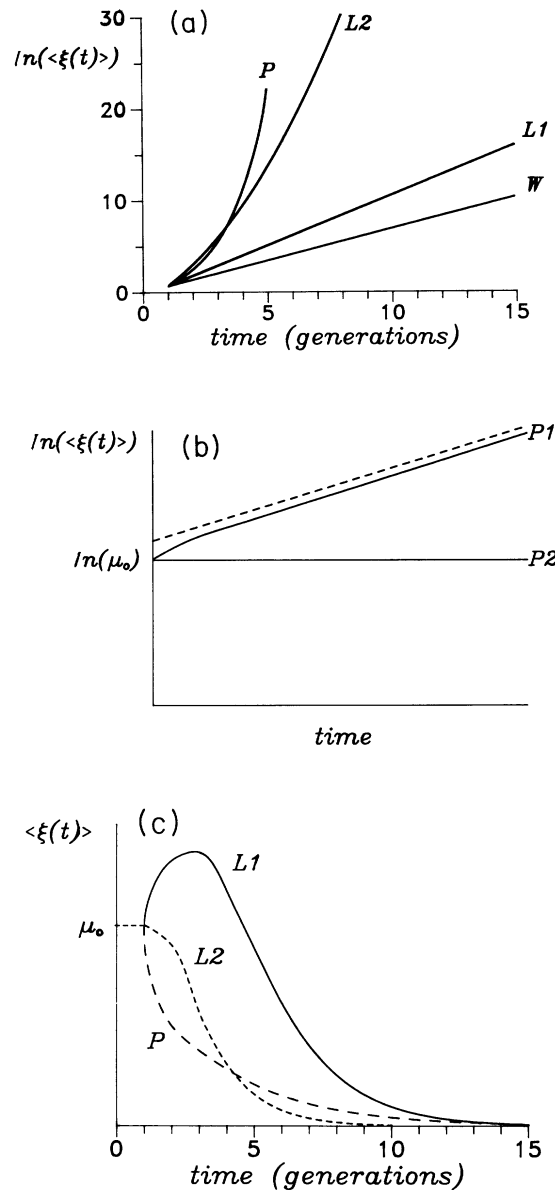


FIG. 1. The growth of the ensemble average of the number of individuals. (a) L1, heredity in the form (4.1) with $\zeta = 1$; L2, heredity in the form (4.1) with $\zeta > 1$; P, heredity in the form (4.2) with $\zeta > 1$; W, Watson-Galton process. All dependencies are plotted in the logarithmic scale. (b) P1, heredity in the form (4.2) with $0 < \zeta < 1$; the dashed line is the asymptote; P2, heredity in the form (4.2) with $\zeta = 0$. The dependencies are plotted in the logarithmic scale. (c) L1 and L2, heredity in the form (4.1) with $0 < \zeta < 1$ ($\zeta_{L1} > \zeta_{L2}$); P, heredity in the form (4.2) with $\zeta < 0$.

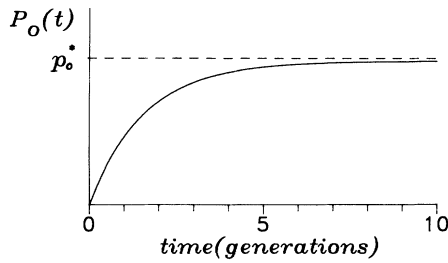


FIG. 2. The growth of the extinction probability with time (qualitative picture).

when and only when $\langle \xi(t) \rangle = 0$. Hence in all cases considered above providing $\langle \xi(t \rightarrow \infty) \rangle \rightarrow 0$ one obtains $p_0^* = 1$. In other cases $0 < p_0^* < 1$ if $\phi_{\mu_0}(0) > 0$.

Using the relations (3.3) and (3.4), it can be directly shown that p_0^* decreases with the increase of ζ (if the increase of ζ leads to the reinforcement of the heredity function growth rate as it is in our examples). Furthermore, if

$$f(k, \zeta \rightarrow \infty) \rightarrow \infty \text{ for any } k > 0 \tag{4.8}$$

then

$$p_0^* \xrightarrow{\zeta \rightarrow \infty} \phi_{\mu_0}(0) .$$

And if

$$f(k, \zeta \rightarrow \infty) \rightarrow \text{const} < \infty \text{ for some } k > 0$$

then

$$p_0^* \xrightarrow{\zeta \rightarrow \infty} \text{const} > \phi_{\mu_0}(0) .$$

For instance, for the heredity in the form (4.2) one obtains

$$p_0^* \xrightarrow{\zeta \rightarrow \infty} \phi_{\mu_0}(0) + \phi_{\mu_0}(1)\phi_1(0)[1 - \phi_1(0)]^{-1} .$$

Obviously in this case [if $\phi_{\mu_0}(0) > 0$]

$$0 < \phi_{\mu_0}(0) < p_0^* < 1$$

(see Fig. 3). For the heredity in the form (4.1) the relation (4.8) is valid and hence p_0^* converges to $\phi_{\mu_0}(0)$ with the increase of ζ (see Fig. 3).

Noting the results illustrated in Figs. 1 and 2, we come to the following qualitative forms of the probability distribution $p_n(t)$ in the long-time limit. For both forms of the heredity function with such a ζ that provides the ensemble average growth at $t \rightarrow \infty$ one obtains the double peak distribution [see Fig. 4(a)]. In this case essentially nonzero probabilities correspond to $\xi(t) = 0$ and to some exponentially (or even more) large number of individuals while the probability of a somewhat intermediate situation is very low.

The same situation arises in simple Watson-Galton processes with $\mu_0 > 1$. However, if the generalized derivative of the heredity function

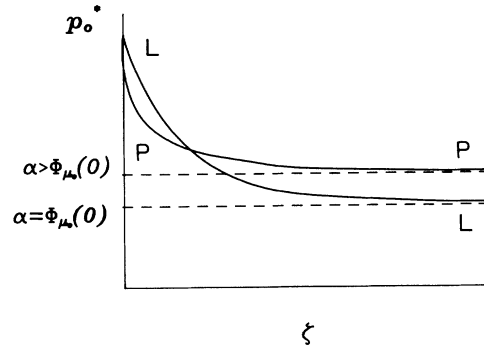


FIG. 3. Reaching the asymptotes (dashed line) α by the full extinction probabilities p_0^* with the increase of the ζ value. L , heredity in the form (4.1); P , heredity in the form (4.2). Both are qualitative pictures.

$$\frac{\partial f(k, \zeta)}{\partial k} > 1 ,$$

then the process with heredity provides the growth faster than the Watson-Galton process. Hence the right-hand peak of the probability distribution $p_n(t)$ of the process with heredity moves to the right faster than one of the Watson-Galton process [see Fig. 4(a)]. As follows from the above considerations, the residual of the peaks velocities is at least exponential. Thus there is an exponentially fast “stratification” between the population branches which are open to correlated fluctuations (variability-heredity) and those which are not.

When the heredity function provides $\langle \xi(t) \rangle = \text{const}$, the probability distribution $p_n(t)$ in the long-time limit constitutes a certain smooth distribution with the average value μ_0 and with $0 < p_n(0) < 1$ [see Fig. 4(b)].

When $\langle \xi(t \rightarrow \infty) \rangle \rightarrow 0$ the limit distribution reduces to

$$p_n(t) = \delta_{n0} , \tag{4.9}$$

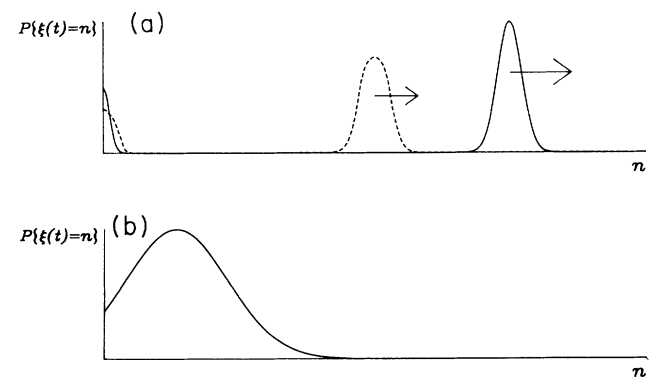


FIG. 4. The qualitative form of the probability distribution $p_n(t) \equiv P\{\xi(t) = n\}$ in the long-time limit. (a) For SBP's providing the growth of the ensemble average; solid line, the process with heredity ($\zeta > 1$); dashed line, the Watson-Galton process. The arrows mark the direction of the right-hand peaks movement. (b) For SBP's providing $\langle \xi(t) \rangle = \text{const}$.

where δ_{ij} is the Kronecker symbol.

An interesting feature of the heredity function (4.1) with $0 < \xi < 1$ is the relay of the increase and the decrease of $\langle \xi(t) \rangle$. As is clear from the above consideration, this leads to the relay of probability distributions. First, the probability distribution similar to that presented in Fig. 4(a) originates and then its collapse to the trivial distribution of the form (4.9) takes place, which implies the full extinction of the population. Such a peculiar behavior of the system is determined by the production of the large number of low-productive individuals in first generations. Initially they cause the maximum of $\langle \xi(t) \rangle$ at some time t , but they yield few descendants; these, in addition, have low productivity. That is why the population eventually comes to extinction.

V. CONCLUSION

The model presented here describes the system with the local selection of its elements due to a certain feature, namely, the fertility. The mechanism of the selection is represented by the correlation of fluctuations postulated in the model as the intrinsic feature of reproducing elements. The mathematical formalism developed in Sec. III allows one to evaluate the main characteristics of the global behavior of multielement systems with such correlations.

For more complicated forms of correlations, Eq. (2.2) should be replaced by the general functional equation. However, if the fertility is inherited, our approach is likely to be valid in this case. Furthermore, one can expect that the global behavior of the system will be the same as that described in Sec. IV, where our approach is applied to one important problem of population dynamics [11].

The existence of local correlated fluctuations significantly affects the global properties of the system. It has been shown that these global properties strongly depend on the form of local correlations characterized by the heredity function. Comparing the illustrative results

presented in Sec. IV with the behavior of various Watson-Galton processes, one can conclude that the model presented above provides the greater variability of evolutionary trends, which are strictly governed by the heredity function. In most cases the existence of correlated fluctuations given by such a function in the form (2.2) leads to the sharpening of two attractors (growth and extinction) of the system.

As is clear from the results obtained, in some cases correlated fluctuations can lead the population to the regime of the stationary stochastic process. Furthermore, the existence of correlated fluctuations can determine the relay of the "prosperity" and the extinction of the population.

Situations similar to those discussed above appear in various population models taking into account the influence of the resources provision, concurrent struggle, environmental conditions, etc. [12]. This implies that the explicit form of correlated fluctuations in the system given by the relation (2.2) can be interpreted not only in terms of intrinsic properties of individuals but also in terms of the joint effect of innate and environmental impacts. This fact constitutes the evidence of the general nature of the model presented above.

The approach developed is convenient from the mathematical point of view. It provides a simple way to investigate the global behavior of the system using explicit iterative relations (3.2) and (3.3). These relations allow one to carry out numerical calculations as well as to develop various analytical approximations and even precise evaluations concerning the evolutionary trends of multielement systems characterized by correlated fluctuations.

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