Photon number distribution of detuned two-mode vacuum and excited squeezed states

M. Zahler and Y. Ben-Aryeh
Department of Physics, Technion–Israel Institute of Technology, Haifa 32000, Israel
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Lie-group methods are used for analyzing the photon number distribution of detuned two-mode squeezed states. The effects of the detuning on the vacuum and excited two-mode squeezed states are analyzed and found to be similar to those obtained previously for the degenerate case. Correlations between the radiations in the two modes are described and found to decrease as a function of the detuning parameters.

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I. INTRODUCTION

The problem of detuned squeezed states (SS's) was first treated by Charmichael, Milburn, and Walls [1] and has been developed further in various works [2]. We have already analyzed the influence of the deviation from resonance on the SS photon number distribution [3] in the degenerate case and have found that the squeezing properties are decreased as much as the detuning from resonance is increased. In addition, the Schleich-Wheeler oscillations [4] of the excited SS photon distribution are smeared by the cavity-mode bandwidth. This process is similar to the "phase-jitter" [5] mechanism of Klauder, McCall, and Yurke, where the phase difference between the squeezed and the coherent fields cannot be locked. However, Klauder, McCall, and Yurke referred these phase fluctuations to the nonideal coherent source, while in the present situation this is related to the detuning-produced phase. In this paper we expand the previous analysis to the general two-mode SS, which is represented by the Hamiltonian [2]

$$H(t)=\hbar\Omega_1^0 a_1^\dagger a_1+\hbar\Omega_2^0 a_2^\dagger a_2 - i\frac{\hbar\kappa\varepsilon}{2}(a_1 a_2 e^{i(2\Omega t+\phi)}-a_1^\dagger a_2^\dagger e^{-i(2\Omega t+\phi)}) . \quad (1.1)$$

The classical parameters of this model are ε, the constant pump amplitude (assumed to be undepleted); ϕ, the phase difference between the pump field and the total phase of the two squeezed modes, and κ describes an effective macroscopic nonlinear coupling strength. The quantum-mechanical properties of the squeezed radiation are presented via a_i and $a_i^\dagger$ $(i=1,2)$, i.e., the annihilation and creation operators of the cavity signal and idler modes, respectively. 2Ω and Ω_i^0 are the pump and the cavity-mode frequencies, respectively. Our model neglects the damping mechanism, but we permit small detuning from resonance. Following the ordinary method of nearly resonant interaction [6] we define a unitary operator:

$$\tilde{U}(t)=e^{-i(\Omega_1 a_1^\dagger a_1+\Omega_2 a_2^\dagger a_2)t} , \quad (1.2)$$

which connects between the state vectors of the Schrödinger picture (SP) and the rotated picture (designated by a tilde):

$$|\psi_{\mathrm{SP}}\rangle=\tilde{U}(t)|\tilde{\psi}\rangle . \quad (1.3)$$

Using the operator-algebra theorems [6], the Hamiltonian in the rotated picture is found to be time independent:

$$\tilde{H}=\tilde{U}(t)^\dagger H(t)\tilde{U}(t) =\hbar\Omega_1^0 a_1^\dagger a_1+\hbar\Omega_2^0 a_2^\dagger a_2 - i\hbar\frac{\kappa\varepsilon}{2}(a_1 a_2 e^{i\phi}-a_1^\dagger a_2^\dagger e^{-i\phi}) . \quad (1.4)$$

Now, with a definition of the detuned two-mode squeezing operator as the evolution operator in the rotated picture, we get

$$|\tilde{\psi}(t)\rangle=S_D|\tilde{\psi}(0)\rangle , \quad (1.5)$$

where the SP evolution operator is written as

$$U(t)=\tilde{U}(t)S_D(t) . \quad (1.6)$$

By using Eq. (1.6), the evolution equation can be easily developed [3] as

$$\frac{d}{dt}S_D=\left[i(\Delta\Omega_1 a_1^\dagger a_1+\Delta\Omega_2 a_2^\dagger a_2) - \frac{s}{2}(a_1 a_2 e^{i\phi}-a_1^\dagger a_2^\dagger e^{-i\phi})\right]S_D(t) , \quad (1.7)$$

where

$$\Delta\Omega_i\equiv\Omega_i-\Omega_i^0, \quad i=1,2 \quad (1.8)$$

are the detuning frequencies and

$$s\equiv\kappa\varepsilon \quad (1.9)$$

is the squeezing frequency.

The solution of Eq. (1.7), under the initial condition

$$S_D(0)=1 , \quad (1.10)$$

is

$$S_D(t)=\exp[i(\Delta\Omega_1 a_1^\dagger a_1+\Delta\Omega_2 a_2^\dagger a_2)t - (st/2)(a_1 a_2 e^{i\phi}-a_1^\dagger a_2^\dagger e^{-i\phi})] . \quad (1.11)$$

So that the resonant $(\Delta\Omega_i=0)$ expression for the two-mode squeezing operator is just a special case of Eq. (1.11). The detuned degenerate-squeezing operator [3] is obtained from (1.11) by assuming $a_1 = a_2$, $\Delta\Omega_1 + \Delta\Omega_2 = \Delta\Omega$.

The Hamiltonian (1.1) describes the nondegenerate parametric down-conversion (PDC) process [7] inside a cavity. The cavity is used in order to avoid the frequency broadening of its output signal and idler field (due to parametric fluorescence) and the spatial dispersion (light cone). In the present nondegenerate case, two down-converted photons with frequencies Ω_1 and Ω_2 ($\Omega_1 + \Omega_2 = 2\Omega$), which are fixed by the phase-matching conditions, are assumed to leave the cavity with a direction parallel to the pump. However, the cavity-mode frequencies Ω_1^0 and Ω_2^0 might be slightly deviated from Ω_1 and Ω_2 due to uncontrollable broadening mechanisms (e.g., mechanical and thermal vibrations of the cavity mirrors which charge the cavity length). In the present article we analyze the effects of this detuning on the photon statistics (i.e., photon number distribution) of the produced squeezed light. In Sec. II we discuss briefly the effects of detuning on nondegenerate vacuum squeezed states. The effects of detuning on the interference with coherent light is discussed in Sec. III. For simplicity we neglect in the present work damping mechanism [2] and phase-mismatching problems [8,9].

II. DETUNED TWO-MODE SQUEEZED VACUUM STATES

We would like to show how the detuning term $i(\Delta\Omega_1 a_1^\dagger a + \Delta\Omega_2 a_2^\dagger a_2)$ in the exponent of the detuned nondegenerate squeezing operator controls the photon distribution. Our first algebraic goal is to arrange the evolution operator to a normal ordered (NO) form, by using the Lie algebra methods (see the Appendix). We exploit here the fact that our squeezed vacuum state (SVS) was a vacuum state initially.

We define dimensionless parameters:

$$\tau = st \quad \text{(the squeezing parameter)}\ , \tag{2.1}$$

$$\delta_i = \frac{\Delta\Omega_i}{s}, \quad i = 1,2 \quad \text{(the detuning parameter)}\ , \tag{2.2}$$

$$\Delta_+ = \delta_1 + \delta_2\ , \tag{2.3}$$

$$\Delta_- = \delta_2 - \delta_1\ , \tag{2.4}$$

and

$$\xi = (1 - \Delta_+^2)^{1/2} \quad \text{(the threshold parameter)}\ . \tag{2.5}$$

Using the results developed in the Appendix we obtain the detuned squeezed vacuum state (DSVS)

$$|\mathrm{DSVS}\rangle = \sum_{n=0}^{\infty} \frac{[\tanh(r)]^n e^{-in(\phi + 2\Omega t - \theta_D)} |n\rangle_1 |n\rangle_2}{\cosh(r)}\ , \tag{2.6}$$

where we use the Carmichael, Milburn, and Walls [1] transformation rule

$$\tanh\left[\frac{\tau}{2}\right] \to \tanh(r) e^{i\theta_D}\ , \tag{2.7}$$

$$\sinh(r) \equiv \frac{\sinh\left[\frac{\xi\tau}{2}\right]}{\xi}\ ,$$

$$\theta_D \equiv \arg\left[\cosh\left[\frac{\xi\tau}{2}\right] + i\left[\frac{\Delta_+}{\xi}\right]\sinh\left[\frac{\xi\tau}{2}\right]\right]\ . \tag{2.8}$$

From (2.6), (A10), and (A11), the chaotic photon-pair probabilities are immediately given by

$$P_{n_1 = n_2 = n} = \frac{\left[\dfrac{\sinh^2\left[\frac{\xi\tau}{2}\right]}{\xi^2}\right]^n}{\left[\cosh^2\left[\frac{\xi\tau}{2}\right] + \left[\frac{\Delta_+}{\xi}\right]^2 \sinh^2\left[\frac{\xi\tau}{2}\right]\right]^{n+1}}\ . \tag{2.9}$$

From (2.9) it is obvious that the detuning phase θ_D does not appear and hence does not affect the photon-pair distribution. However, we show in the next section that this phase affects the *excited* squeezed photon distribution. The effects of the detuning parameter Δ_+ on the photon number distribution for the nondegenerate SVS are very similar to those analyzed by us [3] for the degenerate SVS. Therefore we do not repeat here the description of these properties.

III. DETUNED EXCITED TWO-MODE SQUEEZED STATES

The two-mode detuned excited squeezed states (DESS's) are defined as

$$|\mathrm{DESS}\rangle = D(\alpha_1) D(\alpha_2) S_D |0\rangle\ , \tag{3.1}$$

where $D(\alpha_1)$ and $D(\alpha_2)$ are the coherent state (CS) displacement operators

$$D(\alpha_i) = e^{\alpha_i a_i^\dagger - \alpha_i^* a_i}\ . \tag{3.2}$$

S_D is the detuned two-mode squeezing operator given according to (1.11) and (2.1)–(2.4) by [10]

$$S_D(t) = \exp\left[\left[\frac{\tau}{2}\right][i\Delta_+(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) - (e^{i\phi} a_1 a_2 - e^{-i\phi} a_1^\dagger a_2^\dagger)]\right]\ . \tag{3.3}$$

Our aim is now to study the effects of the detuning on the photon number distribution of DESS's. For this purpose we rearrange the operatoric product $D(\alpha_1) D(\alpha_2) S_D$ in a NO form as we did in the previous section for S_D. Here the relevant Lie groups are found [6] to be D-6. Its algebraic representation and the Baker-Campbell-Hausdorf (BCH) relations are developed in the Appendix. By using these results we get for the NO expression for the two-mode DESS's:

$$|\mathrm{DESS}\rangle = e^{D+E} e^{A a_1^\dagger a_2^\dagger + B a_1^\dagger + C a_2^\dagger}|0\rangle , \tag{3.4}$$

where the parameters A, B, C, D, and E are given in (A20).

The expansion of (3.4) can be written as

$$|\mathrm{DESS}\rangle = e^{D+E}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{q=0}^{m}\frac{\sqrt{m!(n-q)!}}{q!(n-m)!(m-q)!}A^{m-q}B^{q}C^{n-m}|m\rangle_1|n-q\rangle_2 . \tag{3.5}$$

The amplitudes of the DESS's for k_1 and k_2 photons in the first and second modes, respectively, are given by

$$\begin{aligned} C_{k_1k_2} &= \langle k_1,k_2|\mathrm{DESS}\rangle \\ &= e^{D+E}\sum_{n=\max\{k_1,k_2\}}^{n=k_1+k_2}\left[\frac{\sqrt{k_1!k_2!}}{(n-k_2)!(n-k_1)!(k_1+k_2-n)!}\right]A^{k_1+k_2-n}B^{n-k_2}C^{n-k_1} . \end{aligned} \tag{3.6}$$

Here the summation is, respectively, from k_1 or k_2 when $k_1>k_2$ or $k_2\ge k_1$ up to $n=k_1+k_2$.

Let us see how the detuning phase (2.8) enters into Eq. (3.6). We substitute

$$\alpha_1 = |\alpha_1|e^{-i\zeta_1}, \quad \alpha_2 = |\alpha_2|e^{-i\zeta_2}, \quad \psi=\zeta_1+\zeta_2-\phi , \tag{3.7}$$

where ζ_1 and ζ_2 are the phases of the coherent fields in the two modes, respectively. By using these definitions Eq. (3.6) is transformed into the form (for the case $k_1>k_2$):

$$\begin{aligned} C_{k_1k_2} &= \sqrt{k_1!k_2!}\exp\left[-\left[\frac{|\alpha_1|^2+|\alpha_2|^2}{2}-|\alpha_1||\alpha_2|\tanh(r)e^{i(\theta_D+\psi)}\right]\right][\cosh(r)]^{-1} \\ &\times\sum_{l=0}^{k_2}\frac{e^{-il(\theta_D+\psi)}[\tanh(r)]^{k_2-l}[|\alpha_1|-|\alpha_2|\tanh(r)e^{i(\theta_D+\psi)}]^{l+k_1-k_2}}{l!(l+k_1-k_2)!(k_2-l)!}[|\alpha_2|-|\alpha_1|\tanh(r)e^{i(\theta_D+\psi)}]^{l} . \end{aligned} \tag{3.8}$$

Here we have ignored out of the summations (n independent) phase factors that are cancelled for the probabilities $|C_{k_1k_2}|^2$. From (3.8) we immediately find that the probabilities depend explicitly on the phase factor $\theta_D+\psi$. This is in contrast to the SVS in which the phases were cancelled for the photon number distribution. That is an expansion to the same result obtained for the degenerate case [3].

We arrange the summation of (3.6) according to the following three cases.

(a) By substituting $l=n-k_1$, for $k_1>k_2$, we get

$$\begin{aligned} C_{k_1k_2} &= e^{D+E}\sqrt{k_1!k_2!} \\ &\times\sum_{l=0}^{k_2}[l!(l+k_1-k_2)!(k_2-l)!]^{-1} \\ &\times A^{k_2-l}B^{l+k_1-k_2}C^{l} . \end{aligned} \tag{3.9}$$

(b) By substituting $l=n-k_2$ for $k_2>k_1$, we get

$$\begin{aligned} C_{k_1k_2} &= e^{D+E}\sqrt{k_1!k_2!} \\ &\times\sum_{l=0}^{k_1}[l!(l+k_2-k_1)!(k_1-l)!]^{-1} \\ &\times A^{k_1-l}B^{l}C^{l+k_2-k_1} . \end{aligned} \tag{3.10}$$

(c) By substituting $l=n-k$ for $k_2=k_1=k$, we get

$$C_{kk} = e^{D+E}k!\sum_{l=0}^{k}[(l!)^2(k-l)!]^{-1}A^{k-l}B^{l}C^{C} . \tag{3.11}$$

Explicit expressions for the amplitude $C_{k_1k_2}$ are obtained by substituting the parameters A, B, C, D, E from (A20) into Eqs. (3.9)–(3.11). By using the definitions (3.7) in a similar way to the derivation of (3.8) we find that the phase dependence of the photon number distribution enters only via the phase $\theta_D+\psi$. Equations (3.9)–(3.11) are convenient for numerical analysis especially for low values of k_1 and k_2.

Equation (3.4) can be expanded also in another form, which leads to analytical expressions for the amplitude $C_{k_1k_2}$,

$$|\mathrm{DESS}\rangle = e^{\gamma}\sum_{n=0}^{\infty}\left[\frac{1}{n!}\right]A^{n}\left[a_1^\dagger+\frac{C}{A}\right]^{n}\left[a_2^\dagger+\frac{B}{A}\right]^{n}|0\rangle , \tag{3.12}$$

where

$$\gamma = D+E-\frac{BC}{A} . \tag{3.13}$$

By using the binomial expansions in (3.12) and after straightforward (although tedious) algebra we get

$$C_{k_1k_2}=\langle k_1k_2|\mathrm{DESS}\rangle$$
$$=\frac{e^{\gamma}}{\sqrt{k_1!k_2!}}\left[\frac{A}{C}\right]_{k_1}\left[\frac{A}{B}\right]^{k_2}\times\sum_{n=\max(k_1,k_2)}^{\infty}\left[\frac{n!}{(n-k_1)!(n-k_2)!}\right]\times\left[\frac{CB}{A}\right]^{n} . \quad (3.14)$$

By rearranging the summation of (3.14) we get analytical expressions for the above three cases.

(a) For $k_1>k_2$, we substitute $l=n-k_1$, and after a straightforward algebra we get

$$C_{k_1,k_2}=e^{\gamma}\left[\frac{k_1!}{k_2!}\right]^{1/2}\left[\frac{A^{k_2}B^{k_1-k_2}}{(k_1-k_2)!}\right]\times{}_1F_1\left[k_1+1,k_1-k_2+1;\frac{BC}{A}\right] , \quad (3.15)$$

where ${}_1F_1$ is the confluent hypergeometric function [11]:

$${}_1F_1(u+1,v+1;w)=\left[\frac{u!}{v!}\right]\sum_{l=0}^{\infty}\left[\frac{1}{l!}\right]\left[\frac{(l+u)!}{(l+v)!}\right]w^{l} . \quad (3.16)$$

(b) For $k_2>k_1$ we substitute $l=n-k_2$, and then we get

$$C_{k_1,k_2}=e^{\gamma}\left[\frac{k_{2!}}{k_{1!}}\right]^{1/2}\left[\frac{A^{k_1}C^{k_2-k_1}}{(k_2-k_1)!}\right]\times{}_1F_1\left[k_2+1,k_2-k_1+1;\frac{BC}{A}\right] . \quad (3.17)$$

(C) For $k_2=k_1$, we get

$$C_{k,k}=e^{\gamma}A^{k}{}_1F_1\left[k+1,1;\frac{BC}{A}\right] . \quad (3.18)$$

Using the transformation (1.3) from the rotating picture into the SP the DESS is given as

$$|\mathrm{DESS}\rangle=\sum_{k_1,k_2=0}^{\infty}C_{k_1,k_2}e^{-i(k_1\Omega_1+k_2\Omega_2)t}|k_1\rangle_1|k_2\rangle_2 . \quad (3.9)$$

The photon number distribution of the $|\mathrm{DESS's}\rangle$ are given as

$$P(k_1,k_2)=|C_{k_1,k_2}|^2 . \quad (3.20)$$

The effects of the detuning on the Schleich-Wheller oscillations for the two-mode DESS's are similar to those treated by us for the one-mode DESS [3]. Other effects related to these oscillations have also been treated in recent articles [12,13,14]. We would like to discuss here the effect of the detuning on the correlation between the number of photons in the two modes. In Fig. 1(a)–1(e) we show the probability $P_{k_1,k_2}=|C_{k_1k_2}|^2$ as a function of number of photons k_1 and k_2 in the two modes for five different values of the detuning Δ_+ with the common parameters $|\alpha_1|=|\alpha_2|=3$, $\tau=2.0$, $\psi=0$. We find that zero or small values of detuning ($\Delta_+<1$) the excited SS shows a strong correlation between the number of photons in the two modes. For larger values of detuning ($\Delta_+\gg 1$) these correlations vanish.

For example, in Fig. 1(e) (for $\Delta_+=2$) the probability $P(k_1,k_2)$ corresponds already to a multiplication of two separate distributions of Poissonian shape (including small oscillations).

IV. SUMMARY AND CONCLUSIONS

In the present article we have analyzed the detuning effects on the photon number distribution of *two-mode,* vacuum, and excited SS's. The use of the Lie-group methods, and the BCH relations, which were developed in our previous work [3] for the degenerate SS's, have been generalized here to the two-mode SS. The present generalizations are important since most of the experiments on SS's correspond to two-mode SS's.

We have developed analytical expressions for the amplitudes of the two-mode DSVS and DESS's. The main physical effects of the present two-mode case are similar to those described by us for the degenerate case [3].

One of the most important properties of SS's are the correlation between the radiations in the two modes [15]. We have calculated the photon number distribution as a function of the detuning between the frequencies of the down-converted photons and the cavity two-mode frequencies. We have shown that the correlations between the two modes are decreasing as a function of the detuning parameters.

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APPENDIX: NORMAL ORDERING OF THE TWO-MODE EVOLUTION OPERATORS

1. DSVS

Since we would like to operate with the evolution operator $S_D(t)$ of (1.11) on the vacuum state, it is worthwhile to rearrange it in a NO form. For this purpose we define the operators:

$$\hat{x}_1 = a_1^\dagger a_1 + a_2^\dagger a_2 + 1 \, ,$$
$$\hat{x}_2 = a_2^\dagger a_2 - a_1^\dagger a_1 \, , \tag{A1}$$
$$\hat{x}_3 = a_1 a_2 , \quad \hat{x}_4 = a_1^\dagger a_2^\dagger \, .$$

These operators are elements of a closed Lie algebra, generating the group [6] SL(2,$\mathbb{R}$). They obey the following commutation relations:

$$[\hat{x}_1,\hat{x}_2] = [\hat{x}_3,\hat{x}_2] = [\hat{x}_4,\hat{x}_2] = 0 \, ,$$
$$[\hat{x}_1,\hat{x}_3 =] = -2\hat{x}_3 , \quad [\hat{x}_1,\hat{x}_4] = 2\hat{x}_4 \, , \tag{A2}$$
$$[\hat{x}_3,\hat{x}_4] = \hat{x}_1 \, .$$

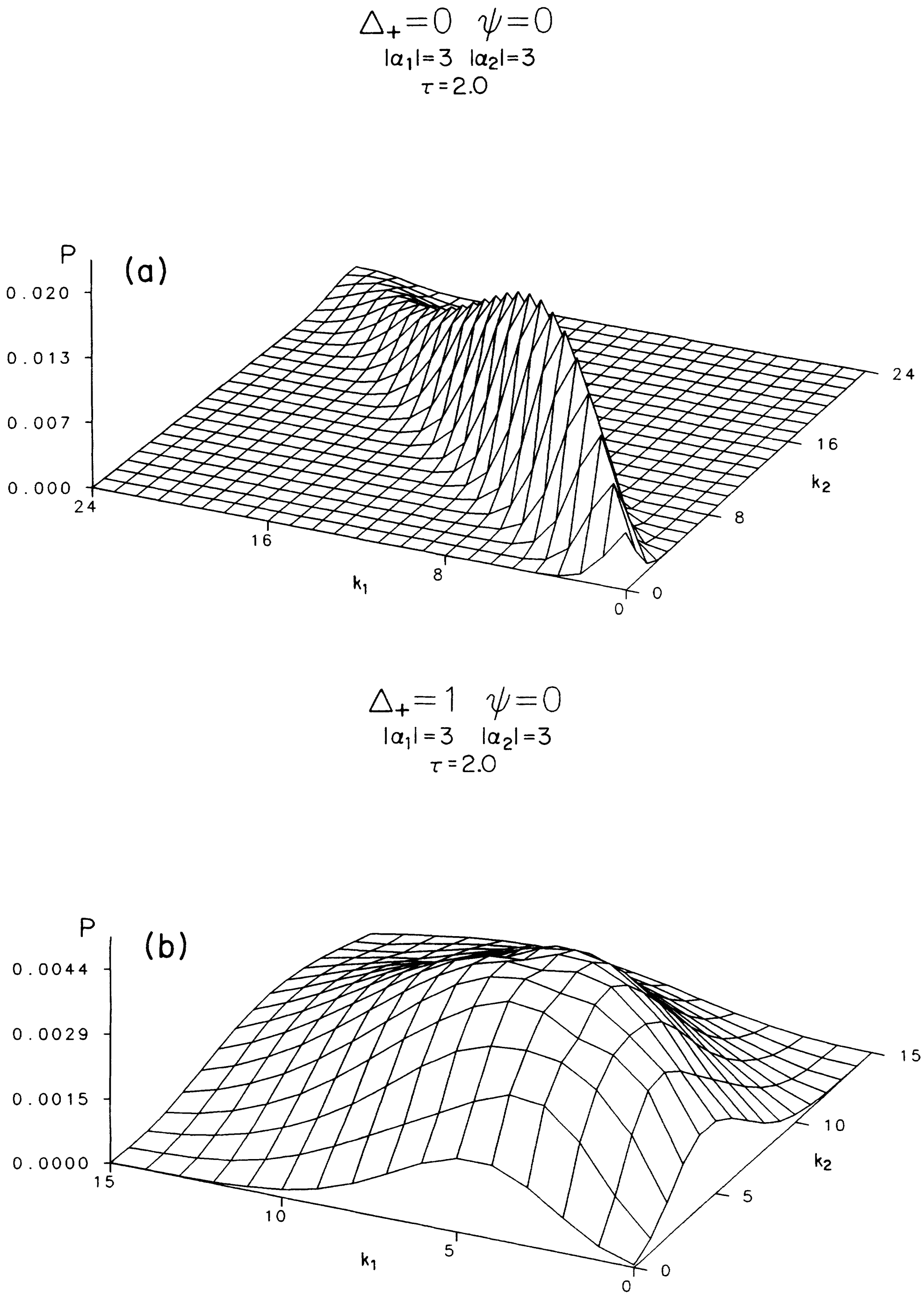

FIG. 1. Photon number distribution P as a function of the number of photons k_1 and k_2 for two-mode DESS's. (a)–(e) The detuning values $\Delta_+ = 0.0$, 1.0, 1.3, 1.6, and 2.0, respectively, while their common parameters are $|\alpha_1| = |\alpha_2| = 3$, $\tau = 2.0$, $\Delta_- = 0$, $\psi = 0$ [these parameters are defined in Eqs. (1.1), (2.1)–(2.4), and (3.2)]. The photon number probabilities have been calculated according to Eqs. (3.7)–(3.9) and (A20).

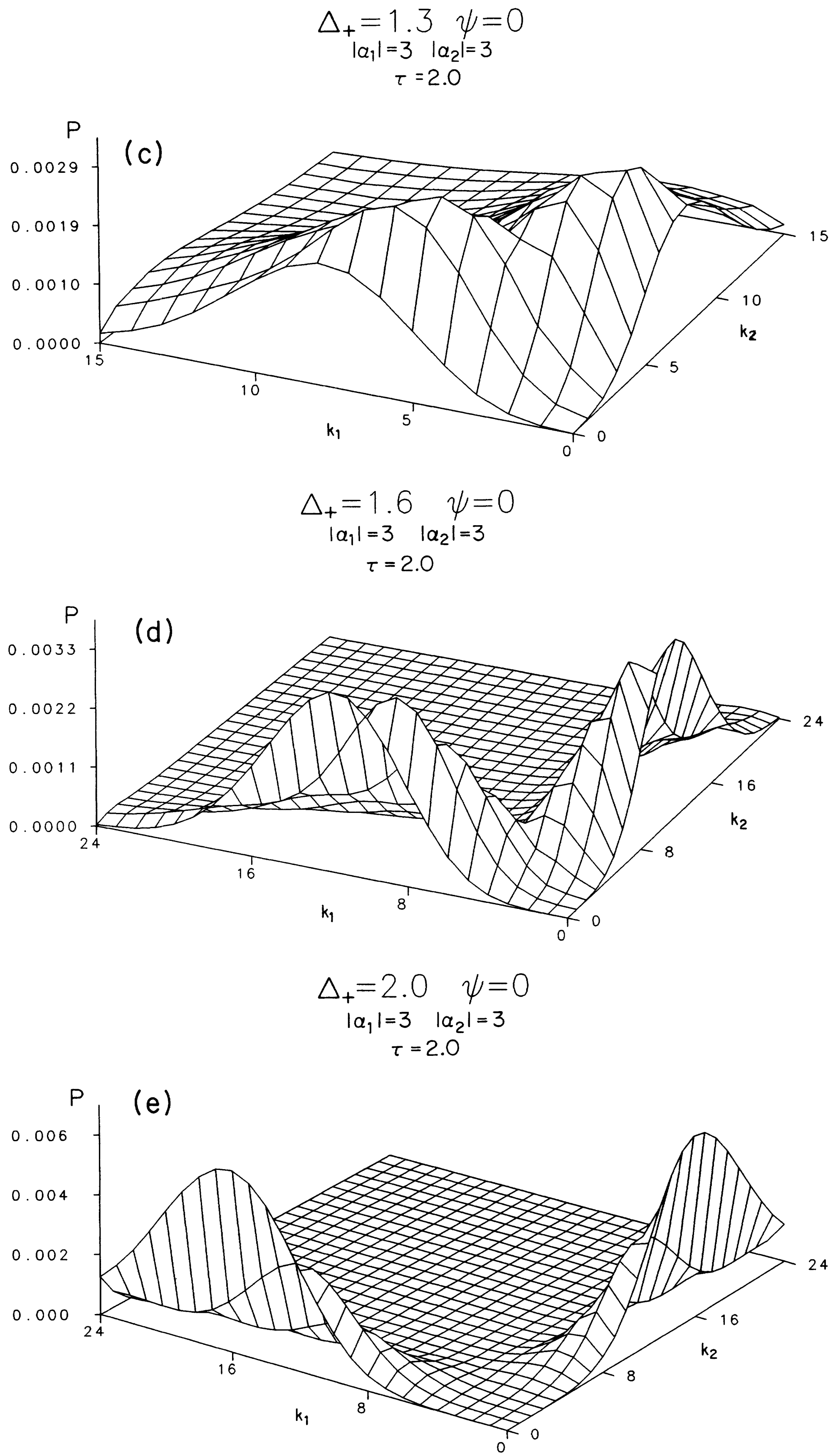

FIG. 1. (*Continued*).

We use the following Lie algebra matrix representations:

$$\hat{x}_1 \to X_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} , \quad \hat{x}_2 \to X_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I , \quad \hat{x}_3 \to X_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \quad \hat{x}_4 \to X_4 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} . \tag{A3}$$

By using the definitions (2.1)–(2.4) and (A1) the detuned two-mode squeezing operator of Eq. (1.11) can be written as [10]

$$S_D(\tau) = \exp\left[\left(\frac{i\tau}{2}\right)\Delta_-\hat{x}_2\right] \times \exp\left[\frac{i\tau}{2}(\Delta_+\hat{x}_1 - e^{i\phi}\hat{x}_3 + e^{-i\phi}\hat{x}_4)\right] . \tag{A4}$$

The NO form of S_D can be written as

$$S_D(\tau) = \exp\left[\left(\frac{i\tau}{2}\right)\Delta_-\hat{x}_2\right]\exp(\alpha\hat{x}_4)\exp(\beta\hat{x}_1)\exp(\gamma\hat{x}_3) . \tag{A5}$$

Our purpose now is to find the relations between the NO parameters α,β,γ and the physical parameters, i.e., Δ_+, ϕ, and τ. These relations are the Baker-Campbell-Hausdorff (BCH) relations [16]. The BCH theorem enables us to substitute the Lie algebra operators with their related matrix representations in order to find the desired relations. Using this theorem we can write the BCH equation as

$$\exp(\alpha\mathbf{X}_4)\exp(\beta\mathbf{X}_1)\exp(\gamma\mathbf{X}_3) = \exp\left[\frac{i\tau}{2}(\Delta_+\mathbf{X}_1 - e^{i\phi}\mathbf{X}_3 + E^{-i\phi}\mathbf{X}_4)\right] . \tag{A6}$$

By expanding (A6) and by following a straightforward algebra, we get

$$\begin{bmatrix} e^{-\beta} & \gamma e^{-\beta} \\ -\alpha e^{-\beta} & e^{\beta} - \alpha\gamma e^{-\beta} \end{bmatrix} = \cosh\left(\frac{\xi\tau}{2}\right)\mathbf{I} + \frac{\sinh\left(\frac{\xi\tau}{2}\right)}{\xi}\mathbf{A} , \tag{A7}$$

where

$$(1-\Delta_+^2) = \xi^2, \quad \mathbf{A} = \begin{bmatrix} -i\Delta_+ & -e^{i\phi} \\ -e^{-i\phi} & i\Delta_+ \end{bmatrix} . \tag{A8}$$

From (A7) we get

$$e^{-\beta} = \cosh\left(\frac{\xi\tau}{2}\right) - i\Delta_+\sinh\left(\frac{\xi\tau}{2}\right)\Big/\xi , \quad \alpha = \left[\frac{\sinh(\xi\tau/2)}{\xi}\right]\left[\cosh\left(\frac{\xi\tau}{2}\right) - i\Delta_+\sinh\left(\frac{\xi\tau}{2}\right)\Big/\xi\right]^{-1} e^{-i\phi} , \tag{A9}$$

We use the following transformations:

$$\frac{\sinh\left(\frac{\xi\tau}{2}\right)}{\xi} \equiv \sinh(r) , \quad \theta_D \equiv \arg\left[\cosh\left(\frac{\xi\tau}{2}\right) + i\Delta_+\frac{\sinh(\xi\tau/2)}{\xi}\right] , \tag{A10}$$

from the transformations (A10) and the relations (A9), we get

$$e^{-\beta} = \cosh(r)e^{-i\theta_D} , \quad \cosh^2 r = \cosh^2\left(\frac{\xi\tau}{2}\right) + \left(\frac{\Delta_+}{\xi}\right)^2\left[\frac{\sinh^2(\xi\tau/2)}{\xi^2}\right] , \quad \alpha = \tanh(r)e^{i(\theta_D-\phi)}, \quad \gamma = -\alpha e^{2i\phi} . \tag{A11}$$

By using the NO form of S_D given by (A5), and by using Eqs. (A1) and (A11) the DSVS in the rotated frame is expanded into the form:

$$|\mathrm{DSVS}\rangle = e^{\alpha a_1^\dagger a_2^\dagger}e^{\beta}|0\rangle = \frac{e^{i\theta_D}}{\cosh(r)}\sum_{n=0}^{\infty}\tanh^n(r)e^{i(\theta_D-\phi)n}|n\rangle_1|n\rangle_2 . \tag{A12}$$

The states $|n\rangle_1|n\rangle_2$ represent states with the same number of photons n in the two modes, which are denoted by the subscripts 1 and 2. The SVS given by (A12) is an eigenstate of the operator $\hat{X}_2$ with a zero eigenvalue. Therefore this state is not changed by multiplying it on the left side with the operator $\exp[(i\tau/2)\Delta_-\hat{X}_2]$. By using the transformation (1.3) from the rotated picture into the SP the DSVS of (A12) is transformed into Eq. (2.6).

2. DESS

In this case the operatoric product $D(\alpha_1)D(\alpha_2)S_D$ is needed to be reorganized in a NO form. The relevant eight algebraic generators of the Lie group are

$$\hat{x}_1 = a_1^\dagger a_1 + a_2^\dagger a_2^\dagger + 1, \quad \hat{x}_2 = a_1 a_2, \quad x_3 = a_1^\dagger a_2^\dagger , \quad \hat{x}_4 = a_1, \quad \hat{x}_5 = a_2, \quad \hat{x}_6 = a_1^\dagger, \quad x_7 = a_2^\dagger, \quad \hat{x}_8 = \hat{I} . \tag{A13}$$

These operators obey the following commutation relations:

$$
\begin{aligned}
&[\hat{x}_1,\hat{x}_2]=-2\hat{x}_2,\quad [\hat{x}_1,\hat{x}_3]=2\hat{x}_3,\quad [\hat{x}_2,\hat{x}_3]=\hat{x}_1\ ,\\
&[\hat{x}_1,\hat{x}_4]=-\hat{x}_4,\quad [\hat{x}_1,\hat{x}_5]=-\hat{x}_5,\quad [\hat{x}_1,\hat{x}_6]=\hat{x}_6\ ,\\
&[\hat{x}_1,\hat{x}_7]=\hat{x}_7,\quad [\hat{x}_2,\hat{x}_6]=\hat{x}_5,\quad [\hat{x}_2,\hat{x}_7]=\hat{x}_4\ ,\\
&[\hat{x}_3,\hat{x}_4]=-\hat{x}_7,\quad [\hat{x}_3,\hat{x}_5]=-\hat{x}_6\ ,\\
&[\hat{x}_4,\hat{x}_6]=[\hat{x}_5,\hat{x}_7]=\hat{x}_8\ .
\end{aligned}
\tag{A14}
$$

We use 6×6 matrix representation for these operators:

$$
\begin{aligned}
&\hat{x}_1\to\mathbf{X}^{(1)}\equiv\{\mathbf{X}^{(1)}_{22}=\mathbf{X}^{(1)}_{33}=-\mathbf{X}^{(1)}_{44}=-\mathbf{X}^{(1)}_{56}=1\}\ ,\\
&\hat{x}_2\to\mathbf{X}^{(2)}\equiv\{\mathbf{X}^{(2)}_{43}=\mathbf{X}^{(2)}_{52}=-1\}\ ,\\
&\hat{x}_3\to\mathbf{X}^{(3)}\equiv\{\mathbf{X}^{(3)}_{25}=\mathbf{X}^{(3)}_{34}=1\}\ ,\\
&\hat{x}_4\to\mathbf{X}^{(4)}\equiv\{\mathbf{X}^{(4)}_{51}=\mathbf{X}^{(4)}_{62}=-1\}\ ,\\
&\hat{x}_5\to\mathbf{X}^{(5)}\equiv\{\mathbf{X}^{(5)}_{41}=\mathbf{X}^{(5)}_{63}=-1\}\ ,\\
&\hat{x}_6\to\mathbf{X}^{(6)}\equiv\{\mathbf{X}^{(6)}_{21}=-\mathbf{X}^{(6)}_{65}=1\}\ ,\\
&\hat{x}_7\to\mathbf{X}^{(7)}\equiv\{\mathbf{X}_{31}=-\mathbf{X}_{64}=1\}\ ,\\
&\hat{x}_8\to\mathbf{X}^{(8)}\equiv\{\mathbf{X}_{61}=-2\}\ .
\end{aligned}
\tag{A15}
$$

We use here a short notation by which we give in the curled brackets the matrix elements that are different from zero (all other matrix elements vanish).

By using the definitions (2.1)–(2.4) and Eqs. (A13), (3.1), and (3.2), we get [10]

$$
\begin{aligned}
|\mathrm{DES}\rangle=&D(\alpha_1)D(\alpha_2)S_D|0\rangle\\
&\times\exp(\alpha_1\hat{x}_6-\alpha_1^*\hat{x}_4)\exp(\alpha_2\hat{x}_7-\alpha_2^*\hat{x}_5)\\
&\times\exp\left[\left[\frac{\tau}{2}\right](i\Delta_+\hat{x}_1-e^{i\phi}\hat{x}_2+e^{-i\phi}\hat{x}_3)\right]|0\rangle\ .
\end{aligned}
\tag{A16}
$$

The NO form of the product $D(\alpha_1)D(\alpha_2)S_D$ can be written as

$$
D(\alpha_1)D(\alpha_2)S_D=e^{A\hat{x}_3}E^{B\hat{x}_6}e^{C\hat{x}_7}e^{D\hat{x}_8}e^{E\hat{x}_1}e^{F\hat{x}_4}e^{G\hat{x}_5}e^{H\hat{x}_2}\ .
\tag{A17}
$$

Our purpose is now to find the relations between the NO parameters A,B,C,D,E,F,G,H and the physical parameter α_1, α_2, Δ_+, ϕ, and τ. By substituting the Lie algebra matrix representations we can write the BCH equation as:

$$
\begin{aligned}
&\exp(\alpha_1\mathbf{X}_6-\alpha_1^*\mathbf{X}_4)\exp(\alpha_2\mathbf{X}_7-\alpha_2^*\mathbf{X}_5)\exp\left[\left[\frac{\tau}{2}\right](i\Delta_+\mathbf{X}_1-e^{i\phi}\mathbf{X}_2+E^{-i\phi}\mathbf{X}_3)\right]\\
&\qquad=\exp(A\mathbf{X}_3)\exp(B\mathbf{X}_6)\exp(C\mathbf{X}_7)\exp(D\mathbf{X}_8)\exp(E\mathbf{X}_1)\exp(F\mathbf{X}_4)\exp(G\mathbf{X}_5)\exp(H\mathbf{X}_2)\ .
\end{aligned}
\tag{A18}
$$

By expanding (A18) and by following a straightforward lengthy algebra we get the matrix identity $\mathbf{P}\equiv\mathbf{Q}$, where $\mathbf{P}$ and Q represent, respectively, the left- and right-hand sides of (A18):

$$
\begin{aligned}
\mathbf{P}\equiv\{&\mathbf{P}_{11}=1;\mathbf{P}_{21}=\alpha_1;\mathbf{P}_{31}=\alpha_2;\mathbf{P}_{41}=\alpha_2^*;\mathbf{P}_{51}=\alpha_1^*\ ;\\
&\mathbf{P}_{22}=\mathbf{P}_{33}=\cosh(r)e^{i\theta_D};\mathbf{P}_{44}=\mathbf{P}_{55}=\cosh(r)e^{-i\theta_D};\mathbf{P}_{24}=\mathbf{P}_{35}=\sinh(r)e^{-i\phi}\ ;\\
&\mathbf{P}_{42}=\mathbf{P}_{53}=\sinh(r)e^{i\phi};\mathbf{P}_{62}=\alpha_1^*\cosh(r)e^{i\theta_D}-\alpha_2\sinh(r)e^{i\phi}\ ;\\
&\mathbf{P}_{63}=-\alpha_1^*\sinh(r)e^{i\phi}+\alpha_2^*\cosh(r)e^{i\theta_D};\mathbf{P}_{64}=\alpha_1^*\sinh(r)e^{-i\phi}-\alpha_2\cosh(r)e^{-i\theta_D}\ ;\\
&\mathbf{P}_{65}=-\alpha_1\cosh(r)e^{-i\theta_D}+\alpha_2^*\sinh(r)e^{-i\phi};\mathbf{P}_{66}=1\}\\
\equiv\mathbf{Q}\equiv\{&\mathbf{Q}_{11}=\mathbf{Q}_{66}=1;\mathbf{Q}_{21}=B-AGe^{-E};\mathbf{Q}_{31}=C-AFe^{-E};\mathbf{Q}_{41}=-Ge^{-E};\mathbf{Q}_{51}=-Fe^{-E};\mathbf{Q}_{22}=\mathbf{Q}_{33}=e^{E}-AHe^{-E}\ ;\\
&\mathbf{Q}_{44}=\mathbf{Q}_{55}=e^{-E};\mathbf{Q}_{24}=\mathbf{Q}_{35}=Ae^{-E};\mathbf{Q}_{42}=\mathbf{Q}_{53}=-He^{-E}\ ;\\
&\mathbf{Q}_{61}=-2D+(GC+BF)e^{-E};\mathbf{Q}_{62}=HCe^{-E}-F;\mathbf{Q}_{63}=HBe^{-E}-G;\mathbf{Q}_{64}=-Ce^{-E};\mathbf{Q}_{65}=-Be^{-E};\}\ .
\end{aligned}
\tag{A19}
$$

In deriving the matrix elements of $\mathbf{P}$ we used the transformation (2.8). We have used here again a short notation by which we give in the curled brackets the matrix elements, which are different from zero (all other matrix elements vanish).

By comparing the matrix elements in $\mathbf{P}$ and $\mathbf{Q}$, we get the relations

$$
\begin{aligned}
&e^{-E}=\cosh(r)e^{-i\theta_D},\quad A=\frac{\sinh(r)e^{-i\phi}}{e^{-E}}=\tanh(r)e^{i(\theta_D-\phi)},\quad H=-Ae^{2i\phi}\ ,\\
&B=\alpha_1-\alpha_2^*\tanh(r)e^{i(\theta_D-\phi)},\quad C=\alpha_2-\alpha_1^*\tanh(r)e^{i(\theta_D-\phi)},\quad D=-\tfrac{1}{2}(|\alpha_1|^2+|\alpha_2|^2)-2\alpha_1^*\alpha_2^*\tanh(r)e^{i(\theta_D-\phi)}\ ,\\
&F=\frac{-\alpha_1^*}{e^{-E}}=\frac{-\alpha_1^*e^{i\theta_D}}{\cosh(r)},\quad G=\frac{-\alpha_2^*}{e^{-E}}=\frac{-\alpha_2^*e^{i\theta_D}}{\cosh(r)}\ .
\end{aligned}
\tag{A20}
$$

By substituting (A17) into (3.1) [with the parameters of (A20)] we get the NO expression for the two-mode DESS as given in (3.4).

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