Structures in random fields: Gaussian fields

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We present two alternative methods for evaluating the probability densities of structures defined by d degrees of freedom in random fields. For Gaussian random fields, both differentiable and nondifferentiable, the application of these methods is considered in detail. The relationship between structures, maxima, and connected regions above a certain level is investigated. The connected correlation functions of these objects are expressed in terms of the autocorrelation of the field.

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I. INTRODUCTION

The assessment of the probability density of "structures" in random fields is a question of fairly general interest which is relevant to almost any branch of science. In this work we present a procedure for computing these probability densities, which is formally valid for any random field, but we shall only develop it explicitly for Gaussian random fields (GRF). As we shall find, the relevant properties of these fields are simple enough to allow a general analytical study of the most salient features of the expressions for the probability densities of structures. This general treatment is essentially homogeneous for differentiable and nondifferentiable GRF's, unlike most works on the topic, which use fundamentally different techniques for either case.

The fundamental content of this work is the result of applying the ideas and computational techniques developed in connection with point distributions [1–3], to continuous fields. In Ref. [3] we have already considered the nondifferentiable GRF, which is the high-density limit of a Poissonian distribution of points, and used some of its properties to compute certain coefficients appearing in the expressions for the probability density of clusters. However, it is only in this work that the proper rationale for some of the computational recipes presented there may be found.

Central to this work is the concept of "structure," which is homologous, in continuous fields, to the concept of cluster of type 2 (Ref. [3]) in Poissonian distributions. It is this definition of "structure" that allows a homogeneous treatment of all kinds of GRF's. At the same time, this is the definition which is most meaningful from a physical point of view.

Antecedents to this work may be found in a previous paper [4] by the present author. There we considered the particular case in which the structures were cylindrical filaments, and roughly outlined the general computational procedure. However, although this procedure was essentially correct, it contained a misconception that led us to miss a practically constant numerical factor which, as we shall find here, may be computed easily. We shall also find that in most cases it is possible to use a much simpler computational procedure than that described in Ref. [4].

The approach assumed in this work is quite different from the standard ones. In fact it has been conceived and developed independently of previous works in the field. The contact with those works is at most peripheral and reduced to the cross-checking of some expressions in the cases and regimes in which both approaches overlap. This is the reason why we make only passing reference to the extensive literature on the field. On the other hand, although, as we have said, the main ideas of the work originated from previous works on point distributions, we have tried to present it in a manner as self-contained as possible. However, knowledge of the mentioned works will redound to its fuller appreciation.

We have said that some of the ideas portrayed in this work are nonstandard. In this situation we could have developed a formalism and notation that would have allowed us to present this work in a more straightforward manner. However, since this work is intended for application in a variety of fields, each one, perhaps, with its peculiar formalism, we have preferred to sacrifice precision and conciseness to the detailed understanding of the working of the ideas in question. Once these ideas are assimilated it should be easy to frame them within the formalism of each particular field.

All the results presented in this work admit a formal demonstration, unless otherwise stated. But to avoid the use of a particular formalism and for brevity, we shall generally substitute it by a verbal argument. In Sec. II we define the concept of "structure," discuss its advantages, and present the general descriptions of the methods to be used. Section III will be devoted to the relatively simple regime of rare structures in GRF's (high-threshold limit). In Sec. IV we shall consider the general case for GRF's; we first develop the analytical treatment and afterwards give some recipes for numerical computations. Section V is devoted to the study of the correlations of the structures and other general considerations.

II. DEFINITION OF "STRUCTURE" AND GENERAL DESCRIPTION OF THE METHOD

In Ref. [3] we presented a discussion on the concept of structure in the context of point distributions. In that

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context the structures were clusters of points. The possibility of extending this concept to use it in the context of continuous random fields has been discussed in Ref. [4], so we shall not reproduce those discussions here. Instead, we shall explain, by means of some examples, why the approach implied by this concept to the problems to which it applies is more physically meaningful than more standard ones, and then we shall give the general definition of structure.

Consider the case of an electric component through which a current is passing. Assume that the intensity of the current is, as a function of time, a homogeneous GRF $I(t)$. The element will fuse if the mean value of the intensity within an interval Δt is above a given threshold $\bar{I}_0$. In this situation one may be interested in weighting the mean life of the element against the surely increasing price of rising $\bar{I}_0$. In the standard approximations to this problem one starts by constructing with $I(t)$ a new GRF $\bar{I}(t)$ which at each time t is given by the mean value of $I(t)$ over an interval of length Δt centered on t; this operation is called filtering (see Ref. [4]). Then one may compute the probability per unit time of excursion sets of the field $\bar{I}(t)$ above the level $\bar{I}_0$, which in this example is numerically equal to that of upcrossing [5] or downcrossing points. However, those points or sets are very strongly correlated and to obtain the probability of having one (one is enough for the element to fuse) or more within a given span of time we shall need all their correlations [6,7]. The mere probability density of upcrossing points is not enough to obtain the mean life of the element, which is the relevant physical quantity. To this purpose, some knowledge of their distribution in time is required. When one studies this distribution, one finds that the points form very tight clusters of size less than Δt. These clusters contain typically several points and the mean distance between them is much greater than Δt. This conspicuous property of the distribution of upcrossing points is what gives rise to the concept of structure. Each of the clusters of points is identified with a structure. To compute the mean life of the element we will need the correlations of the structures, but they are much milder than the correlations of the points within the structures themselves. If the structures are rare enough their correlations may be neglected or, at most, only the two-point correlation function will be required.

From the above discussion it could seem that the concept of structure is merely an instrument to separate the clustering of the upcrossing points into two convenient and qualitatively different hierarchical levels. But, actually, what we call structure is the underlying global prop-

[redacted]

certain threshold. Obviously, provided we can find such point, we shall find a nonzero measure set satisfying the same conditions. The main question which the concept of structure has been designed to answer is which points correspond to really distinct underlying structures.

The standard procedure to deal in general with the above problem is as follows: the field $f(\mathbf{X})$ is filtered with the window function corresponding to the global property in question. The mean density and clustering properties of the maxima above the given threshold of the resulting field, $\bar{f}(\mathbf{X})$, are then obtained and used to compute the above-mentioned probability. Alternatively, one may use the mean density of connected regions above the given threshold and the clustering properties of their representative (with some criteria) points. These approaches, however, would be very complicated even for the simplest problems and, anyway, their validity is restricted to certain types of structures only (which we shall characterize by $L=0$). Furthermore, neither the concept of local maxima nor the number density of connected regions may be defined for a nondifferentiable field.

In our approach, we take advantage of the fact that the filtering of $f(\mathbf{X})$ on scale s built this scale in the autocorrelation of $\bar{f}(\mathbf{X})$. In other words, we use the fact that the values of $\bar{f}(\mathbf{X})$ at points whose distance from each other is much less than s are generated by essentially the same values of $f(\mathbf{X})$. This leads us to an associating criteria, different from the one implied by connectedness: all the points lying within a distance approximately equal to s (more precision later) and such that $\bar{f}(\mathbf{X}) \geq \bar{f}_0$, form an associated region at threshold $\bar{f}_0$. Each associated region corresponds to a structure. However, it should not be forgotten that the structure itself is something of linear dimensions approximately equal to s, while the associated region has, for a GRF, linear dimensions approximately equal to s/ν for differentiable fields and approximately equal to s/ν^2 for nondifferentiable fields ν being the value of the threshold in units of the rms value of $\bar{f}(\mathbf{X})$.

We may illustrate what we have said with another practical example. A widely used model [8] for the large-scale structure of the universe assumes that galaxies form at the peaks above some threshold of the mass density field (which is assumed to be a GRF) smoothed in an appropriate scale. Clearly, this model has been chosen merely as an expedient way of picking out a point process, without much physical grounds. Any physically grounded model should incorporate the following fact: assume that it may be shown that galaxies can only form at places where the total mass within a certain sphere is above some threshold (in more realistic models the proba-

is, the number of peaks and connected regions within it, as well as their height and size distributions and correlations, is irrelevant for the first-order (in the sense that only the mean density and not the distribution of matter within the sphere is taken into account) model we are considering. Furthermore, this configuration is the result of filtering with the sphere in question the matter distribution inside the sphere and it its immediate neighborhood. So, it will not even be very helpful in making higher-order models; the relevant property to this purpose being the distribution of smaller-scale structures within and around the sphere.

We shall find that, for a differentiable GRF, the ratios between the number densities of associated regions, connected regions and peaks, goes rather quickly to 1 as the threshold increases. For the model we have mentioned, the number density of peaks is only about 10% higher than that of associated regions, that is, of "galaxies." So, taking into account the fact that we are using a rather gross model, its conclusions will probably not be modified very much by changing the number density of peaks by that of associated regions. However, this example serves well the purpose of illustrating our point, since it is easy to realize that by adding more and more power on the small scale, the ratio between these number densities for a fixed threshold can be made arbitrarily large. Taking this process to the limit we shall obtain a nondifferentiable field for which neither the local maxima nor the number of connected regions are defined (talking improperly: their number density is infinite). But there is no reason to believe that the validity of the first-order approximation we are considering is restricted to differentiable fields. In fact its validity depends on the distribution of matter on scales smaller, but comparable to, that of the sphere in question, and not on the distribution on much smaller scales. So, we realize that if we want to apply the model in question to any possible situation in which it could be valid, this can be done only by using the concept of structure or its counterpart for the filtered field, the associated regions. The reason for this is that only these concepts allow us to represent properly, in all possible situations, the underlying global physical property which is relevant to our first-order model.

Consider a random field $f(\mathbf{X})$ with $\mathbf{X}\in\mathbb{R}^N$ and a closed and non-self-intersecting family of surfaces in $\mathbb{R}^N$, connected or not, represented by $S(\mathbf{X},\mathbf{Y},\beta)$, where S is obviously not a function, but a set, the surface. Here $\mathbf{X}$ is a parameter which represents the position of the surface in $\mathbb{R}^N$, using to this purpose the position of a representative point. $\mathbf{Y}$ stands for a parameter belonging to some region $\mathbb{R}_L$ of an L-dimensional manifold and β stands for all the fixed parameters entering the definition of S. All the values of $\mathbf{X}_1$ and $\mathbf{Y}_1$, such that the integral of $f(\mathbf{X})$ over the interior of $S(\mathbf{X}_1,\mathbf{Y}_1,\beta)$, which we call $\bar{f}(\mathbf{X}_1,\mathbf{Y}_1)$, is above some level, are said to correspond to a d-dimensional ($d=N+L$) S structure at the quoted level. Two pairs of values, $\mathbf{X}_1,\mathbf{Y}_1$, $\mathbf{X}_2,\mathbf{Y}_2$, are said to correspond to different S structures if and only if their corresponding S do not intersect. We have separated all the d degrees of freedom (DOF) of the S structure into N translationals (in $\mathbb{R}^N$) and L "interior" (other than translational) degrees of freedom (IDOF). We could have represented all DOF by a single d-dimensional variable $\mathbf{X}$ with $\mathbf{X}\in\mathbb{R}^N\times\mathbb{R}_L$. This we shall do when the mentioned separation between the DOF is irrelevant. However, as we shall find, this difference is sometimes important.

In the model we considered before, the surfaces are spheres, the parameter $\mathbf{X}$ is the position of their centers (or another representative point) in $\mathbb{R}^3$; there are no $\mathbf{Y}$ parameters, and β corresponds to the radius of the sphere. The corresponding spherical structures only have translational DOF (note that with respect to the family S, $\mathbf{X}$, $\mathbf{Y}$, and β are just parameters, but with respect to the corresponding structures $\mathbf{X}$ and $\mathbf{Y}$ are DOF while β remains a parameter). In the example considered in Ref. [4] the surfaces were cylinders parametrized by their position and orientation in $\mathbb{R}^3$, as well as by their radius r and length L. The corresponding cylindrical structures (filaments) have five DOF: three translational plus two rotational ($\mathbf{Y}\in[0,\pi/2]\times[0,2\pi]$); the parameters are r and L. We could introduce an additional DOF by making r/L variable within certain limits e_1,e_2 and using the volume of the cylinder as a parameter, in this case $\mathbf{Y}\in[0,\pi/2]\times[0,2\pi]\times[e_1,e_2]$.

The criteria we have used to decide which structures are distinct, namely, their nonoverlapping, may seem somewhat arbitrary. One could wonder why not count as distinct structures those whose fractional overlapping is smaller than a given one, or those such that the correlation of their corresponding fields is smaller than a certain value. But, as we have discussed in some detail in Refs. [3] and [4], whenever the concept of structure has a clear physical meaning, it is quite sharply defined and the size of the associated regions are substantially smaller than the structures themselves. In these situations the dependence of the number density of structures on the particular criteria chosen is entirely negligible. On the other hand, when the structures are very common, they physically merge with each other and their exact definition is to a large extent arbitrary. However, the definition that we have chosen is the most expedient and its interpretation in mathematical terms is clear and simple, even when the physical one is not.

In this work we shall use the concept of structure only in the sense of a single global property of $f(\mathbf{X})$. However, this is only a first step in the use of this concept. A more detailed use of this concept will deal with the distribution of the field within the structure. For example, in the case of cylindrical filaments, we could start by setting constraints on the masses in each of its halves. The treatment of this sort of problem will be a straightforward generalization of the one presented in this work, although considerably more complicated.

The essential idea on which the method presented here will be built was introduced by the authors of Ref. [1]. This idea is contained in a simple expression which relates the probability that a body that has been placed at random within a Poissonian distribution contains K points P_K and the probability density D_K of distinct clusters of K points

$$D_K=P_K/\langle\Delta V\rangle\ , \tag{1}$$

where $\langle \Delta V \rangle$ is the mean connected volume in which the body defining the cluster may be moved while it still contains the same K points. The probability density $D_d(\bar{f}_0)$ of structures above level $\bar{f}_0$, obeys a similar relationship,

$$D_d(\bar{f}_0)=\frac{A(\bar{f}_0)}{\langle \Delta V \rangle}, \quad A(\bar{f}_0)=\int_{\bar{f}_0}^{\infty} P(\bar{f})d\bar{f} , \tag{2}$$

where $P(\bar{f})$ is the probability distribution of the values of the filtered field $\bar{f}(\mathbf{X})$ (now $\mathbf{X}$ stands for all DOF). $\langle \Delta V \rangle$ represents here the mean volume of the associated regions above $\bar{f}_0$. The rationale for this expression, and for all similar ones, is quite simple $D_d(\bar{f}_0)$ is the mean number of structures per unit of volume. The associated region of a structure is the set of all points, corresponding to this structure, such that $\bar{f}(\mathbf{X})$ is above $\bar{f}_0$. So, $\langle \Delta V \rangle D_d(f_0)$ is the fraction of volume such that $\bar{f}(\mathbf{X})$ is above $\bar{f}_0$, which is simply the probability of $\bar{f}(\mathbf{X})$ being above $\bar{f}_0$.

When expression (2) is applied to a homogeneous field of $\mathbb{R}^N$ with no IDOF, ΔV represents simply a volume in $\mathbb{R}^N$ and $D_N(\bar{f}_0)$ is a constant density in $\mathbb{R}^N$. In general, however, when the structure has L IDOF, ΔV represents a volume in $\mathbb{R}^N \times \mathbb{R}_L$. One may wonder about the meaning of volume in $\mathbb{R}_L$, that is, about which is the appropriate measure associated with the IDOF. In Ref. [4] we discussed this question. However, this would only be an interesting question if we were to consider the field $\bar{f}$ as a d-dimensional one ($d=N+L$) in $\mathbb{R}^N \times \mathbb{R}_L$, and we were interested in the probability density of the structures in this space. But, usually, we shall be interested only in the density in the reference space $\mathbb{R}^N$ (usually $N=1,2,3$). So, we integrate the density over the IDOF. Furthermore, if we allow for inhomogeneity we may express the density of structures in $\mathbb{R}^N$ in the form

$$D_N(\bar{f}_0,\mathbf{X})=\int_{\mathbb{R}_L} \frac{A(\bar{f}_0,\mathbf{X},\mathbf{Y})}{\langle \Delta V \rangle(\bar{f}_0,\mathbf{X},\mathbf{Y})} \prod_{i=1}^{L} dy_i . \tag{3}$$

This expression is valid when the fractional change of A over the volume $\langle \Delta V \rangle$ is small. With the definition of ΔV that we shall adopt, ΔV is not really an invariant with respect to transformations of $\mathbf{Y}$. But, $A/\langle \Delta V \rangle$ is a relative density of weight 1 and, consequently, D_N is an invariant with respect to $\mathbf{Y}$ and a density with respect to $\mathbf{X}$ (coordinates in $\mathbb{R}^N$ are assumed to be Cartesian). When the fractional change of A over $\langle \Delta V \rangle$ is not small one can no longer consider the associated region of the structures to be punctual. So we cannot define a local density D_d. What we can do is to obtain the mean number of structures within some region. This may be done by a procedure which, formally, is only slightly more complicated than the one implied by (3), but we shall not present it here, since it is only of limited interest.

In expressions (2) and (3) we have used the mean volume of an associated region to compute D_d. We shall see now that there is an alternative procedure which is in general much more simple and useful. In this procedure, instead of using the volume of the associated regions above $\bar{f}_0$, we shall use the volume of the subsets of these regions for which $\bar{f}(\mathbf{X})$ is between $\bar{f}_0$ and $\bar{f}_0+\Delta\bar{f}$, $\Delta\bar{f}$ being an arbitrarily small quantity. Calling this volume $\Delta V'(\Delta\bar{f})$ we may write instead of (2)

$$D_d(\bar{f}_0)=\lim_{\Delta\bar{f}\to 0} \frac{P(\bar{f}_0)\Delta\bar{f}}{\langle \Delta V'(\Delta\bar{f}) \rangle}=\frac{P(\bar{f}_0)}{\langle \Delta V' \rangle} ,$$
$$\langle \Delta V' \rangle \equiv \lim_{\Delta\bar{f}\to 0} \frac{\langle \Delta V'(\Delta\bar{f}) \rangle}{\Delta\bar{f}} . \tag{4}$$

The homolog of expression (3) is

$$D_N(\bar{f}_0,\mathbf{X})=\int_{\mathbb{R}_L} \frac{P(\bar{f}_0,\mathbf{X},\mathbf{Y})}{\langle \Delta V' \rangle(\bar{f}_0,\mathbf{X},\mathbf{Y})} \prod_{i=1}^{L} dy_i .$$

Whether we use (3) or (4), the essential problem that we must solve to obtain D_d is the computation of $\langle \Delta V \rangle$ or $\langle \Delta V' \rangle$. From a purely formal point of view, a straightforward way of obtaining $\langle \Delta V \rangle$ (the same is valid, *mutati mutandis*, for $\langle \Delta V' \rangle$) is the following: we set some criteria to select a representative point in each associated region, and compute the probability that $\bar{f}(\mathbf{X})$ is above $\bar{f}_0$ at a point whose coordinate difference with respect to that point is $\Delta\mathbf{X}$ (now $\mathbf{X}$ stands for all DOF; $\mathbf{X}\in\mathbb{R}^K\times\mathbb{R}_L$), which we call $P_r(\bar{f}_0,\Delta\mathbf{X})$ (the possible dependence on $\mathbf{X}$ shall be omitted). Then we may write

$$\langle \Delta V \rangle=\int_0 P_r(\bar{f}_0,\Delta\mathbf{X}) \prod_{i=1}^{d} d\Delta x_i , \tag{5}$$

where 0 represents the region such that for $\mathbf{X}\in 0$ the S's corresponding to $\mathbf{X}$ (representative point) and to $\mathbf{X}+\Delta\mathbf{X}$, intersect.

However, to express P_r in terms of the correlations of $\bar{f}(\mathbf{X})$ is an extremely complex problem. P_r', the homolog of P_r when computing $\langle \Delta V' \rangle$, is more directly related to the correlations of $\bar{f}(\mathbf{X})$ than P_r, but still, the integral equations expressing this relation are complex enough to make this procedure quite inconvenient in general. A more useful procedure consists in obtaining the probability distributions of ΔV, $P(\Delta V)$. If we knew all the moments, $\langle (\Delta V)^n \rangle$, of this distribution we could write

$$P(\Delta V)=\frac{e^{-\Delta V/\langle \Delta V \rangle}}{\langle \Delta V \rangle} \left[\sum_{i=0}^{\infty} C_i L_i \left[\frac{\Delta V}{\langle \Delta V \rangle} \right] \right] ,$$

where

$$C_i=\frac{1}{(i!)^2} \left[\sum_{j=0}^{i} A_j^i \frac{\langle (\Delta V)^j \rangle}{(\langle \Delta V \rangle)^j} \right]$$

with

$$L_n(x)=\sum_{i=0}^{n} A_i^n x^i , \tag{6}$$

$L_n(x)$ being the nth-order Laguerre polynomial.

Denoting by $P(\bar{f}_2,\ldots,\bar{f}_n/\bar{f}_1,\bar{f}_0)$ the probability of $\bar{f}_2,\ldots,\bar{f}_n$ being above $\bar{f}_0$ conditional to $\bar{f}_1$ being above $\bar{f}_0$, where $\bar{f}_j$ is the value of the field at $\mathbf{X}^j$ ($\mathbf{X}$ includes all DOF), it is quite simple to show that for $n\geq 2$

$$I_{n-1}\equiv\frac{\langle(\Delta V)^n\rangle}{\langle\Delta V\rangle}=\int_{[0\times\cdots\times 0]_{n-1}}P(\bar f_2,\ldots,\bar f_n/\bar f_1,\bar f_0)\prod_{j=2}^{n}\prod_{i=1}^{d}d\Delta x_i^j,\quad \Delta x_i^j\equiv x_i^j-x_i^1 \tag{7}$$

where $[\ ]_{n-1}$ indicates that the Cartesian product is taken $n-1$ times. We shall find that the probability distributions $P(\bar f_2,\ldots,\bar f_n/\bar f_1,\bar f_0)$ [or rather, the $P'(\bar f_2,\ldots,\bar f_n/\bar f_1,\bar f_0)$ corresponding to $\Delta V'$] allow a relatively simple expression in terms of the correlations of $\bar f(\mathbf{X})$. We may then compute, in principle, all moments of $P(\Delta V)$ with $n\geq 2$. However, $\langle\Delta V\rangle$, which is the moment that we need, cannot be obtained directly by means of an integral over 0 of a simple expression in terms of the correlations of $\bar f(\mathbf{X})$. Using all expressions (7), with $n\geq 2$, we may obtain, by means of expression (6), $P(\Delta V)$ with $\langle\Delta V\rangle$ as parameter. But, if we used $P(\Delta V)$ to compute $\langle\Delta V\rangle$ we will not obtain an equation for $\langle\Delta V\rangle$, but an identity, since the procedure for obtaining $P(\Delta V)$ implied by expression (6) is valid for any values of the moments. To be able to obtain $\langle\Delta V\rangle$ we need some additional information independent of the higher-order moments. Usually, this information is presented in the form of restrictions on the possible forms of $P(\Delta V)$. If we knew that $P(\Delta V)$ has to belong to some n-parametric family, we shall need to compute only up to the $n+1$th moment to determine it. In fact, in most examples in this work, we shall only need I_1. We shall then write

$$\langle\Delta V\rangle=\frac{I_1}{g},\quad g\equiv\frac{\langle(\Delta V)^2\rangle}{\langle\Delta V\rangle^2}\ . \tag{8}$$

Knowledge of fairly general properties of the field in question is usually enough to compute g. The relations obeyed by $\langle\Delta V'\rangle$ are formally the same as those obeyed by $\langle\Delta V\rangle$

$$\begin{aligned}&\langle\Delta V'\rangle=\frac{I_1'}{g'},\quad g'\equiv\frac{\langle(\Delta V')^2\rangle}{\langle\Delta V'\rangle^2}\ ,\\ &I_1'=\int_0 P'(\bar f_2/\bar f_1,\bar f_0)\prod_{i=1}^{d}d\Delta x_i\ ,\quad \Delta x_i\equiv\Delta x_i^2\ .\end{aligned} \tag{9}$$

$P'(\bar f_2/\bar f_1,\bar f_0)$ is the probability distribution of $\bar f_2$, conditional to $\bar f_1=\bar f_0$, evaluated at $\bar f_2=\bar f_0$.

So far, we have described the procedure for obtaining the probability density of structures in any random field. From now on, we shall restrict our attention to GRF's. In Ref. [4] we used expression (3) to obtain the probability density of cylindrical filaments in a GRF ($d=5$). But there, instead of computing $\langle\Delta V\rangle$ for the whole five-dimensional space, we made the following approximation:

$$\begin{aligned}D_3(\nu)&=\tfrac{1}{2}\int_\Omega\frac{\frac{1}{2}\mathrm{erfc}(\nu)}{\langle\Delta V\rangle}d\Omega\\ &\simeq\frac{1}{2}\frac{\mathrm{erfc}(\nu)}{\langle\Delta V\rangle_3}\frac{1}{2}\int_\Omega\frac{1}{\langle\Delta V\rangle_2}d\Omega\\ &=\bar D_3(\nu)\frac{2\pi}{\langle\Delta V\rangle_2}\ ,\end{aligned}$$

where the subindexes 2 and 3 refer, respectively, to rotational and translational DOF. $\bar D_3(\nu)$ is the probability density of cylindrical filaments with a fixed orientation. Furthermore, we implicitly set $g=1$, which is very far from the truth; in this example we have $g=4\frac{64}{13}$. However, apart from these details, we do not have much to add to the description given in Ref. [4] of the method in question. Expression (18) in this reference gives an accurate approximation for $P(\bar f_2/\bar f_1,\bar f_0)$ (the notation was different there)

$$P(\bar f_2/\bar f_1,\bar f_0)\simeq\tfrac{1}{2}\mathrm{erfc}\left[A(1-B)\frac{\bar f_0}{\sigma}\right]+\frac{\exp\left[-\dfrac{\bar f_0^2A^2(1-C)C^{1/2}}{\sigma^2}\right]\mathrm{erfc}\left[\dfrac{\bar f_0}{\sigma}\left[1-\dfrac{C}{B}\right]\dfrac{AB}{C^{1/2}}\right]}{\sqrt{2\pi}(\bar f_0/\sigma)\mathrm{erfc}(\bar f_0/\sqrt{2}\sigma)} \tag{10}$$

with

$$\begin{aligned}&A\equiv[2(1-\sigma_{12}^2/\sigma^4)]^{-1/2}\ ,\\ &B\equiv\frac{\sigma_{12}}{\sigma^2}\ ,\\ &c\equiv\frac{2(BA)^2}{1+2(BA)^2}\ .\end{aligned}$$

σ is the variance of $\bar f$, which, in principle, may depend on $\mathbf{X}$; σ_{12} is the correlation between the values of $\bar f$ at $\mathbf{X}^1$ and at $\mathbf{X}^2=\mathbf{X}^1+\Delta\mathbf{X}$, which may also depend on $\mathbf{X}$. Denoting by $W(\mathbf{K},\mathbf{X},\beta)$ the window function corresponding to $S(\mathbf{X},\beta)$, which is simply the Fourier transform of the function whose value is 1 inside and 0 outside S, we have for σ,σ_{12},

$$\begin{aligned}&\sigma_{12}(\mathbf{X},\Delta\mathbf{X})=\int_{\mathbb{R}^N}W(\mathbf{K},\mathbf{X},\beta)W(\mathbf{K},\mathbf{X}+\Delta\mathbf{X},\beta)P(\mathbf{K})d\mathbf{K}\ ,\\ &\sigma(\mathbf{X})=\sigma_{12}(\mathbf{X},\mathbf{0})\ ,\end{aligned} \tag{11}$$

where $P(\mathbf{K})$ is the power spectrum of $f(\mathbf{X})$.

We must remember that $\mathbf{X}$ stands here for all DOF, but the Fourier transform refers only to the translational ones; in $\mathbb{R}^N$ (usually $N=1,2,3$). It must also be remembered that this approximation assumes that the scale of variation of σ_{12} with respect to $\mathbf{X}$ is much larger than with respect to $\Delta\mathbf{X}$.

Expression (10) is quite complex, and it is not even ex-

act. The expressions required to compute higher-order moments of $P(\Delta V)$, which one needs sometimes, are very complicated. So, as we have anticipated, the procedure which makes use of $\langle \Delta V' \rangle$ will be much more convenient in all but two cases: the high-level case, and the case of one-dimensional differentiable GRF's. In the first case the procedure is no more complex than that based on $\langle \Delta V' \rangle$, while in the latter case this last procedure is, as we shall see, impossible to use. For high $\bar{f}_0$, expression (10) is reduced to

$$P(\bar{f}_2/\bar{f}_1,\bar{f}_0) \underset{\bar{f}_0\to\infty}{\simeq} \operatorname{erfc}\left[\left[1-\frac{\sigma_{12}}{\sigma^2}\right]\frac{\bar{f}_0}{2\sigma}\right] . \tag{12}$$

This is the expression we shall use in Sec. III.

To use the procedure based on $\langle \Delta V' \rangle$ we may need to obtain some of the joint conditional distributions P'. Since the set of fields $\bar{f}_1,\ldots,\bar{f}_n$ follow a Gaussian multivariate distribution, obtaining P' is immediate,

$$P'(\bar{f}_2,\ldots,\bar{f}_n/\bar{f}_1,\bar{f}_0)=\frac{\exp(-Q)}{[(2\pi)^n\Delta]^{1/2}} \times\left[\frac{\exp}{\sqrt{2\pi}}(-\bar{f}_1^2/\sigma^2)\right]^{-1},$$

$$Q\equiv\tfrac{1}{2}\sum_{i,j=1}^{n}(\mathbf{C}^{-1})_{ij},\quad C_{ij}\equiv\langle\bar{f}_i\bar{f}_j\rangle,\quad \Delta\equiv\det\mathbf{C} . \tag{13}$$

$\mathbf{C}$ is called the covariance matrix. For $n=2$ and 3, which are the values we shall use, we find

$$P'(\bar{f}_2/\bar{f}_1,\bar{f}_0)=\frac{1}{2\sqrt{\pi}}\frac{1}{\sqrt{W(1-W/2)}} \times\exp\left[-\frac{\nu^2}{4}\frac{W}{1-W/2}\right],$$

with

$$\nu\equiv\frac{\bar{f}_0}{\sigma},\quad W\equiv 1-\frac{\sigma_{12}}{\sigma^2},$$

$$P'(\bar{f}_2,\bar{f}_3/\bar{f}_1,\bar{f}_0)=\frac{1}{2\pi\Delta^{1/2}}\exp\left[-\frac{\nu^2}{\Delta}W_1W_2W_3\right], \tag{14}$$

$$\Delta=\tfrac{1}{2}\sum_{i,j,k=1}^{3}(W_j+W_k-W_i)W_i|\delta_{123}^{ijk}|-2W_1W_2W_3,$$

with

$$W_i\equiv\tfrac{1}{2}\sum_{j,k=1}^{3}\left[1-\frac{\sigma_{jk}}{\sigma^2}\right]|\delta_{123}^{ijk}| . \tag{15}$$

We shall use these last expressions for computing $\langle(\Delta V')^3\rangle$.

III. HIGH-THRESHOLD LIMIT

In many of the problems in which one may be interested in the probability density of structures, these structures are quite rare and sharply defined. For GRF's we may consider as rare structures those for which $\nu\gtrsim 4$. In this section we shall obtain the asymptotic expressions for $D_d(\nu)$ when $\nu\to\infty$. In the following section we shall find that the fractional difference between the asymptotic and the real expression is of order ν^{-2}.

To obtain the asymptotic expressions we only need to know the first-order dependence of $W(\Delta\mathbf{X})$ on $\Delta\mathbf{X}$. At this point we find a qualitative difference between differentiable and nondifferentiable GRF's. In the former case all the first derivatives of $W(\Delta\mathbf{X})$ are zero, unlike in the latter case. The origin of this difference may be illustrated with the case of one-dimensional fields. For nondifferentiable fields the correlation between the filtered fields at two points is at first order proportional to the fractional overlapping of the intervals of length L centered at them (for a white-noise spectrum (see Ref. [3]), the proportionality constant is 1). This implies that the derivative of W, which is equal to minus the derivative of the normalized autocorrelation, is a constant different from zero. On the other hand, it is trivial to show that the first derivative of the correlation of a differentiable field is zero. So, at first order, $W\propto\Delta X$ in the former case, while in the latter $W\propto(\Delta X)^2$.

In general, for nondifferentiable GRF's W is, to first order, proportional to the nonshared volumes inside the S corresponding to $\bar{f}_1$ and $\bar{f}_2$. A very general expression for $W(\Delta\mathbf{X})$ is

$$W(\Delta\mathbf{X})\simeq W_0(\Delta\mathbf{X})=\left[\sum_{i=1}^{J}U_i^2(\Delta X_j)^2\right]^{1/2} +\sum_{i=J+1}^{d}U_j|\Delta X_i| . \tag{16}$$

The U_i are some constant coefficients. A typical situation is that in which $K_i=U_0$ for $i=1,\ldots,J$, in which case the first J DOF corresponds to an isotropic subspace.

For differentiable fields, since the first derivatives of W are zero, we have

$$W_0(\Delta\mathbf{X})=\sum_{i=1}^{d}U_i(\Delta X_i)^2 . \tag{17}$$

In this case there is no formal difference between the two types of coordinates considered in the previous case. A common situation, as much in differentiable as in nondifferentiable GRF's, is that in which there are pairs (or multiples) of coordinates such that for $\Delta X_j\le\Delta X_i$ $W(\Delta X)$ is independent of ΔX_j. These situations, which are particularly simple to handle, together with those expressed by (16) and (17), practically exhaust all the possible situations one may find.

The coefficients U_i may be computed in general by deriving expression (11) with respect to ΔX_i, once (16) or twice (17); derivation under the integral sign being possible only for differentiable fields. In some cases, however, simple geometrical considerations are sufficient (see Ref. [3]).

As we shall see shortly, the computation of I_1 and I_1' in (8) and (9) at the high-threshold limit is very simple. It is the computation of g and g' which requires some attention. We shall start by dealing with this problem in the case of nondifferentiable GRF's.

The computation of g and g' for differentiable GRF's is, as we shall find, both mathematically and conceptually

quite simple. In the case of nondifferentiable GRF's, the computation of these quantities is still fairly simple, but we cannot say the same about its conceptual foundation. The reason for this is that in this computation we have to use some knowledge about $P(\Delta V)$ which has been obtained through our work on Poissonian distributions (Ref. [3]) and the nondifferentiable GRF which is their high-density limit. Whatever we may say about $P(\Delta V)$ in this last case may also be said about any nondifferentiable GRF, since at the high-ν limit the shape of $P(\Delta V)$ does not depend on the spectrum. So all dimensionless ratios associated with $P(\Delta V)$, as g and g' will be independent of the spectrum.

We cannot discuss here the details concerning the formation of structures in nondifferentiable GRF's that would be required for a proper understanding of the forthcoming computation. What we shall do instead is to state the results needed to this purpose. One of these results is that the volume ΔV in a d-dimensional problem is proportional (with some geometrical coefficient) to the product $\prod_{i=1}^{d} Z_i$, where the Z_i's are independent quantities, each following essentially the same probability distribution as the volume (length) of an associated region in a one-dimensional problem. This last quantity may again be expressed as the product of two quantities independently distributed: the distance between the extremes of an associated region X_i and a factor f_i which accounts for the fact that the associated region occupies only a fraction of this distance. The probability distribution of X_i is given in Ref. [3] [in the form of P_i, see expression (9)]

$$P(X_i)=\sqrt{u/\pi}\,e^{-uX_i}\frac{1}{\sqrt{X_i}}\ , \tag{18}$$

where u is a constant. The probability distribution of f_i is

$$P(f_i)=1,\quad P(f_i)=f_i^{-2/5}\ \text{ for } i>1\ . \tag{19}$$

ΔV is then proportional to $\prod_{i=1}^{d} X_i f_i$ and the joint probability distribution of all these quantities is simply the product of all their probability distributions. So dropping the scale u, which is irrelevant here, we have for g

$$\begin{aligned} g &\equiv \frac{\langle (\Delta V)^2\rangle}{\langle \Delta V\rangle^2} \\ &= \frac{\int \prod_{i=1}^{d} P(X_i)P(f_i)(X_i f_i)^2 dX_i df_i}{\left[\int \prod_{i=1}^{d} P(X_i)P(f_i)X_i f_i dX_i df_i\right]^2} \\ &= \prod_{i=1}^{2}\left[\frac{\int P(X_i)P(f_i)f_i^2 X_i^2 df_i dX_i}{\int P(X_i)P(f_i)f_i X_i df_i dX_i}\right]^{(d-2)(i-1)+1} \\ &= 4\left(\tfrac{64}{13}\right)^{d-1}\ . \end{aligned} \tag{20}$$

The computation of g' is a little more complicated. Now we have for $\Delta V'$

$$\begin{aligned} &\Delta V' \propto X_1 \prod_{i=2}^{d} X_i f_i \quad \text{for } d>1\ , \\ &\Delta V' = X_1 \quad \text{for } d=1\ , \\ &P(X_1,X_i,f_i)=e^{-X_1}X_1^{2(d-1)}\prod_{i=2}^{d}\frac{e^{-X_i}}{\sqrt{\pi}}X_i^{-1/2}P(f_i)\ , \end{aligned} \tag{21}$$

where all the scales in P have been dropped. Only the first factor in $\Delta V'$ differs from those in ΔV. The origin of the other factors is the same as in (20), and their derivation is relatively simple. The situation is quite different in regards to the factor X_1 and its corresponding distribution $P(X_1)$. This is the only distribution in (21) that depends on d, and although one can justify it to some extent, it has essentially been chosen because it renders the correct value of g'. However, it may be worth mentioning that $P(X_1)$ leads to the exact distribution at least for one dimension and in the large-d limit. In this latter case we must have $g'(d)=g(d-1)$, and a similar relation for higher-order moments,

$$\begin{aligned} g' &= \frac{\int P(X_1,X_i,f_i)(\Delta V')^2\left[\prod_{i=2}^{d} dX_i df_i\right]dX_1}{\left[\int P(X_1,X_i,f_j)\Delta V'\left[\prod_{i=2}^{d} dX_i df_i\right]dX_1\right]^2} \\ &= \frac{d}{d-\frac{1}{2}}\left[\frac{64}{13}\right]^{d-1}\ . \end{aligned} \tag{22}$$

We may check that the probability distribution given in (21) is exact, as there are grounds for believing, by computing with it the dimensionless quantity

$$h'\equiv\frac{\langle\Delta V'\rangle\langle(\Delta V')^3\rangle}{\langle(\Delta V')^2\rangle^2}=\left[\frac{845}{432}\right]^{d-1}\left[\frac{d+\frac{1}{2}}{d}\right]\ . \tag{23}$$

At the end of this section we shall use (14) and (15) to compute h' for a particular example, so as to be able to check (23).

To compute I_1, we only need to insert expression (16) for W into the high-threshold limit expression for $P(\bar f_2/\bar f_1,\bar f_0)$, (12), and to substitute it in (7),

$$I_1=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\operatorname{erfc}\left[\frac{\nu}{2}W_0^{1/2}(\Delta\mathbf{X})\right]\prod_{i=1}^{d}d\Delta x_i\ , \tag{24}$$

where the integration region 0 in (7) has been substituted by $\mathbb{R}^d$, since in this limit the integral converges well inside 0, and the exact shape and size of 0 are irrelevant. Using now the following multiple integral identity:

$$\begin{aligned} &\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f[(\Delta x_1^2+\cdots\Delta x_J^2)^{1/2}+|\Delta x_{J+1}|+\cdots|\Delta x_d|]\prod_{i=1}^{d}d\Delta x_i=\Omega(J)2^{d-J}\frac{(J-1)!}{(d-1)!}\int_0^{\infty}f(s)s^{d-1}ds\ , \\ &\Omega(J)\equiv\frac{2\pi^{J/2}}{\Gamma(J/2)}\ , \end{aligned} \tag{25}$$

where $\Omega(J)$ is the total solid angle in J dimensions. Knowing that

$$\int_0^\infty \mathrm{erfc}(\sqrt{s})s^{d-1}ds=\frac{(2d-1)!!}{2^d d} \tag{26}$$

we obtain immediately I_1, and using it in expression (8), together with g [given by (20)], we obtain $\langle \Delta V\rangle$. Inserting it into expression 2 with $A(f_0)=\frac{1}{2}\mathrm{erfc}(\nu/\sqrt{2})$, we find for $D_d(\nu)$

$$D_d(\nu)\simeq\frac{\left[\prod_{i=1}^{d}U_i\right]2^{J-1}d!(\frac{16}{13})^{d-1}}{\Omega(J)(2d-1)!!(J-1)!}\nu^{2d}\mathrm{erfc}\left[\frac{\nu}{\sqrt{2}}\right]. \tag{27}$$

To obtain the probability density $D_N(\nu)$ in the reference space $\mathbb{R}^N$ we only need to integrate over the IDOF as indicated in (3). The dependence of $D_d(\nu)$ on $\mathbf{X}$ enters through ν, which is just $\bar{f}_0/\sigma(\mathbf{X})$ and through the U_i.

A particularly simple case is that in which $d=N=J$, that is, all the DOF are translational and isotropic. $D_N(\nu)$ is then the probability density of N spherical structures in $\mathbb{R}^N$. In this case we have

$$\begin{aligned}U_i=U(N)&=\frac{\Omega(N)(N-1)S}{N\Omega(N-1)},\\ D_N(\nu)&\simeq\frac{(\frac{32}{13})^{N-1}N}{(2N-1)!!\Omega(N)}\left[\frac{\Omega(N)(N-1)S}{N\Omega(N-1)}\right]^N\nu^{2N}\\ &\quad\times\mathrm{erfc}\left[\frac{\nu}{\sqrt{2}}\right],\end{aligned} \tag{28}$$

where S is a quantity depending on the spectrum, which for a white noise is equal to 1. Expression (27) has been used in Ref. [3] to compute some coefficients. But there we did not include the factor $(\frac{16}{13})^{d-1}$, so that those coefficients were reduced by this factor. To compute I_1', we use expression (14) in the limit of small W,

$$I_1'=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{e^{-(\nu^2/4)W_0(\Delta\mathbf{X})}}{\sqrt{W_0(\Delta\mathbf{X})}}\prod_{i=1}^{d}d\Delta x_i. \tag{29}$$

Using (25) again, we have

$$I_i'=\frac{1}{2\sqrt{\pi}}\frac{\Omega(J)2^{3d-J-1}}{\left[\prod_{i=1}^{d}U_i\right]\nu^{2d-1}}\frac{(J-1)!}{(d-1)!}\int_0^\infty e^{-s}s^{d-1}\frac{ds}{\sqrt{s}} \tag{30}$$

and since

$$\frac{1}{2\sqrt{\pi}}\int_0^\infty e^{-s}s^{d-1}\frac{ds}{\sqrt{s}}=\frac{(2d-3)!!}{2^d}.$$

Substituting I_1', and g' [as given by (22)] into (9) and using (4) with $P(\bar{f}_0)=e^{-\nu^2/2}/\sqrt{2\pi}$, we find

$$D_d(\nu)\simeq\frac{\left[\prod_{i=1}^{d}U_i\right]d!2^j(\frac{16}{13})^{d-1}}{\Omega(J)(2d-1)!!(J-1)!}\nu^{2d-1}\frac{e^{-\nu^2/2}}{\sqrt{2\pi}}\equiv D_d(A), \tag{31}$$

noting that

$$\mathrm{erfc}(x)\underset{x\to\infty}{\sim}\frac{1}{\sqrt{\pi}}\frac{e^{-x^2}}{x}\left[1+O\left[\frac{1}{x^2}\right]\right]. \tag{32}$$

We realize that at the limit at which they are valid, expressions (27) and (31) are equal. In Sec. IV of Ref. [3] we used expression (27) to compute some geometrical coefficients, but we did not explain there the rationale of the method.

We shall now compute g and g' for differentiable GRF's. In this case, unlike in the previous one, ΔV, and $\Delta V'$ cannot be expressed as the product of d quantities independently distributed. The reason is quite clear: in the previous case the level sets were not manifolds, but fractals of dimension $d-\frac{1}{2}$. Their geometry is so complex and it fluctuates so much from one region to another that, for a fixed value of the global maxima of the region, its length along a particular DF fluctuates by a considerable amount. This fluctuation is large enough to prevent any correlation between the length along the different DOF. Hence, these lengths are not correlated with each other. In the present case, however, for ν high enough, the associated regions are made up of a single connected region containing just one maxima. The lengths of a region along different DOF are functions of the height of this maxima and, hence, they are totally correlated with each other. For a region whose maxima have a height $\nu+s$ the length of the region along the ith DOF, $2L_i$, is given by

$$\tfrac{1}{2}\lambda_i L_i^2\simeq s,\quad L_i\propto\left[\frac{s}{\lambda_i}\right]^{1/2},$$

where λ_i is the second derivative of $\bar{f}$ (in units of its rms value) with respect to x_i. ΔV is then proportional to

$$\Delta V\propto\prod_{i=1}^{d}\left[\frac{s}{\lambda_i}\right]^{1/2}. \tag{33}$$

The probability distribution $P(s,\lambda_i)$ will be considered in more detail in the following section. But in the limit we are considering the probability distribution of λ_i is arbitrarily peaked (for an arbitrarily high ν) around its mean value, which obviously depends on ν. So only s and its probability distribution are relevant to the computation of g. $P(s)$ has the general form [see (75)]

$$\begin{aligned}P(s)&\propto\{(\nu+s)^d+O[(\nu+s)^{d-2}]\}e^{-(\nu+s)^2/2}\\ &\simeq\nu^d\left[1+d\frac{s}{\nu}-\frac{s^2}{2}+O(\nu^{-4})\right]e^{-\nu s}.\end{aligned} \tag{34}$$

But at the high-ν limit, and neglecting irrelevant scales we have

$$P(s)\propto e^{-s},$$

$$\begin{aligned}g=\frac{\langle(\Delta V)^2\rangle}{\langle\Delta V\rangle^2}&=\frac{\left[\int_0^\infty e^{-s}ds\right]\left[\int_0^\infty e^{-s}s^d ds\right]}{\left[\int_0^\infty e^{-s}s^{d/2}ds\right]^2}\\ &=\frac{\Gamma(d+1)}{\Gamma^2\left[\frac{d}{2}+1\right]}.\end{aligned} \tag{35}$$

To compute g' we must take into account that $\Delta V'$ is the limit of the ratios between the volume of a region whose field values are between ν and $\nu+\Delta\nu$, and $\Delta\nu$ itself. Now, this volume must be proportional to the sum over all DOF of the product of the thicknesses of this region along each DOF multiplied by the projected area [$(d-1)$-dimensional] of the level set $\bar{f}(\mathbf{X})=\nu$ on to the subspace in which this DOF is constant. For the mentioned thicknesses ΔL_i and areas A_i, we have

$$\Delta L_i \sim \frac{\Delta\nu}{(\lambda_i s)^{1/2}}, \quad A_i \propto \frac{s^{(d-1)/2}}{\prod_{j(\neq i)} \lambda_j^{1/2}} ,$$

so making $\Delta\nu$ go to zero we have for $\Delta V'$

$$\Delta V' \propto \frac{s^{d/2-1}}{\prod_{j=1}^{d} \lambda_i^{1/2}} . \tag{36}$$

Recalling what was said about $P(s,\lambda_i)$ at the high-ν limit, we find

$$g' = \frac{\left[\int_0^\infty e^{-s} ds\right]\left[\int_0^\infty e^{-s} s^{d-2} ds\right]}{\left[\int_0^\infty e^{-s} s^{d/2-1} ds\right]^2} = \frac{\Gamma(d-1)}{\Gamma^2(d/2)} . \tag{37}$$

From this expression we see the peculiarity of the one-dimensional case, since $\Gamma(x)\to\infty$ as $x\to 0$. So, as we have anticipated, the method given by (4) cannot be used in this case. The case $d=2$ is also peculiar, since from (36) we see that $\Delta V'$ does not depend on s, so that the nth moment is simply $\langle \Delta V' \rangle^n$.

In an equal fashion we may compute h'

$$h' \equiv \frac{\langle \Delta V' \rangle \langle (\Delta V')^3 \rangle}{\langle (\Delta V')^2 \rangle^2} = \frac{\Gamma(d/2)\Gamma(\frac{3}{2}d-2)}{\Gamma^2(d-1)} . \tag{38}$$

We shall later have the opportunity of checking this result.

To compute I_1 in the differentiable case, we only need to insert (17) into (24), and use (25) to transform the integral

$$I_1 = \frac{\Omega(d)2^d}{\left[\prod_{i=1}^{d} U_i\right]^{1/2}} \nu^d \int_0^\infty \operatorname{erfc}(s) s^{d-1} ds ,$$

$$\int_0^\infty \operatorname{erfc}(s) s^{d-1} ds = \frac{\Gamma\left[\frac{d+1}{2}\right]}{\sqrt{\pi} d} . \tag{39}$$

So, inserting this result and (35) in (2), we find

$$D_d(\nu) \simeq \frac{\left[\prod_{i=1}^{d} U_i\right]^{1/2}}{\Omega(d)2^d} \frac{\sqrt{\pi} d!}{\Gamma\left[\frac{d+1}{2}\right]} \frac{d}{\left[\Gamma\left[\frac{d}{2}+1\right]\right]^2} \times \nu^d \tfrac{1}{2}\operatorname{erfc}\left[\frac{\nu}{\sqrt{2}}\right] . \tag{40}$$

For N-dimensional spheres in N-dimensional homogeneous and isotropic GRF's ($d=N$; $L=0$)

$$U_i = U(N), \quad \left[\prod_{i=1}^{N} U_i\right]^{1/2} = [U(N)]^{N/2} ,$$

$$U(N) \equiv \frac{1}{2}\frac{d^2 W}{dr^2}(r) = -\frac{1}{2}\frac{1}{\sigma^2}\frac{d^2\sigma_{12}}{dr^2}(r) .$$

Since the window functions corresponding to two S's displaced (in $\mathbb{R}^N$) with respect to each other by an amount $\Delta\mathbf{X}$, differ only in the factor $\exp[-i\mathbf{K}\cdot\Delta\mathbf{X}]$, it is easy to show, by differentiating twice with respect to r ($\equiv|\Delta\mathbf{X}|$) under the integral sign in (11), that

$$U(N) = \frac{1}{2}\frac{\langle K^2 \rangle(N)}{N} ,$$

$$\langle K^J \rangle(N) \equiv \frac{\int P_R(K) K^{N+J-1} dK}{\int P_R(K) K^{N-1} dK} . \tag{41}$$

Generally we shall simply write $\langle K^J \rangle$; the value of N being obvious from the context. $P_R(K)$ is the power spectrum of $\bar{f}(\mathbf{X})$, that is, of the field which results from filtering $f(\mathbf{X})$ using the window function corresponding to an N sphere of radius R,

$$P_R(K) = P(K) W^2(K,R) ,$$

$$W(K,R) = (-i)^{-N} \frac{\Omega(N)}{(\sqrt{2\pi})^N} \frac{d^{N-1}}{dK^{N-1}} \frac{(1-e^{-iKR})}{K} . \tag{42}$$

For the particularly interesting cases $N=1,2,3$, we have

$$D_1(\nu) \simeq \frac{\langle K^2 \rangle^{1/2}}{2\sqrt{2\pi}} \operatorname{erfc}\left[\frac{\nu}{\sqrt{2}}\right] ,$$

$$D_2(\nu) \simeq \frac{\langle K^2 \rangle}{8\pi} \nu^2 \operatorname{erfc}\left[\frac{\nu}{\sqrt{2}}\right] ,$$

$$D_3(\nu) \simeq \frac{\left\langle \frac{K^2}{3} \right\rangle^{3/2}}{2(2\pi)^{3/2}} \nu^3 \operatorname{erfc}\left[\frac{\nu}{\sqrt{2}}\right] . \tag{43}$$

At the high-ν limit there is just one connected region in each associated region. So the probability density of N-spherical structures in isotropic N-dimensional fields is simply the probability density of connected regions (excursion sets) of the filtered field ($\bar{f}(\mathbf{X})$) above level ν. Expression (40) must then display the same asymptotic behavior as the standard expression [9] for the probability density of excursion sets, as is obvious in (43).

Computing I_1' and using it together with (37), in (9) and (4) we find for $d>1$

$$I_1' = \frac{2^{d-1}}{2\sqrt{\pi}} \frac{\Omega(d)}{\left[\prod_{i=1}^{d} U_i\right]^{1/2}} \nu^{d-1} \int_0^\infty e^{-s} s^{d-1} \frac{ds}{\sqrt{s}}$$

$$= \frac{\Omega(d)}{2\sqrt{\pi}} \frac{2^{d-2}}{\nu^{d-1}} \Gamma\left[\frac{d-1}{2}\right] ,$$

$$D_d(\nu) \simeq \frac{\sqrt{\pi}\left[\prod_{i=1}^{d} U_i\right]^{1/2} \Gamma(d-1)}{\Omega(d)2^{d-3}\Gamma\left[\frac{d-1}{2}\right]\Gamma^2\left[\frac{d}{2}\right]} \frac{e^{-\nu^2/2}}{\sqrt{2\pi}} \equiv D_d(A) . \tag{44}$$

Now, since

$$\frac{2^3(d-2)!}{\Gamma\left[\frac{d-1}{2}\right]\Gamma^2\left[\frac{d}{2}\right]}=\frac{dd!}{\Gamma\left[\frac{d+1}{2}\right]\Gamma^2\left[\frac{d}{2}+1\right]} .$$

We realize that (40) and (41) have the same asymptotic values; that is, they are equal to the extent at which they are valid. From expression (44) we see that the method given by (4) is not valid for $d=1$. This we have already seen in the expression for g' (37) which contains $\Gamma(d-1)$. Now we find it in the fact that the correlations between arbitrarily close points, which are arbitrarily large, cause in this case the logarithmic divergence of the integral in I_1', as we may see from the presence of the factor $\Gamma[(d-1)/2]$. One might expect $D_1(\nu)$ to be equal to the limit of (44) for $d\to 1$. Since the limit of the ratio of the two mentioned factors is $\frac{1}{2}$, this is actually true. However, the meaning of the mentioned limit is not clear since, to admit a proper interpretation, d must be an integer. So, although this procedure seems fairly plausible and may be much simpler than that given by (2), we must remember that in the one-dimensional case, only this last procedure is entirely justified.

Before closing this section, we shall illustrate with a few examples the use of (15) for the computation of h' at the high-ν limit. As we have seen, h' is not necessary to compute $D_d(\nu)$ at this limit, but it may be useful in more general cases. Furthermore, the value of h' obtained in this way will be useful in checking the validity of the distributions $P(\Delta V')$ given in (21) and in (37).

Using the definition of h' and expression (7) (or rather the homologous expression for $\Delta V'$), we have

$$h'\equiv\frac{\langle\Delta V'\rangle\langle(\Delta V')^3\rangle}{\langle(\Delta V')^2\rangle^2}=\frac{I_2'}{(I_1')^2} . \tag{45}$$

The computation of I_1' has already been carried out in computing $D_d(\nu)$. To compute I_2' we use (15) in the form that it takes at the low-W_i limit,

$$\Delta\underset{W_i\to 0}{\sim}\frac{1}{2}\sum_{i,j,k=1}^{3}(W_j+W_k-W_i)W_i|\delta_{123}^{ijk}| .$$

For circular structures in two-dimensional differentiable GRF's, we have

$$\begin{aligned} &W_i=Ur_i^2,\quad \mathbf{r}_i\equiv\frac{1}{2}\sum_{j,k=1}^{3}(\mathbf{X}_j-\mathbf{X}_k)\delta_{123}^{ijk},\quad r_i\equiv|\mathbf{r}_i| ,\\ &W_3=Ur_3^2,\quad W_2=Ur_2^2 ,\\ &W_1=U(r_3^2+r_2^2-2r_3r_2\cos\theta) , \end{aligned} \tag{46}$$

θ being the angle between $\mathbf{r}_2$ and $\mathbf{r}_3$. We have then for I_2'

$$I_2'=\frac{(2\pi)}{2\pi}\frac{1}{2}\int_{-\pi}^{\pi}\int_0^{\infty}\int_0^{\infty}\left[\frac{e^{-(\nu^2/4)[U(r_3^2+r_2^2-2r_3r_2\cos\theta)/(1-\cos^2\theta)]}}{[U^2r_3^2r_2^2(1-\cos^2\theta)]^{1/2}}\right]r_2r_3dr_2dr_3d\theta=\frac{\pi^2}{U^2\nu^2} ,$$

$$I_1'=\frac{\pi}{U\nu},\quad h'=1 . \tag{47}$$

For spherical structures in three dimensions the W_i are, again, those given in (46). θ is, as before, the angle between $\mathbf{r}_2$ and $\mathbf{r}_3$. We then have

$$\begin{aligned} I_2'&=\frac{4\pi(2\pi)}{2\pi(2)}\int_0^{\pi}\int_0^{\infty}\int_0^{\infty}Cr_3^2r_2^2\sin\theta\,d\theta\,dr_2dr_3\\ &=\frac{6\pi^2}{\nu^4} ,\\ I_1'&=\frac{4\sqrt{\pi}}{\nu^2},\quad h'=\frac{3\pi}{8} , \end{aligned} \tag{48}$$

where C is the parenthesis in the integrand in (47). We see that in both cases, h' agrees with the value given by (38). This is hardly surprising, since the derivation of $P(\Delta V')$ leading to (36) does not present doubts. The case of nondifferentiable GRF's is more interesting because, as we have said, the derivation of (21), which we used to obtain (23), was much more involved than in the differentiable case. So any test of the validity of (21), independent of the one provided by g', is to be welcomed. The simplest case that we may use to this purpose is the one corresponding to d spherical structures with $d>1$; the one-dimensional case giving no new information. We now have

$$\begin{aligned} &W_3=Ur_3 ,\\ &W_2=Ur_2 ,\\ &W_1=U(r_3^2+r_2^2-2r_3r_2\cos\theta)^{1/2} ,\\ &\Delta=U^2\{4r_3r_2-[r_3+r_2-(r_3^2+r_2^2-2r_3r_2\cos\theta)^{1/2}]^2\} . \end{aligned}$$

After some manipulations, I_2' may be reduced to

$$\begin{aligned} I_2'(d)&=\left[\int_{-1}^{1}\int_0^{1}\frac{\{4x-[1+x-(1+x^2-2xy)^{1/2}]^2\}^{2d-3/2}}{x^d(1+x^2-2xy)^{d-1/2}(1-y^2)^{(3-d)/2}}dx\,dy\right]\\ &\quad\times\frac{[\Omega(d)]^2(d-2)!!(2(d-1)]!}{U^{2d}\nu^6(d-3)!!\pi^2}(2\pi)^{P(d)},\quad P(d)=\begin{cases}1 & \text{for } d \text{ odd}\\ 0 & \text{for } d \text{ even} ,\end{cases}\\ I_1'(d)&=\frac{2^{d-1}\Omega(d)(2d-3)!!}{U^d\nu^{2d-1}},\quad h'(d)=\frac{I_2'(d)}{[I_1'(d)]^2} . \end{aligned} \tag{49}$$

This result agrees with (23) within the accuracy of our numerical evaluations of $I_2'(d)$ ($\simeq 0.1\%$ for $d \leq 12$). If more accuracy is required for h', (49) must be used.

From the above examples, one may realize how much work may be saved just by using the knowledge about $P(\Delta V')$ that may be obtained through general considerations. We may also note the following mathematical curiosity: the form of probability distributions $P(\Delta V')$, as much for the differentiable as for the nondifferentiable case, depends, at the high-ν limit, only on d. So any dimensionless ratio, like g' or h', is a unique function of d which may be obtained immediately from $P(\Delta V')$. Now, since all these quantities may be expressed algebraically in terms of the I_i', which are given by $2i$-multiple integrals, for each of these quantities we have an identity relating algebraically integrals of different multiplicity. The details of the particular example that one may use, which are contained in the functions W_i, cancel out, to leave the function of d corresponding to the quantity that one is computing. This fact may not be easy to prove in an independent manner, and it provides a useful tool to compute some multiple integrals. For example, we could have obtained the exact value of I_2' [in (49)]; by using (23) (assuming it is exact).

We have seen in this section that the differentiable and nondifferentiable GRF's display a sharply different behavior: the linear dimensions of their associated regions scale with ν^{-1} in the former case and with ν^{-2} in the latter. One could wonder how is it possible that a qualitative property translates into a quantitative one. After all, by adding more and more power on small scales, we can make a differentiable field look increasingly like a nondifferentiable one. But this is true only for a fixed value of ν. No matter how much the field may resemble a differentiable one at some level, its differentiable character will dominate for sufficiently high levels. The clear cut difference between the two types of fields appears only in their corresponding asymptotic expressions.

IV. THE GENERAL PROCEDURE

For moderate values of ν, the approximations used in the previous section are no longer justified and a more detailed analysis of the problem is required. To this purpose we shall study separately how the computations of I_1' and g' are affected by the finiteness of ν. Then we shall present a general expression for $D_d(\nu)$ which contains some coefficients, and show how they may be evaluated using the results of these studies.

When we computed I_1' in Sec. III we made two approximations: the functions $W(\Delta\mathbf{X})$ were substituted by their lowest-order approximations, and expression (14) was replaced by the integrand in (29). We shall start with this last question, using the full expression (14) while keeping expressions (16) and (17) for $W_i(\Delta\mathbf{X})$.

Forgetting about overall constants, which are given in (27), we have for nondifferentiable GRF's

$$I_1'(\nu) \propto \int_0^1 \frac{e^{-u^2(\nu^2/2)}}{(1+u^2)^d} u^{2(d-1)} du \ ,$$

$$I_1'(\nu) = I_1'(A)\left[1 + \sum_{i=1}^{E[(\nu^2+1)/2]} \frac{(-1)^i (d+2i-3)!!\, d(d+2)\cdots[d+2(i-1)]}{2^i i!\, \nu^{2i}} + R + D\right], \quad R = \left[\frac{1}{\nu^2}\right]^{E[(\nu^2+1)/2]} . \tag{50}$$

where $I_1'(A)$ stands for the asymptotic form of I_1' [given by (30)], and D stands for all exponentially decaying terms (EDT's), which are irrelevant to the computing procedure we shall present here. To obtain (50) we have used (25) to transform the relevant multiple integral into a single one, and made the change of variable: $u^2 = s/(1-s/2)$. For differentiable fields we proceed in a similar manner (now we have $u^2 = s^2/1 - s^2/2$) and find

$$I_1'(\nu) \propto \int_0^1 \frac{e^{-u^2(\nu^2/2)}}{(1+u^2)^{d/2}} u^{d-2} du \ ,$$

$$I_1'(\nu) = I_1'(A)\left[1 + \sum_{i=1}^{E[(\nu^2+1)/2]} \frac{(-1)^i (2i+2d-3)!!\, d(d+1)\cdots(d+i+1)}{i!\, \nu^{2i}} + R + D\right], \quad R \approx \left[\frac{1}{\nu^2}\right]^{E[(\nu^2+1)/2]} \tag{51}$$

For d odd, $I_1'(\nu)$ may be expressed as a finite combination of elementary functions, but, to our purpose, what we need are the expansions in powers of ν^{-2}. The fact that these series do not converge and, hence, for a finite value of ν have some irreducible error, is irrelevant to our purpose.

Now we shall consider how I_1' changes when approximations to $W_i(\Delta\mathbf{X})$ more accurate than (16) and (17) are used. Writing W in the form

$$W(\Delta\mathbf{X}) = W_0(\Delta\mathbf{X}) + \sum_{i,j=1}^{d} U_{ij} |(\Delta x_i)^{2-P}| |(\Delta x_j)^{2-P}| + \sum_{i,j,k=1}^{d} U_{ijk} |(\Delta x_i)^{2-P}| |(\Delta x_j)^{2-P}| |(\Delta x_k)^{2-P}| + O(|\Delta\mathbf{X}|^{4(2-P)}) \ , \tag{52}$$

where the quantity P is equal to 1 for nondifferentiable fields and to 0 for differentiable fields. For nondifferentiable fields expression (52) is true only provided that the surfaces defining the structures are differentiable at the points at which they intersect when they experience an infinitesimal displacement with respect to each other. If this were not

true for displacements along the ith DOF, (52) could contain terms which are odd with respect to Δx_i.

For differentiable fields the terms of different orders in the expansion for $W(\Delta\mathbf{X})$ are alternatively positive and negative. For nondifferentiable fields the sign of terms of order higher than 2 is not constrained, but in both cases the U_{ij} are negative. This means that when we use (52) to compute I_1' we obtain a larger value than in (50) and (51). At least in the differentiable case, this enlargement of I_1' is always more than enough to compensate for the opposite effect implied in (51). So, the actual value of I_1' is larger than $I_1'(A)$,

$$
\begin{aligned}
&I_1'(\nu)\frac{1}{2\sqrt{\pi}}\int_0 \frac{\exp\left[-\frac{\nu^2}{4}\frac{W}{1-W/2}\right]}{[W(1-W/2)]^{1/2}}\prod_{i=1}^{d} d\Delta x_i = I_1'(A)\left[1+\frac{A_1}{\nu^2}+\frac{A_2}{\nu^4}+O\left[\frac{1}{\nu^6}\right]+D\right],\\
&A_1=-[Z_1(4^P d^2+2^P d-2)+2^{P-1}d(2^P d-1)],\\
&Z_1\equiv\frac{1}{d(d+1)}\left[\sum_{i\neq j}^{d}\overline{U}_{ij}+2\sum_{i=1}^{d}\overline{U}_{ii}\right],\quad \overline{U}_{ij}\equiv\frac{U_{ij}}{U_iU_j}\\
&A_2=[8Z_1^2-16(Z_1+Z_2)-6]\frac{(2^P d+3)!!}{(2^P d-2)!!8}+[6(Z_1-\tfrac{1}{2})^2+8(Z_1-Z_2)]\frac{(2^P d+2)!!}{4(2^P d-2)!!}+8(Z_1+\tfrac{1}{2})^2\frac{(2^P d+5)!!}{16(2^P d-2)!!},\\
&Z_2=\frac{1}{d(d+1)(d+2)}\left[\sum_{\substack{i,j,k=1\\ i\neq j\neq k\neq i}}^{d}\overline{U}_{ijk}+6\sum_{\substack{i,j=1\\ i\neq j}}^{d}\overline{U}_{iij}+6\sum_{i=1}^{d}\overline{U}_{iii}\right],\quad \overline{U}_{ijk}\equiv\frac{U_{ijk}}{U_iU_jU_k},
\end{aligned}
\tag{53}
$$

where, as on previous occasions, the region 0 [see (5)] has been changed by $\mathbb{R}^d$; only the EDT's are affected by this change. To derive this expression we have inserted (52) in the integrand, expanded it in powers of ν^{-2}, and integrated each term using the multiple-integral transformation (25).

In those differentiable cases in which the $\overline{U}_{ij}$ are independent of the indexes, which we shall call isotropic cases (even when the values of the U_i's may be different), the value of Z_n is given by

$$
\begin{aligned}
Z_n&=(-1)^n\frac{\langle(\cos\theta)^{2(n+1)}\rangle\langle K^{2(n+1)}\rangle}{[2(n+1)]![U(d)]^{n+1}}\\
&=\frac{(-1)^n(2d)^{n+1}}{\sqrt{\pi}(2n+2)!}\frac{\Gamma\left[\frac{2n+3}{2}\right]}{\Gamma\left[n+\frac{d}{2}+1\right]}\Gamma\left[\frac{d}{2}\right]\frac{\langle K^{2(n+1)}\rangle}{\langle K^2\rangle^{n+1}},\\
Z_1&=-\frac{1}{2}\left[1-\frac{2}{d+2}\right]\frac{\langle K^4\rangle}{\langle K^2\rangle^2},\\
W(r)&=\sum_{i=0}^{\infty}Z_i[U(d)r^2]^{i+1},\quad Z_0=1.
\end{aligned}
\tag{54}
$$

$U(d)$ and $\langle K^n\rangle$ are as in (41), and $\langle\cos^n\theta\rangle$ is the mean over all the d-dimensional solid angles of the nth power of the cosine of the angle between a given axis and an isotropically distributed d-dimensional vector. It is clear (Chebyschev's inequality) that $|Z_1|\geq[1-2/(d+2)]/2$. So, as we anticipated, $A_1\geq 0$.

We must now study the dependence of g' on ν. To this purpose one must use all the knowledge about $P(\Delta V')$ that may be derived from the general properties of the field, so as to be able to constrain it to belong to the smallest possible family. We shall find that in the differentiable case P may be, in principle, determined entirely, by general arguments, while in the nondifferentiable case the most one can do on general grounds is to constrain it to belong to a monoparametric family. This difference is an instance of a deeper one which will manifest itself again in the fact that while in the former case $D_d(\nu)$ depends on the Z_n's only through EDT's, in the latter this restriction does not hold. Both differences of behavior stem from the contrivances displayed by the fluctuations of the values of the field and its derivatives in differentiable fields, which renders them essentially simpler objects than nondifferentiable fields; the mentioned contrivances lacking any meaning in these last cases.

We shall first consider the nondifferentiable cases. In these cases, the origin of the ν dependence of g' may be traced back to two different sources: one of them is present even in the one-dimensional case, while the other is present only in multidimensional cases, where it is superposed on the previous one. This superposition translates into the fact that the factor carrying the dependence on ν of g' is the product of the factors corresponding to the two mentioned sources. This fact may be shown by considering the high-density limit of Poissonian distributions of points (Ref. [3]).

The first factor is the one that carries all the dependence of g' on ν in the one-dimensional case. So, we may obtain it by considering this case. Comparing expression (50) in this case with the exact expression for $D_1(\nu)$ in a white noise GRF (see Ref. [3]) we find that, except for EDT's, $g'(\nu)$ is given by

$$
\begin{aligned}
&g'(\nu)=2G(\nu),\\
&G(\nu)\equiv\left[1+\sum_{i=1}^{\infty}\frac{(-1)^i(2i-1)!!}{\nu^{2i}}\right].
\end{aligned}
\tag{55}
$$

We have used (50) instead of (53) since for the case we are considering all Z_n vanish. On the other hand, it is not difficult to realize (using arguments of the sort used in Ref. [3]) that the Z_n may enter the factor in $g'(\nu)$ we are

considering only through EDT's. So in any nondifferentiable GRF the factor relevant to our method is $G(\nu)$.

To obtain the contribution to the ν dependence of g' coming from the second source we mentioned, we shall write $g'(\nu)$ in the form

$$g'(\nu)=g'(A)F(\nu)G(\nu)\ ,$$
$$g'(A)=\frac{d}{d-\frac{1}{2}}\left(\tfrac{64}{13}\right)^{d-1}\ . \tag{56}$$

In the same manner as $g'(A)$ is essentially the product of $d-1$ equal factors, with $g(A)$ being equal to the dth power of this factor, $F(\nu)$ is essentially the product of $d-1$ equal functions of ν; the dth power of it giving the ν dependence of $g(\nu)$. However, although this fact could be used, especially for large d, to obtain a simple approximation to $F(\nu)$ it is clear that this nice dependence of $F(\nu)$ on dimensionality cannot be exact. To obtain $F(\nu)$ properly we need to obtain the expansion of $h'(\nu)$ in powers of ν^{-2}. To this purpose we only need operating with I_2' in an analogous manner as we did in (53) with I_1'. We shall not present this expansion here, but we shall show how its coefficients are related to those of $F(\nu)$. To this purpose we must have some insight as to its origin. Considering a two-dimensional field and ΔV instead of $\Delta V'$ we realize that the form assumed in (20) for ΔV can only be true for very high ν. The linear dimensions of the associated regions are, for ν high enough, proportional to $l\nu^{-2}$, l being the linear dimensions of the body defining the structure. If we were to measure ΔV by displacing the body from left to right through the associated region and measuring at each point its length in the orthogonal direction, the situation represented by (20) corresponds to one in which all these lengths fluctuate simultaneously, following the same distribution as the total length from left to right but independent of it. But, as ν decreases the length of the associated region becomes a sizable fraction of the length of the body. So as we move it from left to right the lengths in the orthogonal direction are increasingly less correlated with that corresponding to the initial position. We may qualitatively describe this situation as being intermediate between one in which ΔV may be expressed as in (20), and another in which ΔV is the sum of two independently distributed volumes ΔV_1 and ΔV_2, each being distributed as in (20). A simple model for this mixed situation would be one in which with probability P we have the first situation, and with probability $1-P$ we have the second. This is a good model which we shall consider later on. But to obtain the exact value of the coefficients of $F(\nu)$ we cannot indulge in modeling. From a careful consideration of the circumstances we have mentioned we learn that $\Delta V'=\Delta V_1'+r\Delta V_2'$, r being a function of ν. There is a little bit of modeling in this expression, but it does not affect the first coefficients and, in general, we shall regard it as exact. To obtain $r(\nu)$ we compute $h'(\nu)$ using the above expression for $\Delta V'$ and the probability distributions of $\Delta V_1', \Delta V_2'$ [which are both given by (21)] and equate it to its value as given by (45). This leads to a quartic equation for $r(\nu)$ which, with the forthcoming definitions, renders the following expressions for the first couple of coefficients in the expansion of $r(\nu)$:

$$g'[h'-\bar{h}(\nu)]r^4(\nu)+[h'g'+3-4\bar{h}(\nu)]r^3(\nu)$$
$$+\left[6-\frac{4\bar{h}(\nu)}{g'}-2g'\bar{h}(\nu)\right]r^2(\nu)$$
$$+[h'g'+3-4\bar{h}(\nu)]r(\nu)+g'[h'-\bar{h}(\nu)]=0\ ,$$
$$\bar{h}(\nu)\equiv\frac{h'(\nu)}{K(\nu)}=\frac{I_2'(\nu)}{[I_1'(\nu)]^2K(\nu)}$$
$$\equiv h'\left[1+\frac{h_1}{\nu^2}+\frac{h_2}{\nu^4}+O(\nu^{-6})\right]\ ,$$
$$r(\nu)\equiv\frac{r_1}{\nu^2}+\frac{r_2}{\nu^4}+O(\nu^{-6})\ , \tag{57}$$
$$r_1=\frac{h'g'h_1}{h'g'+3-4h'}\ ,$$
$$r_2=\frac{h_2h'g'+4h'r_1h_1-r_1^2\left[6-\frac{4h'}{g'}-2g'h'\right]}{h'g'+3-4h'}\ .$$

g' and h' stand for $g'(A)$ and $h'(A)$, and are given by (22) and (23) [or (49)]. $K(\nu)$ correspond to $(h')^{-1}I_2'(\nu)[I_1'(\nu)]^{-2}$ in a one-dimensional white-noise GRF [as $G(\nu)$ in (56)]. $F(\nu)$ is then given by

$$F(\nu)=\left[1-\left[1-\frac{1}{g'}\right]\frac{2r(\nu)}{[1+r(\nu)]^2}\right]$$
$$=\left[1-\left[1-\frac{1}{g'}\right]\left[\frac{2r_1}{\nu^2}+\frac{2(r_2-2r_1^2)}{\nu^4}\right]+O(\nu^{-6})\right]\ . \tag{58}$$

In the differentiable cases $g'(\nu)$ is simpler than in the previous case, but its derivation is much less straightforward. Of course, we could, as in the nondifferentiable case, use the expansion of $h'(\nu)$ to obtain the coefficients in the expansion for $g'(\nu)$, a possibility that we shall later consider in more detail. But now $g'(\nu)$ may be obtained by means of general considerations and we think that stressing this fact is both interesting and expedient; interesting because it will help to point out and illuminate some remarkable properties of differentiable GRF's; expedient because it enables us to bypass the considerable amount of work implied in the more straightforward procedure. To this purpose, however, we shall need to discuss in some detail the behavior of differentiable GRF's around maxima. This we have already done in deriving (35) and (37). But there we used approximations which are valid only for very high ν: the fluctuations of the λ_i's around their mean were neglected, and only the second derivatives of the field were taken into account. To see how the situation is modified for moderate values of ν, we shall first consider the joint distribution of the value of the field and its second derivative in a one-dimensional problem

$$P(\bar{f},\lambda)=\left[\frac{e^{-(\bar{f})^2/2}}{\sqrt{2\pi}}\right]\left[\frac{e^{-(\lambda-R\bar{f})^2/[2(v^2-R^2)]}}{\sqrt{2\pi}(v^2-R^2)^{1/2}}\right],$$
$$R\equiv\frac{\sigma_{12}}{\sigma_1^2}=-\langle K^2\rangle\ ,\quad v\equiv\frac{\sigma_2}{\sigma_1}=\langle K^4\rangle\ , \tag{59}$$

where σ_1 and σ_2 are, respectively, the rms values of $\bar{f}$ (note that in most expressions $\bar{f}$ is expressed in units of its rms value) and λ, and σ_{12} is their correlation; $\langle K^2\rangle$ and $\langle K^4\rangle$ are as defined in (41) for $N=1$. From this expression we learn that the mean value of λ at points where the value of the field is $\bar{f}$ is proportional to $\bar{f}$, and that its rms value is $(V^2-R^2)^{1/2}$. For high $\bar{f}$ this quantity is a small fraction of the corresponding mean value of λ, so that one may neglect the fluctuations of λ, as we did in (35).

A result similar to (59) is found for any derivative of the field. If we could express the volume of the region in terms of the values of these derivatives at the maxima, we could use their joint probability distribution to compute $g'(\nu)$. Obtaining this joint distribution in full may be a rather cumbersome task, but we could make the simplifying assumption that the fluctuations of the derivatives around their mean values are independent. In this case the distribution in question will simply be the product of several distributions of the form of (59). We shall see, however, that, as we said before, this procedure may be avoided by mean of some general arguments.

It may easily be shown that the probability densities of maxima, connected regions, and associated regions differ only in EDT's. The nonexponentially decaying part (also called asymptotic expression, although we have reserved this name for the leading term of this expression) of the probability density of maxima does not depend on Z_n (only derivative up to second order being relevant to it). So the same must hold for $D_d(\nu)$. However, in expression (53) we see that $I_1'(\nu)$ depends on Z_n. $g'(\nu)$ must then be such as to cancel out this dependence. This can only happen because of peculiar relationships between the derivatives of the field in GRF's. For example, the mean value of the fourth derivative at the maxima appears in (53) (Z_1). But, as we may see in (59), this mean value is related to the rms value of the fluctuations of the second derivative around its mean value, which will be responsible for the presence in $g'(\nu)$ of a term of the same order on ν depending on Z_1. The fact that the dependences of $I_1'(\nu)$ and $g'(\nu)$ on the Z_n cancels out (except for EDT's) allows us to obtain their ratio by using the Z_n that are more convenient from a computational point of view. We shall then choose them so as to obtain the simplest expression for the joint probability distribution of the field and its derivatives. We shall restrict our attention to isotropic (in the sense that the $\bar{U}_{ij}$ are equal; but not necessarily the U_i fields). The advantage of these fields is that the relationship between the Z_n and the coefficients appearing in (59) (and similar expressions for higher-order derivatives) is particularly simple. For other fields the procedure will be similar, but a bit more complicated.

For d-dimensional isotropic GRF's we have that the derivatives in a given direction follow distributions like (59) (and similar distributions) but with the corresponding trigonometric factor entering the definitions of R and V (and similar definitions). A comparison between these quantities and the Z_n (54) lead us to the following relation:

$$\left\langle\frac{\partial^{2n}\bar{f}}{\partial x_i^{2n}}\right\rangle_{\bar{f}}=\bar{f}[(2n)!]Z_{n-1}U^n(-1)^n\ ,$$
$$\left\langle\frac{\partial^{2n+1}\bar{f}}{\partial x_i^{2n+1}}\right\rangle_{\bar{f}}=0\ ,$$
$$\left\langle\left[\frac{\partial^n f}{\partial x_i^n}-\left\langle\frac{\partial^n f}{\partial x_i^n}\right\rangle\right]^2\right\rangle_{\bar{f}} \tag{60}$$
$$=[(2n)!]Z_{n-1}U^n-(n!Z_{n/2-1}U^{n/2})^2\ ,$$
$$U\equiv U(d)=\frac{\langle K^2\rangle}{2d}\ ,$$

where the averages are over all points where the value of the field is $\bar{f}$. For i half integer or smaller than zero, Z_i is zero. An interesting direct consequence of these expressions is that the mean value of the field at a distance r from a point (taking as reference the value of the field at this point) is proportional ($\bar{f}$ being the proportionality constant) to $-W(r)$.

From expression (59) we see that as V approaches R, the fluctuations of λ, for a given value of $\bar{f}$, tend to zero. In a d-dimensional isotropic GRF, this happens when $Z_1=-\frac{1}{6}$. It is obvious, however, from the definition of Z_1 and Chebyshev's inequality that this result cannot correspond to a real situation, since we may write

$$Z_1=-\frac{1}{6}\frac{\langle\cos^4\theta\rangle}{\langle\cos^2\theta\rangle^2}\frac{\langle K^4\rangle}{\langle K^2\rangle^2}\ .$$

Even if the power spectrum were a δ distribution, $|Z_1|$ will have to be larger than $\frac{1}{6}$ owing to the trigonometric factor. Nevertheless, although $g'(\nu)$ as a function of the Z_n is not meaningfully defined for values smaller than the ones given by (54) with the factor depending on the power spectrum set equal to 1, there is nothing that prevents us from considering its analytic continuation to these values of the Z_n. That this is so may be realized by a qualitative consideration of $g'(\nu)$ which shows the absence of singularities in the region to which we are extrapolating. The mean values of the derivatives cannot be prescribed arbitrarily; they have to obey the analytical continuation of the relations they obey for possible values of the Z_n. If we make Z_n go to $2(-2)^n[(2n+2)!]^{-1}$, the mean value of the $2n$th derivative of the field in a given direction goes to U^n, and its rms fluctuation goes to zero. This means that the values of the even order derivatives are a unique function of the value of the field. The odd-order derivatives have zero mean, their rms values being U^n. Their dimensionless correlation is 1, so that they all fluctuate simultaneously. It may be worth mentioning that for one-dimensional fields this situation is a possible one; it corresponds to the power spectrum being given by a δ distribution. For the forthcoming discussion it will be convenient to concentrate on this case; the full dimensionality of the problem will be recovered afterwards. So far we have considered the joint distribution of the values

of the field and its derivatives for random points. However, what we need is the joint distribution of these quantities at maxima. To this end, we must, first of all, substitute the first factor in (59) by (34). To obtain the distribution of the values of the derivatives, we must account for the fact that the even-order derivatives are unique functions of $\bar{f}$, which is a simple proportionality, as we see in (60). The probability distribution of the derivatives of the field at random points where the value of the field is ν may be expressed in terms of the probability distribution of these quantities at the closest maxima and the size of the corresponding connected region which, in turn, may also be expressed in term of those quantities. Now since the probability distribution of the second derivative (for example) at points where the value of the field is ν, is simply a δ distribution centered on νR, we must impose these conditions (and all similar ones for higher even-order derivatives) on the distribution at the maxima. The result found is that the mean values of the derivatives at the maxima are proportional to the height of the maxima; the same situation as in (60) (for random points at level ν). The first derivative is, by construction, equal to zero, and, since all odd derivatives are totally correlated, they must also vanish. This last result may seem surprising, since it implies the field being symmetric with respect to the maxima. It must be remembered, however, that the field we are considering is not ergodic, since its power spectrum is not continuous. Expressing formally all that we have said, we have

$$\begin{aligned}
\bar{f}(x)&=\nu+s-\left[\nu\left[1+\frac{s}{\nu}\right]\left[\frac{(xR^{1/2})^2}{2!}-\frac{(xR^{1/2})^4}{4!}+\cdots\right]\right]\\
&=\nu\left[1+\frac{s}{\nu}\right]\cos(xR^{1/2}),\\
\left.\frac{d^{2i}\bar{f}(x)}{dx^{2i}}\right|_{x=L}&=-\nu R^i,\quad \bar{f}(L)=\nu .
\end{aligned}\tag{61}$$

L stands for the distance from maxima of height $\nu+s$ to the closest point where the value of the field is ν. It is clear that the above conditions on the even-order derivatives holds. To deal with the d-dimensional problem we proceed as in (36) but use (61) to obtain the relation between the value of s and the size of the connected region along direction $i, 2L_i$,

$$\Delta V' \propto \Delta L_1 \prod_{i=2}^{d} L_i ,$$

$$\Delta L_1 \propto \left[-\left.\frac{d\bar{f}(x)}{dx}\right|_{x=L_1}\right]^{-1} .$$

To first order in s we have

$$\begin{aligned}
&\bar{f}(x)\simeq\nu\left[1+\frac{s}{\nu}\right]\left[1-\frac{(xR^{1/2})^2}{2!}+\frac{(xR^{1/2})^4}{4!}\right],\\
&L(s)\simeq\frac{2(\nu R)^{-1/2}s^{1/2}}{\left[1+\frac{5}{6}\frac{s}{\nu}\right]^{1/2}},\\
&\Delta L_1\propto\left[L(s)\left[1-\frac{1}{3}\frac{s}{\nu}\right]\left[1+\frac{s}{\nu}\right]\right]^{-1},\\
&\Delta V'\propto[L(s)]^{d-1}\Delta L_1 ,\\
&g'(\nu)\simeq\frac{\int_0^\infty P(s)ds\int_0^\infty P(s)s^{(d-2)}\left[1-(\frac{5}{6}d-\frac{1}{3})\frac{s}{\nu}\right]ds}{\left[\int_0^\infty P(s)s^{(d/2-1)}\left[1-\frac{1}{2}(\frac{5}{6}d-\frac{1}{3})\frac{s}{\nu}\right]ds\right]^2}\simeq g'(A)\left[1-\frac{4d^2-7d+8}{6\nu^2}\right].
\end{aligned}\tag{62}$$

$P(s)$ is as given by (34). The coefficients entering $P(s)$ may be expressed (in Sec. V we shall see how) in terms of smaller-order coefficients in $g'(\nu)$. For example, to obtain the first two coefficients in $g'(\nu)$ only the first term (the explicit one) in (34) is necessary. Combining (62) and (53) with $Z_1=-\frac{1}{6}$ we have

$$\begin{aligned}
D_d(\nu)&=D_d(A)\left[1-\frac{2d^2-3d+6}{6\nu^2}+O(\nu^{-4})+D\right],\\
I_1'(\nu)&=I_1'(A)\left[1-\frac{d(d-2)+1}{3\nu^2}+O(\nu^{-4})+D\right].
\end{aligned}\tag{63}$$

In a similar manner we may obtain all the coefficients in $D_d(\nu)$ by setting

$$Z_n=(-1)^n 2^{n+1}/[2(n+1)]! .$$

However, it is easy to realize that expression (63) for D_d cannot be correct, since it does not render the right expression in the cases $d=1,2,3$ which are known. This is not a drawback to our method since, as we shall find, the information that we have is enough to obtain the coefficients in question. But it is still an intriguing fact that the right answer is not obtained directly.

To obtain the right coefficients we may proceed as we

did before, but we use an expansion with arbitrary coefficients for $\bar{f}(x)$, instead of (61), in which the coefficients are those corresponding to $\cos(x)$. Comparing the coefficients obtained in this manner for the expansion of $D_d(\nu)$ with those of a known case (i.e., $d=3$), we may obtain all coefficients in $\bar{f}(x)$. However, for the first coefficients, this procedure is not even necessary since a mere qualitative consideration of the dependence of the coefficients of $D_d(\nu)$ on d, together with the information provided by the cases $d=1,2,3$ lead us to

$$D_d(\nu)\equiv D_d(A)[E(\nu,d)+D]\ ,$$
$$E(\nu,d)=\left[1+\sum_{i=1}^{d-2}\frac{(-1)^{i}\,i+1}{i!\nu^{2i}}\prod_{j=1}(d-j)\right]\ . \tag{64}$$

The coefficients in E are the so-called geometrical numbers. But, except for the first two coefficients, expression (64) is only a conjectured extrapolation. To show that it is actually correct, we may use the procedure we have just described, or an alternative and simpler one that we shall now proceed to describe. The essential idea of this method is to consider Gaussian fields (isotropic) in $d-1$ dimensions that are the restriction to hyperplanes of d-dimensional ones. Is easy to realize that, in the limit we are considering, the probability density of connected regions (or maximas or structures) in the former case is proportional to the one corresponding to the latter multiplied by the mean length of the region along any DOF, $\langle L(s)\rangle$. A similar relation holds for the probability distribution of the height of maxima, $P_m(\bar{f},\nu)$ [see (75)]. Formally we have

$$E(\nu,d-1)\propto E(\nu,d)\langle L(s)\rangle(\nu)\ ,$$
$$-e^{\nu^2/2}\frac{d}{d\nu}[e^{-\nu^2/2}E(\nu,d-1)]\propto E(\nu,d)\langle \Delta L(s)\rangle(\nu)\ ,$$
$$\langle L(s)\rangle(\nu)\equiv\int_0^\infty P(s)L(s)ds\ ,$$
$$\langle \Delta L(s)\rangle(\nu)\equiv\int_0^\infty P(s)\Delta L(s)ds\ , \tag{65}$$
$$P(s)\equiv P_m(\nu+s,\nu)\ ,\quad L(s)\equiv\bar{f}^{-1}(\nu)\ ,$$
$$\Delta L(s)\equiv\left[\left.\frac{d\bar{f}(x)}{dx}\right|_{x=L(s)}\right]^{-1}\ ,$$
$$\bar{f}(x)=(\nu+s)\left[1-\sum_{i=1}^{\infty}\alpha_i x^{2i}\right]\ .$$

These proportionalities, transformed into equations by factoring out the leading term in each member, render $E(\nu,d),\alpha_i$.

The coefficients in (64) depend only on the dimensionality; the EDT's bear all the dependence on the power spectrum [except for U which appears in $D_d(A)$]. We shall now see how to obtain the EDT's. To this end we shall consider the general expression for $\langle \Delta V\rangle$ in terms of the mean lengths S_i of the associated regions along a set of independent direction,

$$\langle \Delta V\rangle = A_0\left[\prod_{i=1}^{d}S_i(\nu)\right]\times\left[1+\sum_{j=1}^{d-1}\sum_{\{i_1,\ldots,i_j\}}A(i_1,\ldots,i_j)\prod_{K=1}^{j}S_{i_k}(\nu)\right]\ . \tag{66}$$

The second sum in the parentheses is over all the possible choices of j functions $s_i(\nu)$ with i from 1 to d; the A's are some coefficients. These functions are the $\langle \Delta V\rangle$'s in the one-dimensional problems corresponding to the d-independent directions; each one of them being completely characterized by the dependence of $W(\Delta\mathbf{X})$ on the displacement along the corresponding direction. We incur a very small error if we replace (66) by

$$\langle \Delta V\rangle = A_0\prod_{i=1}^{d}S_i(\nu)\left[1+\sum_{i=1}^{d}A_iS_i(\nu)+\sum_{j=2}^{d-1}B_jS^j(\nu)\right]\ , \tag{67}$$
$$S(\nu)\equiv\sqrt{\pi/2}\frac{1}{\nu^P}\operatorname{erfc}\left[\frac{\nu}{\sqrt{2}}\right]e^{\nu^2/2}\left[1-\operatorname{erfc}\left[\frac{\nu}{\sqrt{2}}\right]\right]^P\ ,$$

where, as before, $P=1$ for nondifferentiable fields and $P=0$ for differentiable ones. The B's are coefficients related to the A's in (66). This expression has been obtained by substituting in all but the first term in the sum within the parentheses in (66) the actual S_i by the approximation to them which uses for $W(\Delta x_i)$ only its first-order expression. In this approximation all the S_i are equal except for an irrelevant overall constant. We may now write

$$D_d(\nu)=\frac{1}{2}\frac{\operatorname{erfc}\left[\frac{\nu}{\sqrt{2}}\right]}{\langle \Delta V\rangle(\nu)}\ , \tag{68}$$

where, for convenience, we are considering densities in d dimensions (without integrating over the IDOF). Taking the EDT's out from (68) and comparing with (64) we may obtain A_0, all B_j, and the sum of the A_i. In the "isotropic" cases the A_i are all equal for any set of orthogonal directions. In general, we shall have to use the set of directions corresponding to the eigenaxes of U_{ij} [see (52)]; the corresponding A_i being all equal. By the procedure we have described, we have reduced the d-dimensional problem to d one-dimensional ones. Given the $W(\Delta x_i)$ corresponding to each of the eigenaxes of U_{ij}, we may obtain $S_i(\nu)$ by means of (10) and $g(\nu)$. But now, in obtaining $g(\nu)$ we cannot neglect EDT's, so that one may need some other moments of $P(\Delta V)$ to compute $g(\nu)$ accurately.

It is worth mentioning that the difference among the probability densities of maxima, connected regions, and associated regions enters only through the S_i; the coefficients in (67) and the function S are the same in all cases. We may generalize the definition of S_i in the following manner:

$$\bar{S}_i(\nu)=\frac{2P_i(\nu)}{\operatorname{erfc}\left[\frac{\nu}{\sqrt{2}}\right]}\ , \tag{69}$$

where $P_i(\nu)$ stands for the probability density of either maxima, connected regions, or associated regions in the one-dimensional problem corresponding to the ith direction. In the last two cases $\bar{S}$ may be interpreted as a length (that of the corresponding region); this interpretation is not possible in the case of maxima. However, this fact is irrelevant in regard with the use of (67) and (68) in this case. So by using in these expressions any of the $\bar{S}_i$ as given by (69), we obtain the corresponding probability density in d dimensions. So far we have outlined an analytical procedure for the computation of $D_d(\nu)$. In general this will be the procedure to follow, especially when the problem contains many degrees of freedom. Sometimes, however, if the problem is not too complicated and one is interested only in a particular numerical value, it may be more expedient to compute $h'(\nu_0)$ numerically and use some model for $P(\Delta V)$ to obtain $g'(\nu_0)$. This we have already done for the nondifferentiable case at the beginning of this section. There we wanted the exact values of some coefficients so that we needed to use a rather accurate form for $P(\Delta V)$.

$$h'(\nu_0)=\frac{[1+r(\nu_0)][h'_1 g'^2_1+r^2(\nu_0)h'_2 g'^2_2+3(g'_1+g'_2)]}{[g'_1+r^2(\nu_0)g'_2+2r(\nu_0)]^2},\quad g'(\nu_0)=\frac{g'_1+r^2(\nu_0)g'_2+2r(\nu_0)}{[1+r(\nu_0)]^2}, \tag{71}$$

where g'_2 and h'_2 are given by (22) and (23); g'_1 and h'_1 are computed using the full expression for $P(s)$ [see (34) and (65)] and with $\bar{f}(x)$ given by (65).

V. CORRELATIONS OF STRUCTURES AND CONCLUDING REMARKS

We have already seen that the method for obtaining $D_d(\nu)$ by means of the computation of $\langle \Delta V' \rangle$, although it is not essential to the main ideas portrayed in this work, has proved to be very useful; the general analytical procedure outlined in Sec. IV would not have been possible without it. We shall now see how a problem of considerable interest, that of the correlations between the structures, which is substantially complicated when treated with other methods, admit of a trivial solution when dealt with the method in question. For example, considering isotropic fields and using (14), we may obtain the two-point correlation function by means of the following argument: considering a spherical shell at a distance between r and $r+\Delta r$ from a structure (defined at level ν), we have that the probability for a point within this shell to have a field value between ν and $\nu+d\nu$ is equal to $P'(\bar{f}_2/\bar{f}_1,\nu)d\nu$ (where $\bar{f}_2$ is the field $\bar{f}_1$ displaced by $\mathbf{r}$; $|\mathbf{r}|=r$). This is so because at the center of the shell there is a structure and its associated region, whose dimensions we are neglecting, contains points where $\bar{f}$ is between ν and $\nu+d\nu$; the above result follows then from the definition of $P'(\bar{f}_2/\bar{f}_1,\nu)$. The total volume of the set of points within the shell such that the field takes a value within the mentioned interval is then equal to the product of the volume of the shell by $P'(\bar{f}_2/\bar{f}_1,\nu)d\nu$. To obtain the mean number of structures within the shell we only need dividing this volume by the mean volume of their associated regions, $\langle \Delta V' \rangle$. Normalizing this mean number to the mean number of structures in a randomly chosen volume like that of the shell, we have

$$\begin{aligned}1+\xi_s(r)&=\frac{[\Omega(d)r^{(d-1)}\Delta r]P(\bar{f}_2/\bar{f}_1,\nu)}{\langle \Delta V' \rangle}\\&\quad\times\left[\frac{e^{-\nu^2/2}\Omega(d)r^{(d-1)}\Delta r}{\sqrt{2\pi}\langle \Delta V' \rangle}\right]^{-1},\\ \xi_s(r)&=\frac{1}{\{1-[\xi_i(r)]^2\}^{1/2}}\exp\left[\frac{\nu^2}{2}\frac{\xi_f(r)}{1+\xi_f(r)}\right]-1\\&\underset{\xi_f(r)\to 0}{\simeq}\nu^2\xi_f(r)+\left[\frac{\nu^4}{2}-\nu^2+\frac{1}{2}\right][\xi_f(r)]^2,\end{aligned} \tag{72}$$

where ξ is the notation in cosmology for correlations: $\xi_s(r)$ stands for the two-point correlation between the structures and $\xi_f(r)$ for the dimensionless autocorrelation of the field $\bar{f}(x)[W(r)=1-\xi_f(r)]$. From the definition of structure it follows that $\xi_s(r)=-1$ for $r\le 2r_0$ (r_0 being the radius of the spheres defining the structures). The correlations, in differentiable fields, between maxima or connected regions are, for $r\ge 2r_0$, equal to the correlations of the structures (provided that the distribution of the number of maxima or connected regions in an associated region is not changed by the presence of another structure at a distance r, which is to all purposes true). In these cases, however, for $r\le 2r_0$ the correlations are given by (72) down to distances of the order of the mean size of the connected region. For distances of this order the correlations will go from positive to negative values in a continuous manner, unlike in the former case, approaching -1 as r goes to zero. The detailed form of the correlations for distances of this order could be obtained by using the probability distribution of

In a numerical evaluation of $D_d(\nu_0)$, however, we may use the simplified model that we mentioned there: $\Delta V'=\Delta V'_1$ with probability $(1-P)$ and $\Delta V'=\Delta V'_1+\Delta V'_2$ with probability P, with $\Delta V'_1$ and $\Delta V'_2$ being distributed like $\Delta V'$ in (21). We then find

$$\begin{aligned}P(\nu_0)&=[h'(\nu_0)-h']\left[\frac{h'g'^2+3g'}{(1+g')^2}-h'\right]^{-1},\\ g'(\nu_0)&=\left[1-\frac{P(\nu_0)}{2}\right]g'+\frac{P(\nu_0)}{2},\end{aligned} \tag{70}$$

where, as in (57), g' and h' are given by (22) and (23). For differentiable fields we may use the fact that for low ν_0, when the associated region is made up of more than one connected region, the behavior of the field somewhat resembles that of a nondifferentiable one. So we use a model in which $\Delta V'=\Delta V'_1+r(\nu)\Delta V'_2$; where $\Delta V'_1$ follows the same distribution as $\Delta V'$ in (62) [using expression (65) for $\bar{f}(x)$], and $\Delta V'_2$ is distributed as in (21). We have then

the sizes of connected regions (assuming this question could have any relevance).

In deriving (72) we have taken the associated regions to be punctual. If we wanted to be more precise we could choose some representative point in them and obtain their correlations. In this sense, (72) may be interpreted as the volume weighted mean of the correlation over all possible choices of these points. However, the size of the associated regions ($\simeq r_0/\nu^{1+P}$) is small enough compared with r ($r \geq 2r_0$) so as to make these considerations of little relevance. Except for these reservations of little consequence, (72) is exact. Operating with (15) in a similar manner we find

$$
\begin{aligned}
&1+\sum_{i=1}^{3}\xi_s(r_i)+\zeta_s(r_1,r_2,r_3)=P'(\bar{f}_2,\bar{f}_3/\bar{f}_1,\nu)\left[\frac{e^{-\nu^2/2}}{\sqrt{2\pi}}\right]^{-2}, \\
&\zeta_s(r_1,r_2,r_3)=\frac{\exp\{\nu^2[1-E(r_1,r_2,r_3)]\}}{[\Delta(r_1,r_2,r_3)]^{1/2}}-\left[1+\sum_{i=1}^{3}\xi_s(r_i)\right], \\
&\Delta(r_1,r_2,r_3)\equiv 1+2\prod_{i=1}^{3}\xi_f(r_i)-\sum_{i=1}^{3}[\xi_f(r_i)]^2\ ,E(r_1,r_2,r_3)\equiv[\Delta(r_1,r_2,r_3)]^{-1}\prod_{i=1}^{3}[1-\xi_f(r_i)], \\
&\zeta_s \underset{\xi_f(r_i)\to 0}{\sim} (\nu^4-\nu^2)\sum_{i<j}^{3}\xi_f(r_i)\xi_f(r_j)\simeq\left[1-\frac{1}{\nu^2}\right]\sum_{i<j}^{3}\xi_s(r_i)\xi_s(r_j)\ .
\end{aligned}
\tag{73}
$$

ζ stands for the connected part of the three-point correlation function. Its asymptotic expression for low correlations seems to have the same form as the galaxy three-point correlation function, and it has attracted some attention from a theoretical point of view (Ref. [8]). In general, using the definition of connected correlation functions (Ref. [6] and [7]), we have

$$
\sum_{\{m_i\}\ \text{perm}}\sum(\xi_1^{m_1},\ldots,\xi_{n-1}^{m_{n-1}})+\xi_n=\frac{P'(\bar{f}_2,\ldots,\bar{f}_n/\bar{f}_1,\nu)}{\left[e^{-\nu^2/2}\right]^{n-1}}=\frac{e^{\nu^2[(n-1)/2-E]}}{(\Delta)^{1/2}}\ ,
$$

really different configurations (structures) of the underlying field $f(\mathbf{x})$. In nondifferentiable fields this concept is essential to the very possibility of picking out the mentioned spots. In differentiable fields other configurational properties of the field, like connected regions or maxima (based on differentiability), may be used to this purpose. But, even in this case, the concept of structure represents a more expedient approach in most situations. By its very definition it removes the problem of the high short-distance correlations of the mentioned objects. From a physical point of view the structures are in a one-to-one correspondence with the really different occurrences of the global property in question. In the galaxy clustering model we mentioned in Sec. II, it is clear that two maxima of the filtered field above the prescribed level cannot correspond to two different galaxies if they lay within a distance $2r_0$ from each other ($r_0 \simeq$ filtering scale), since two spheres of radius r_0 centered at them will overlap substantially. When the distance between the maxima is $\gtrsim r_0$, the probability of each of the structures evolving into a different galaxy may still be small, but this will depend on the detailed dynamics of the system. Thus, the role of the concept of structures in this example is to account for the restrictions on the possibility of galaxy formation coming from purely geometrical considerations.

The computational methods we have developed in this work have been designed with the concept of structure in mind. However, as work progressed we have been able to see that these methods are to a large extent relevant to the computation of the probability densities of maxima and connected regions. A set of general considerations allowed us to figure out the relation between these quantities and the probability density of structures. Our general procedure for the computation of the later quantity turns out to be also, after a trivial modification, a general procedure for the computation of the former ones. This means that as work progressed its main stress passed from the concept of structure to the methods developed, in principle, to deal with it. In this sense we have seen how the method based on the computation of $\langle \Delta V' \rangle$, which is merely the result of applying a mathematical trick, proved to be essential to the possibility of treating the problem in general, due to the great simplification implied by it.

The detailed consideration of the Gaussian case is useful, apart from its intrinsic interest, because it illustrates well the general procedure. We have seen how the appropriate use of some general properties of the field simplifies the problem considerably. In fact, many problems will only be treatable after incorporating these general considerations, the straightforward procedure being too envolved. This points to the relevance of the analytical treatment which can be dispensed with only in relatively simple problems. A remarkable fact in this respect is that in the rare structure regime, which is increasingly difficult to compare with the result of Monte Carlo simulations, the application of the methods described here is particularly simple.

The techniques presented in this work allow us to compute the probability density of structures of arbitrary shapes in spaces of any number of dimensions. The case of filamentary structures in three dimensions is an example of some relevance in cosmology. But we have not considered constraints on the distribution of the field within the body defining the structure. In more general definitions of structure, the L IDOF defining it will set constraints not only on the possible shapes of the structures but also on the possible distributions of the field within it, by stating, for example, the values of the integral of the field over a set of different regions within the body in question. The computation of the probability density of these general structures may be dealt with a straightforward generalization of the techniques presented here, although the actual procedure will be considerably more complex. With these generalized techniques, one could, in principle, deal with any possible question concerning random fields.

As an example of the simplification implied by the techniques presented here even in the cases whose solution was already known, we shall briefly consider the obtention of the probability density of maxima in three-dimensional isotropic Gaussian fields. The standard treatment of this problem is quite involved. But using (64), (67), (68), and (69) we may reduce this problem to a one-dimensional one. We then have

$$M_3(\nu)=\frac{1}{2}\frac{\operatorname{erfc}\left[\frac{\nu}{\sqrt{2}}\right]}{A_0 S^3(1+A_1 S+A_2 S^2)}\,,$$
$$M_3(\nu)\underset{\nu\to\infty}{\to}\frac{\left\langle\frac{K^2}{3}\right\rangle^{3/2}}{2(2\pi)^{3/2}}\nu^2\left[1-\frac{1}{\nu^2}\right]e^{-\nu^2/2}\,, \tag{77}$$

where

$$S\equiv S[\nu,U(3),Z_1(3)]\,,$$

with

$$S[\nu,U(1),Z_1(1)]\equiv\frac{2M_1[\nu,U(1),Z_1(1)]}{\operatorname{erfc}\left[\frac{\nu}{\sqrt{2}}\right]}$$
$$\underset{\nu\to\infty}{\to}\frac{\langle K^2\rangle^{1/2}}{\sqrt{2\pi}}\left[1+\sum_i(-1)^i\frac{(2i-1)!!}{\nu^{2i}}\right]^{-1}.$$

M_3 and M_1 are, respectively, the probability density of maxima in three and one dimensions. The latter is a well-known function of ν, $U(1)$, $Z_1(1)$ [$U(d)$, $Z_1(d)$ are given by (41), (54)], on which terms we may express all M_d. Taking the high-ν limit of the first of expressions (77) (using the above high-ν limit for S) and comparing with the second, we obtain immediately A_0, A_1, A_2.

Using the corresponding S function [see (69)] we may obtain the probability density of connected regions or structures. In these cases S may, in principle, depend on all Z_i. But, in the latter case this function is still to be obtained.

At several points in this work we have used general re-

sults which were not explicitly demonstrated. We think that most of these points do not present major difficulties. On the other hand, the simplicity of some other points belies considerable difficulties. Take expression (19), whose meaning is that the fraction of the distance between the extremes of an associated region in one dimension that belongs to the associated region is uniformly distributed between zero and one. This question proved to be a very obdurate one. Its demonstration is, however, quite simple once one knows the lines to follow. Knowing that the distribution must be symmetric with respect to one-half and that it has got to be monotonally decreasing or constant, we find that only the last distribution leads to the correct value for g'. We must also note that the asymmetry between f_1 and all others f_i in this expression is only due to our choice of presentation. In fact, all the f_i are uniformly distributed, and the Z_i are not totally independent. But we have transformed the actual problem into a problem which is to our purposes equivalent to it and allow a simpler description. The direct presentation of the problem would have been more complicated, without adding much insight into it. A proper understanding of this problem will demand a paper for itself, and the same may be said about some other points in this work. The simplest characterization of them valid to our purposes has in all cases been chosen. In regard to the expansion in powers of ν for $I_2'(\nu)$ we must say that the coefficients in it may be evaluated analytically, but the procedure is tricky and lengthy. Thus, one may prefer a numerical evaluation.

As we stated in the Introduction, in this work we have sacrificed explicit rigor to conciseness and scope. In this sense we have always opted for "heuristic" expositions and presented the conclusions omitting the qualifying asides which are usual, and in rigor, necessary in this type of work. The way to account for the deficiencies of this presentation is for the reader to qualify the word "exact." We have called exact all expressions that incorporate all the factors that may be deemed relevant in the contexts one is considering. But in most cases it is possible to consider additional factors which will become relevant in situations which are far from the initial context. For example, we have said that the computational method given by (3) is equivalent to that given by (4). But, considering the case of one-dimensional structures of length L, we realize that, when the typical distance between an upcrossing of level ν and the next downcrossing is $\geq 2L$, both methods give different answers. With the criteria corresponding to the first method we will find that by displacing the interval defining the structures from the first to the second point we shall find at least three nonoverlapping ones above ν, that is, we have at least three structures; with the second we have two nonoverlapping crossing points and, hence, two structures. This is particularly clear if we look to the behaviors of the corresponding expressions for $D_1(\nu)$ when ν goes to $-\infty$. In the first case we obtain (correctly) L^{-1}; in the second we obtain zero, since the expression for $D_1(\nu)$ is even in ν. In higher dimensions, even if we used the first method, the problem that one faces in this limit is that of the random packing or compact packing (depending on the detailed manner in which one searches for the structures) of the bodies defining the structures. It is obvious that these problems are very different from the initial one and, hence, one cannot expect the solution to those problems to be given by expression (8).

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