Bistable spheroidal optical solitons

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The possible forms the intensity dependence of the nonlinear refractive index would have to take for bistable light bullets to exist are examined. Bistable light bullets are propagating spheroidal optical solitons characterized by different sizes and intensity profiles but with the same energy.

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Recently, the possible existence of a so-called "light bullet" has been suggested [1] which could arise when a three-dimensional optical pulse is able to propagate without change in space or time because the natural tendency to spread due to diffraction and anomalous dispersion is balanced by the self-focusing due to an appropriately chosen nonlinear refractive index. In this paper we would like to point out that *bistable* light bullets are possible for certain classes of intensity-dependent refractive indices. Bistable light bullets are propagating spheroidal optical solitons characterized by different sizes and intensity profiles but with the same energy.

Assuming a refractive index of the form $n=n_0+n_2F(|\phi|^2)$, where n_0 and n_2 are real and positive, ϕ is the electric field, and F is an arbitrary function, and making the usual slowly-varying-envelope approximation in the direction of propagation z_1, Maxwell's wave equation yields (subscripts denoting partial derivatives)

$$2ik(\phi_{z_1}+v_g^{-1}\phi_t)+\phi_{x_1x_1}+\phi_{y_1y_1}-kD\phi_{tt}+2k^2(n_2/n_0)F(|\phi|^2)\phi=0 . \quad (1)$$

Here k is the wave number, v_g the group velocity, t the time, x_1 and y_1 the transverse spatial directions, $D=-(\partial v_g/\partial\omega)/v_g^2$ the group-velocity dispersion (GVD). Assuming anomalous GVD, $D<0$. Then using the standard Mollenauer-Stolen-Gordon [2] scaling, viz., $T\equiv\tau^{-1}(t-z_1/v_g)$, $Z\equiv\tau^{-2}|D|z_1$, $X\equiv\tau^{-1}(k|D|)^{1/2}x_1$, $Y\equiv\tau^{-1}(k|D|)^{1/2}y_1$, $E\equiv\tau(kn_2/|D|n_0)^{1/2}\phi$, and $f(|E|^2)\equiv(kn_2\tau^2/|D|n_0)F(|\phi|^2)$, where τ is a temporal scale factor, Eq. (1) takes on the normalized dimensionless form

$$iE_z+\tfrac{1}{2}(E_{XX}+E_{YY}+E_{TT})+f(|E|^2)E=0 . \quad (2)$$

Spherically symmetric solutions of the form $E(Z,R)=\exp(i\delta Z)U(R)$ are sought with $R^2=X^2+Y^2+T^2$ and δ a real, positive constant. Equation (2) then reduces to

$$U_{RR}+(d-1)U_R/R+2U[f(U^2)-\delta]=0 , \quad (3)$$

with the dimension $d=3$ here. Bistable solitary-wave solutions to Eq. (3), having their maximum value at $R=0$ and with U_R, $U_{RR}\to 0$ as $R\to\infty$, have been extensively investigated by Kaplan [3] and Enns *et al.* [4–6] for $d=1$. Kaplan has also briefly discussed the $d=2$ case. In this paper we are interested in what form $f(U^2)$ must take for bistable solitary waves to exist for $d=3$. To generalize the guidelines for "building" bistable solutions which were used [4,5] for $d=1$, we utilize an argument due to Zakharov and Synakh [7] and apply it to the d-dimensional form (3). The energy (first conserved quantity) of a localized solution to Eq. (3) is given by $P=\int_0^\infty|E|^2R^{d-1}dR=\int_0^\infty U^2R^{d-1}dR$. We consider $f(U^2)=U^{2n}=I^n$, where I is the intensity, and assume a solution to (3) of the form $U(R)=\delta^bV(\delta^cR)$. On substitution into (3), we find that $c=\frac{1}{2}$, $b=1/2n$, so that on setting $\zeta\equiv\delta^{1/2}R$ we obtain $P=\delta^{1/n-d/2}\int_0^\infty V^2(\zeta)\zeta^{d-1}d\zeta$. Thus, if P is plotted versus the parameter δ, the slope $dP/d\delta$ is positive for $n<2/d$, zero for $n=2/d$, and negative for $n>2/d$. In $d=1,2,3$ dimensions the critical (zero) slope value of n is 2, 1, and $\frac{2}{3}$, respectively.

Bistability (more generally, multistability) occurs when two or more values of δ are possible for a given value of P. Solitary waves corresponding to different values of δ will in general have different $U(R)$ (or intensity) profiles and sizes. Of particular interest are those ranges of δ for which the solitary waves are "robust" (solitonlike). For $d=1$, Enns *et al.* [4–6] have studied the stability of solitary waves corresponding to various $f(I)$ through numerical collision and switching simulations as well as through an analytic Painlevé analysis [8]. Clearly the model $f=I^n$, corresponding to an n-photon process, does not allow for the possibility of bistability for any d. Bistability is possible, however, if *f(I) is dominated by different n values,* corresponding to the onset of different processes, *in different ranges of I.* Of particular importance are "steplike" $f(I)$, a simple example of which is $f(I)=0$ for $I\le I_0$ and $\Delta(1-\sqrt{I_0/I})$ for $I\ge I_0$. A U-shaped $P(\delta)$ curve will result for any d because $f(I)$ rises very rapidly initially (negative slope) and eventually saturates (positive slope). This may be verified for $d=1$, 2, and 3 by *analytically* integrating Eq. (3) and then calculating P. For example, for $d=3$, setting $\beta\equiv\delta/\Delta$, one finds that

$$\rho\equiv P\Delta^{3/2}I_0=\{\theta_0/[2^{3/2}\beta^{1/2}(1-\beta)^2]\}\times[3\beta\theta_0-\tfrac{1}{2}\beta^2\theta_0+\tfrac{1}{3}\beta^{1/2}\theta_0^2+\tfrac{5}{2}\beta^{1/2}+\tfrac{1}{2}\beta^{3/2}\theta_0^2+\tfrac{1}{2}(1-\beta)^2\theta_0^2] \quad (4)$$

with θ_0 obtained from the transcendental equation

$$\cot[\sqrt{(1-\beta)}\theta_0]-[\beta\sqrt{(1-\beta)}\theta_0]^{-1}=\sqrt{(1-\beta)/\beta}\ . \quad (5)$$

Equation (4) is plotted on a doubly logarithmic scale in Fig. 1, the minimum or critical value $\rho_{cr}\simeq 38.9$ at $\beta_{cr}\simeq 0.08$. What about stability? According to Infeld and Rowlands [9], and as we previously confirmed for $d=1$, *negative-slope solitary waves are always unstable for any d.* From the work of Vakhitov and Kolokolov [10] and Laedke and Spatschek [11] on saturating nonlinearities in plasma physics, the positive-slope solitary waves for $d=3$ are stable against small perturbations.

To create two (or more) *stable* branches we can build on the basic U shape by allowing $f(I)$ to have further intensity dependence in different ranges of I. Two simple examples illustrate the approach. The first possibility is to "splice" a positive slope leg onto the U to form an N-shaped $P(\delta)$ curve. From our guidelines this requires that $f(I)$ be *sublinear* (with slope less than $\frac{2}{3}$ to be precise) *at small I.* Since Zakharov and Kuznetsov [12] have already demonstrated that three-dimensional solitary wave solutions to (3) are stable for $f=U=I^{1/2}$, we consider the "sublinear plus smooth step" (SLSS) model, viz.,

$$f(I)=\begin{cases}\Delta\mu(I/I_0)^{1/2}\ , & I\le I_0\\ \Delta[1-(1-\mu)\sqrt{I_0/I}\ , & I\ge I_0\end{cases} \quad (6)$$

with Δ, μ, and I_0 positive parameters and $0<\mu<1$. For $\mu=0$, this reduces to our previous model. We have solved Eq. (3) numerically for the SLSS model and then calculated the normalized energy $\rho\equiv P\Delta^{3/2}/I_0$, with the result being shown in Fig. 1 for $\mu=0.05$. The inset shows the solitary-wave profiles corresponding to the *energetically degenerate* points a ($\beta_a=0.005$) and b ($\beta_b=0.4$) on the stable lower and upper positive-slope branches. Profile b is considerably narrower than a with a maximum intensity I_{max} over 200 times that of a. Note that for b, $I_{max}\gg I_0$ and the numerically calculated energy curve is close to the analytic U-shaped curve. As $\beta\to 1$, the upper branch becomes more dominant and the two curves should (and do) coalesce. It should also be noted that the sudden drop at $\beta\simeq 0.012$ is an artifact of the discontinuity in slope of our model. Models can be readily created which smooth out this feature.

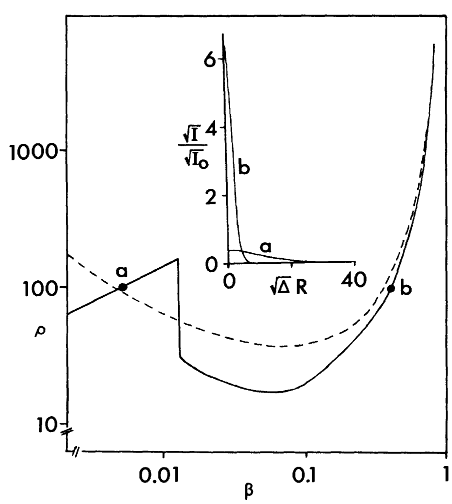

FIG. 1. Normalized solitary-wave energy ρ vs parameter β. Dashed curve, for model $f(I)=0$, $I\le I_0$ and $\Delta(1-\sqrt{I_0/I}\ )$, $I\ge I_0$; solid curve, for SLSS model. Inset: solitary-wave profiles corresponding to points a and b in main figure.

Whether materials can be found with the requisite sublinear response at low I is an open question. However, it should be pointed out that type-B $BaTiO_3$ has been reported to display an $I^{1/2}$ behavior [13,14] at low intensities in its photoinduced absorption (which would correspond to the *imaginary* part of n_2) and to respond on a picosecond time scale [15].

Our second example is based on the fact that many materials (e.g., silica, Rb vapor etc.) have a *Kerr behavior* $[f(I)\propto I]$ *at low intensities.* From our guidelines a saturable $[f(I)\to\text{constant at large }I]$ Kerr model will also yield a U-shaped energy curve. If, as I is further increased, $f(I)$ undergoes a "jump" (e.g., due to a phase transition) and then saturates at a still higher level, a W-shaped (two U's back to back) energy curve should be possible with two negative- and two positive-slope branches. To test this idea, we have numerically integrated Eq. (3) for the "double saturable Kerr" (DSK) model, viz.,

$$f(I)=\begin{cases}1-\exp(-I)\ , & I\le I_0\\ 2-\exp(-I_0)-\exp[-b(I-I_0)]\ , & I\ge I_0\end{cases} \quad (7)$$

taking, e.g., $b=2$ and $I_0=2,5$, and 8. The shape of $f(I)$ is shown in Fig. 2, as well as, for comparison purposes, the SLSS model for $\mu=0.05$, $\Delta=2$. Again, models with similar behavior can be readily created which smooth out the discontinuity in slope. The energy curves for the DSK model are shown in Fig. 3.

For $I_0=2$, a distorted U-shaped energy curve is obtained. Referring to Fig. 2 and recalling our guidelines, this result is easily understood. The "jump" at $I/I_0=1$ occurs before sufficient saturation has taken place, thus wiping out the possibility of a lower positive-slope leg. When I_0 is increased to 5, $f(I)$ has flattened (see Fig. 2)

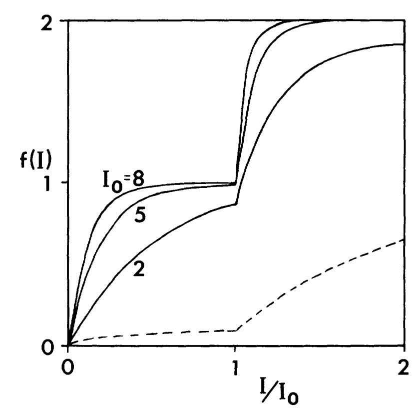

FIG. 2. $f(I)$ discussed in text. Dashed curve, SLSS model for $\mu=0.05$ and $\Delta=2$; solid curves, DSK model for $b=2$ and $I_0=2,5$, and 8.

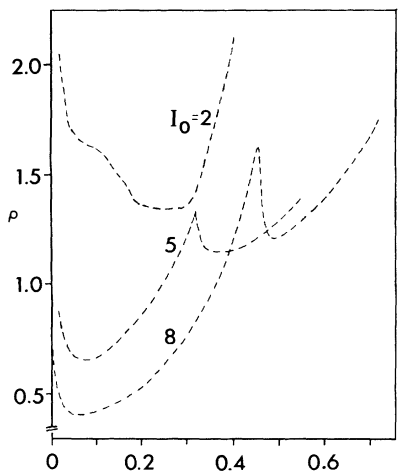

FIG. 3. Solitary-wave energy curves for DSK model for $b=2$ and $I_0=2,5$ and 8.

sufficiently before the jump at $I=I_0$ so that we indeed obtain a (lopsided) W-shaped energy curve. A further increase of I_0 to 8 deepens the "dips" in the W. From stability studies in plasma physics [9,11] on the single saturable Kerr model, the two positive- (negative-) slope branches for $I_0=5$ and 8 are stable (unstable) against small perturbations. The robustness (established, e.g., through collision studies) of the positive-slope solitary waves for this and the other models discussed remains to be established.

Before continuing it should be emphasized that the two models presented here are for illustrative purposes only, to show how bistable three-dimensional solitary waves can be created by applying the guidelines to model $f(I)$. Which material will yield a given $f(I)$ is beyond the scope of this paper. Referring to Fig. 2 it can be seen that even though differently shaped energy curves result, the $f(I)$ profiles needed to create a three-dimensional bistable solitary wave with two stable branches are qualitatively similar, i.e., *two saturable jumps are required.* In the one-dimensional case, only one jump was necessary [4–6]. The jumps could arise due to the onset of higher-order photon processes or phase transitions.

What would the solitary-wave profiles a and b in Fig. 1, e.g., look like in laboratory coordinates? The solitary waves are spherical in terms of the dimensionless coordinates X,Y,T so the widths $\Delta T=\Delta X=\Delta Y$. Thus in laboratory coordinates if the pulse duration is Δt, the width at half maximum in Fig. 1 is W, and F_{sat} is the saturation value of $F(|\phi|^2)$ when $f=\Delta$, then the longitudinal (in the direction of propagation) and transverse widths, W_L and W_T, respectively, are given by

$$W_L=v_g\Delta t=v_g\left[\frac{|D|n_0}{k(n_2F_{\text{sat}})}\right]^{1/2}(2W)\ , \tag{8a}$$

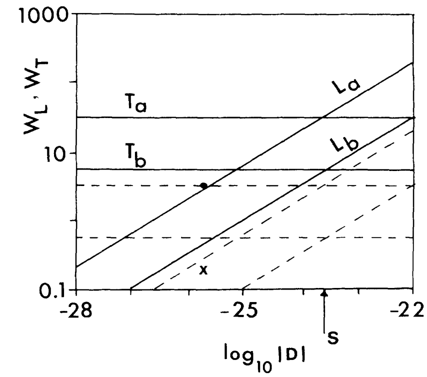

FIG. 4. W_L and W_T for solitary waves a and b in Fig. 1 as a function of $|D|$ for $n_2F_{\text{sat}}=10^{-8}$ (solid lines) and 10^{-6} (dashed lines). T_a stands for the transverse width of a, etc. S labels the value of $|D|$ for which $W_L=W_T$ and the solitary waves are spherical. The dot (cross) labels W_T (W_L) for silica.

$$W_T=\frac{\Delta t}{(k|D|)^{1/2}}=\left[\frac{n_0}{k^2(n_2F_{\text{sat}})}\right]^{1/2}(2W)\ . \tag{8b}$$

Not surprisingly W_L involves the ratio of the GVD to the strength of the nonlinearity whereas W_T is independent of the GVD. To have spherical solitary waves or light bullets in laboratory coordinates requires that the medium parameters satisfy the relation $kv_g^2|D|=1$. For many materials, the anomalous GVD regime is roughly $\lambda=1-10\ \mu$m. For $n_0=1.5$, $\lambda=1\ \mu$m, e.g., this would require $|D|=2.7\times10^{-24}$ s^2/m (indicated by S in Fig. 4). In general the light bullets will be spheroidal, i.e., "flattened" spheres. For silica, e.g., which is Kerr-like and therefore *cannot display bistability,* $|D|=1.8\times10^{-26}$ s^2/m. For, say, $\Delta t=1$ ps, $W_L=0.2$ mm and $W_T=3$ mm. In this case the sphere is highly squashed in the direction of propagation. Since $P(\delta)$ has only negative slope for the Kerr case this solitary wave is not stable, as already noted in Ref. [1].

In Fig. 1 for solitary pulses a and b, $W_a\simeq13$, $W_b\simeq2.2$. For silica, $n_2=1.2\times10^{-22}$ m/V^2, so for $|\phi|\sim10^7-10^8$ V/m, $n_2F(|\phi|^2)=n_2|\phi|^2\sim10^{-8}-10^{-6}$. In the absence of known material parameters, in Fig. 4 we have taken $n_2F_{\text{sat}}=10^{-8}$ and 10^{-6}, $n_0=1.5$, $\lambda=1\ \mu$m (for $\lambda=10\ \mu$m, multiply W_L,W_T by $\sqrt{10}$) and plotted W_L and W_T (in mm) as a function of $|D|$ for solitary pulses a and b. To the left of the point S, $W_T>W_L$, while to the right, $W_T<W_L$. The W_L and W_T values for silica are shown for reference purposes. The bottom axis in Fig. 4 for which $W_L=0.1$ mm corresponds to $\Delta t=0.5$ ps. For shorter pulse durations and therefore smaller widths Eq. (1) becomes inadequate and other contributions (higher-order dispersion, finite response time, self-steepening) must be included [16].

In conclusion we have shown that bistable light bullets may be possible if materials with appropriate $f(I)$ can be found or fabricated.

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