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Analytic evaluation of the self-energy and outer-vertex corrections to the decay rate of orthopositronium in the Fried-Yennie gauge

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In this paper, the order-α contributions of the self-energy and outer-vertex graphs to the decay rate of orthopositronium in the Fried-Yennie gauge are obtained in analytic form.

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This Brief Report describes the analytic calculation of the self-energy and outer-vertex contributions to the decay rate of orthopositronium in the Fried-Yennie gauge. These graphs contribute to the rate at order $(\alpha/\pi)\Gamma_{\rm LO}$, where

$$\Gamma_{\rm LO}=\frac{2}{9\pi}(\pi^2-9)m\alpha^6 \tag{1}$$

is the lowest-order rate [1,2]. These are the first results in a study of the orthopositronium decay rate in the Fried-Yennie gauge. All previous numerical [3–7] and analytical [8–12] work on radiative corrections to orthopositronium decay has been done in the Feynman and Coulomb gauges.

The Fried-Yennie gauge seems optimal for higher-order calculations of positronium decay rates. The Fried-Yennie gauge propagator

$$D^{\mu\nu}_{\rm FY}(k)=\frac{-1}{k^2}\left[g^{\mu\nu}+2\frac{k^\mu k^\nu}{k^2}\right] \tag{2}$$

is covariant and is only slightly more complicated than the Feynman gauge propagator $-g^{\mu\nu}/k^2$. The Fried-Yennie gauge has agreeable infrared properties [13]. The Coulomb gauge shares this good behavior in the infrared, but has a noncovariant propagator which complicates the evaluation of multiloop graphs. The other covariant gauges are poorly behaved in the infrared. Since good infrared behavior and a relatively simple covariant propagator seem essential for multiloop bound-state calculations, we believe that the Fried-Yennie gauge is the only viable gauge for use in calculating the order-$\alpha^2\Gamma_{\rm LO}$ corrections to the decay rates. Calculation of the orthopositronium decay rate to order $\alpha^2\Gamma_{\rm LO}$ is needed to understand the implications of recent high-precision measurements of this rate [14,15].

The orthopositronium decay rate can be written as [7]

$$\Gamma=\frac{m}{2^7\pi^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_2 \sum_{\epsilon_1,\epsilon_2,\epsilon_3}\frac{1}{3}\sum_{\epsilon_m}\frac{1}{3!}|\mathcal{M}|^2\ , \tag{3}$$

where $x_i=\omega_i/m$ is the normalized energy of photon i, ϵ_i is the polarization vector of photon i, and ϵ_m is the orthopositronium spin vector. Energy conservation constrains the normalized photon energies by $x_1+x_2+x_3=2$. The invariant matrix element

$$\mathcal{M}=\mathcal{M}_{\rm LO}+2\mathcal{M}_{\rm SE}+2\mathcal{M}_{\rm OV}+\cdots \tag{4}$$

has contributions from each of the graphs in Fig. 1. The two self-energy graphs contribute equally, as do the two outer-vertex graphs.

The self-energy and outer-vertex graphs differ from the lowest-order graph by radiative corrections on or adjacent to one of the outer vertices. Thus the self-energy and outer-vertex matrix elements can be obtained from the lowest-order matrix element by the replacement of one of the outer-vertex factors (a γ matrix) by a more complicated factor. The lowest-order matrix element is

$$\mathcal{M}_{\rm LO}=-i\pi\alpha^3\sum_{\sigma\in S_3}\frac{x_{\sigma(2)}}{x_1x_2x_3}\mathrm{tr}\left[\gamma\epsilon_{\sigma(3)}(-\gamma R_{\sigma(3)}+1)\gamma\epsilon_{\sigma(2)}(\gamma R_{\sigma(1)}+1)\gamma\epsilon_{\sigma(1)}\begin{bmatrix}0 & \sigma\cdot\hat{\epsilon}_m\\ 0 & 0\end{bmatrix}\right]\ , \tag{5}$$

where the sum is over the 3! permutations of the final photons and the R's are the dimensionless momentum vectors $R_i=N-K_i$ with $N=(1,\mathbf{0})$ and $K_i=(\omega_i/m,\mathbf{k}_i/m)$. The self-energy and outer-vertex matrix elements are obtained by the replacement

$$\gamma\epsilon_{\sigma(1)}\rightarrow\Lambda(x_{\sigma(1)})\epsilon_{\sigma(1)} \tag{6}$$

for "vertex correction" matrices Λ^λ that are functions of the single variable $x_{\sigma(1)}$. These "vertex correction" matrices have the form

$$\Lambda^\lambda(x_i)=\frac{\alpha}{4\pi}[\gamma^\lambda f(x_i)+(\gamma R_i-1)\gamma^\lambda g(x_i)+(\gamma R_i-1)N^\lambda h(x_i)+N^\lambda j(x_i)]\ , \tag{7}$$

where

$$f_{\rm SE}(x)=\frac{12x^2}{(1-2x)^2}\ln(2x)+\frac{6x}{1-2x}\ , \tag{8a}$$

$$g_{\rm SE}(x)=\frac{6x}{(1-2x)^2}\ln(2x)+\frac{3}{1-2x}\ , \tag{8b}$$

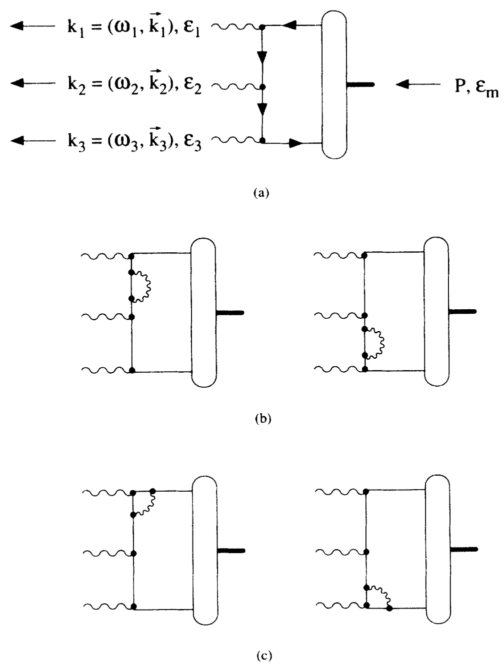

FIG. 1. Contributions to the decay rate of orthopositronium: (a) the lowest-order graph, (b) the self-energy graphs, and (c) the outer-vertex graphs.

$$h_{\rm SE}(x)=0 \ , \tag{8c}$$

$$j_{\rm SE}(x)=0 \ , \tag{8d}$$

for the renormalized self-energy correction, and

$$f_{\rm OV}(x)=\frac{1}{x}\eta(x)+\frac{2-6x-4x^2}{(1-2x)^2}\ln(2x)-\frac{4x}{1-2x} \ , \tag{9a}$$

$$g_{\rm OV}(x)=\frac{-4x}{(1-2x)^2}\ln(2x)-\frac{2}{1-2x} \ , \tag{9b}$$

$$h_{\rm OV}(x)=\frac{1}{x}\left[\frac{1}{x}\eta(x)+\frac{2-10x+8x^2}{(1-2x)^2}\ln(2x)-\frac{2(1-x)}{1-2x}\right] , \tag{9c}$$

$$j_{\rm OV}(x)=\frac{-4x}{(1-2x)^2}\ln(2x)-\frac{2}{1-2x} \ , \tag{9d}$$

for the renormalized outer-vertex correction. The η function

$$\eta(x)=\zeta(2)-Li_2(1-2x) \tag{10}$$

contains a dilogarithm [16]. The fact that the $h_{\rm OV}$ and $j_{\rm OV}$ functions are the same in the Fried-Yennie gauge as in the Feynman gauge [10] is a consequence of gauge invariance.

The order-α corrections to the decay rate come from cross terms in the square of $\mathcal{M}$. The self-energy correction $\Gamma_{\rm SE}$ comes from Eq. (3) by the replacement

$$|\mathcal{M}|^2\rightarrow 2\times 2\,{\rm Re}[(\mathcal{M}_{\rm LO})^*\mathcal{M}_{\rm SE}] \tag{11}$$

and the outer-vertex correction $\Gamma_{\rm OV}$ by the replacement

$$|\mathcal{M}|^2\rightarrow 2\times 2\,{\rm Re}[(\mathcal{M}_{\rm LO})^*\mathcal{M}_{\rm OV}] \ . \tag{12}$$

Since the phase space and $\mathcal{M}_{\rm LO}$ are symmetric under photon interchange, there is no need to symmetrize in $\mathcal{M}_{\rm SE}$ and $\mathcal{M}_{\rm OV}$ as well. Thus the "vertex correction" can be taken to act on photon 1 only. On performing the polarization sums, the spin sum, the resulting trace [17], and the x_2 integration, one obtains

$$\begin{aligned}\Gamma_{\rm SE}=\frac{m\alpha^7}{6\pi^2}\int_0^1 dx\frac{1}{x^2}[&P_f(x)f_{\rm SE}(x)+P_g(x)(-2x)g_{\rm SE}(x)\\&+P_h(x)(-2x)h_{\rm SE}(x)\\&+P_j(x)j_{\rm SE}(x)]\end{aligned} \tag{13}$$

(where $x=x_1$) for the self-energy correction and a similar form for the outer-vertex correction. The P factors are [10]

$$P_f(x)=\left[\frac{-8}{2-x}+12-5x+x^2\right]\ln(1-x)+8x-3x^2 \ , \tag{14a}$$

$$P_g(x)=\left[\frac{8}{(2-x)^2}-\frac{18}{2-x}+10+x\right]\ln(1-x)+\frac{4x}{2-x}+x \ , \tag{14b}$$

$$P_h(x)=\left[\frac{-4}{(2-x)^2}+\frac{6}{2-x}-\frac{3x}{2}\right]\ln(1-x)-\frac{2x}{2-x}+3x+\frac{x^2}{2} \ , \tag{14c}$$

$$P_j(x)=\left[\frac{-8}{(2-x)^2}+\frac{14}{2-x}-8x+2x^2\right]\ln(1-x)-\frac{4x}{2-x}+7x-3x^2 \ . \tag{14d}$$

Integration of Eq. (13) yields the final results

$$\begin{aligned}\Gamma_{\rm SE}&=\frac{m\alpha^7}{\pi^2}[-\tfrac{13}{54}\zeta(3)+\tfrac{461}{108}\zeta(2)\ln(2)-\tfrac{251}{72}\zeta(2)-\tfrac{29}{6}\ln(2)+\tfrac{9}{2}]\\&=\frac{m\alpha^7}{\pi^2}(-0.007\,132\,904)\\&=\Gamma_{\rm LO}\frac{\alpha}{\pi}(-0.036\,911\,113)\end{aligned} \tag{15}$$

and

$$\begin{aligned}\Gamma_{\rm OV}&=\frac{m\alpha^7}{\pi^2}[-\tfrac{88}{54}\zeta(3)-\tfrac{299}{216}\zeta(2)\ln(2)+\tfrac{49}{18}\zeta(2)+\tfrac{13}{6}\ln(2)-2-\tfrac{1}{6}R]\\&=\frac{m\alpha^7}{\pi^2}(0.732\,986\,380)\\&=\Gamma_{\rm LO}\frac{\alpha}{\pi}(3.793\,033\,599) \ .\end{aligned} \tag{16}$$

The outer-vertex result contains the degree-four [18] quantity

$$R = \int_0^1 dx \frac{\ln(1-x)}{2-x}[\zeta(2) - Li_2(1-2x)]$$
$$= -1.743\,033\,833\,7(3)\ , \qquad (17)$$

which has not been evaluated analytically [19].

The self-energy and outer-vertex graphs are the first orthopositronium decay graphs to be evaluated in the Fried-Yennie gauge. A numerical evaluation of the remaining order-α graphs in the Fried-Yennie gauge is in progress. When this work is completed, we will have a new and independent result for the orthopositronium decay rate to order α. This result will be of use in higher-order calculations of the decay rate using the Fried-Yennie gauge.

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