Propagation of electromagnetic solitary waves in dispersive nonlinear dielectrics

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We have derived the wave equation of electromagnetic field coupling with TO phonons in second-order nonlinear dielectrics. If the loss of the medium can be ignored, theoretical calculation shows that the propagation of electromagnetic pulses without distortion in the dispersive medium is possible due to the dependence of the index of refraction on the electromagnetic field. This electromagnetic field is governed by a Boussinesq equation that has soliton solutions. If the loss of the medium is sufficiently strong that the damping distance is much smaller than the signal width, we find electromagnetic shock-wave solutions. Possible experiments in $LiTaO_3$ are discussed.

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I. INTRODUCTION

The propagation of electromagnetic signals has been studied extensively in many nonlinear systems. Hasegawa and Tappert [1,2] first predicted the optical envelope soliton in glass fibers with dispersion and nonlinearity, and Mollenauer, Stolen, and Gordon demonstrated this later in their experiment [3]. This leads to the possibility of tens of gigabits per second data transmission rates for optical communication. The electromagnetic shock wave and soliton in nonlinear transmission lines were studied in both theory [4,5] and experiment [6]. Recently, due to the rapid development in the generation and detection of terahertz electromagnetic signals [7–9], the study of the ultrashort- and ultrahigh-amplitude electromagnetic signal propagating in nonlinear dispersive dielectrics becomes possible in experiments. In this paper, we discuss the electromagnetic signal coupling with TO phonons (polariton) in a nonlinear medium [10].

If the crystal medium has a second-order nonlinearity and the electromagnetic field is coupled to the TO phonon of the crystals, the electric displacement $D(\omega)$ is given by

$$D(\omega)=\epsilon(\omega)E(\omega)+\int d^{(\omega)}E(\omega-\omega')E(\omega')d\omega' , \qquad (1.1)$$

where

$$\epsilon(\omega)=\epsilon(\infty)+[\epsilon(0)-\epsilon(\infty)]\frac{\omega_{TO}^2}{\omega_{TO}^2-\omega^2-i\omega g} . \qquad (1.2)$$

Here $\epsilon(0)$ is the dc dielectric constant, $\epsilon(\infty)$ is the optical dielectric constant, ω_{TO} is the transverse optic lattice resonant frequency, g is the damping rate, and $d^{(\omega)}$ is the nonlinear susceptibility.

If $\omega \ll \omega_{TO}$, the dielectric constant of Eq. (1.2) can be written as

$$\epsilon(\omega)=\epsilon(\infty)+[\epsilon(0)-\epsilon(\infty)]\left[1+\frac{\omega^2}{\omega_{TO}^2}+i\frac{\omega g}{\omega_{TO}^2}\right] . \qquad (1.3)$$

By taking the inverse Fourier transformation and assuming $d^{(\omega)}$ is independent of ω ($d^{(\omega)}$ should depend on ω according to Miller's rule, but if the nonlinear term is much smaller than the linear term and $\omega \ll \omega_{TO}$, this is a good approximation), Eqs. (1.1) and (1.3) give D in the time domain:

$$D(t)=\int D(\omega)e^{i\omega t}dt$$
$$=\epsilon(0)E(t)-\frac{S}{\omega_{TO}^2}\frac{\partial^2 E}{\partial t^2}-\frac{Sg}{\omega_{TO}^2}\frac{\partial E}{\partial t}+dE^2 ,$$

where $S=\epsilon(0)-\epsilon(\infty)$ and $d=d^{(\omega)}$.

Putting the $D(t)$ into Maxwell's wave equation, we have

$$\frac{\partial^2 E}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\epsilon(0)E-\frac{S}{\omega_{TO}^2}\left[\frac{\partial^2 E}{\partial t^2}+g\frac{\partial E}{\partial t}\right]+d(E^2)\right] , \qquad (1.4)$$

where $c'^2=c^2/\epsilon(0)$, $S=\epsilon(0)-\epsilon(\infty)$.

Equation (1.4) is the basic equation of this paper because it describes the electromagnetic field in this type of medium. In the right side of Eq. (1.4), the second term is the dispersion term, the third term is the dissipation term, and the last one is the nonlinear term. Since this equation is a nonlinear dispersive wave equation with damping, it is expected to have solitary-wave solutions if the damping term can be ignored or shock-wave solutions if the dispersion term can be ignored.

II. SOLITON SOLUTIONS IN THE ZERO-DAMPING CASE

First we consider the zero-damping case. So Eq. (1.4) becomes

$$\frac{\partial^2 E}{\partial x^2}=\frac{1}{c'^2}\frac{\partial^2 E}{\partial t^2}-\frac{S}{\omega_{\mathrm{TO}}^2 c^2}\frac{\partial^4 E}{\partial t^4}+\frac{d}{c^2}\frac{\partial^2}{\partial t^2}(E^2)\ . \tag{2.1}$$

Equation (2.1) is almost the same as the regular Boussinesq equation

$$u_{xx}+u_{xxxx}+6(u^2)_{xx}=u_{tt} \tag{2.2}$$

which describes a shallow-water wave propagating in both directions [11] and a one-dimensional nonlinear lattice [12]. But there are two differences between them: (i) In (2.1), the nonlinear and dispersion terms are in the form of time derivatives instead of space derivatives for (2.2). (ii) The signs of the nonlinear and dispersion terms are opposite and this leads to (2.1) being a well-posed equation for linear dispersion; so we call (2.1) a well-posed Boussinesq equation. We want to emphasize that the well-posed Boussinesq equation rarely arise in physical problems and actually this is the first case we are aware of. In Appendix A, we give the N-soliton solution of Eq. (2.1) and the inverse scattering transformation for the initial-condition problem of (2.1). In the following, we will discuss the electromagnetic soliton governed by (2.1).

A. Basic properties of a single electromagnetic soliton

Assume Eq. (2.1) has the traveling-wave solution $E(x-vt)$, where v is the velocity of the traveling wave. Let $\xi=x-vt$, and introduce the following parameters:

$$\alpha=-\frac{2\omega_{\mathrm{TO}}^2 d}{v^2 S},\quad \beta=\frac{v^2}{c'^2}-1,\quad u=\frac{\beta c^2\omega_{\mathrm{TO}}^2}{v^4 S}\ . \tag{2.3}$$

By assuming the boundary condition $E=E_\xi=0$ at $\xi=\pm\infty$, which is due to the solitary-wave property, we find the solitary-wave solution:

$$E=\frac{3u}{\alpha}\mathrm{sech}^2\left[\frac{\sqrt{u}}{2}\xi\right]\ . \tag{2.4}$$

More detailed study of Eq. (2.1) in Appendix A shows that (2.1) has an N-soliton solution and (2.4) is the one-soliton solution. Figure 1(a) schematically shows this single soliton with the normalized time and amplitude. From (2.4), we obtain some basic information about the electric soliton.

(i) The velocity of soliton v. By the definition $E=E(x-vt)$, v is the velocity of the soliton. From the solution (2.4), $u>0$ must be satisfied in order to have a finite solution. But

$$u=\frac{\beta c^2\omega_{\mathrm{TO}}^2}{v^4 S}=\frac{\left[\frac{v^2}{c'^2}-1\right]c^2\omega_{\mathrm{TO}}^2}{v^4 S}\ .$$

This leads to the requirement for the soliton velocity to be superphotonic:

$$v>c'=\frac{c}{\sqrt{\epsilon(0)}}\ . \tag{2.5}$$

(ii) Amplitude of soliton E_0. From the solution (2.4), it is easy to see that

$$E_0=\frac{3u}{\alpha}=-\frac{3\left[\frac{v^2}{c'^2}-1\right]c^2}{2v^2 d}\ . \tag{2.6}$$

(iii) Width of the soliton.

$$\Delta=\frac{2}{\sqrt{u}}=\frac{2v^2 S^{1/2}}{\left[\frac{v^2}{c'^2}-1\right]^{1/2} c\,\omega_{\mathrm{TO}}}\ . \tag{2.7}$$

Now the solution (2.4) can be reduced to

$$E=E_0\mathrm{sech}^2\left[\frac{1}{\Delta}(x-vt)\right]\ , \tag{2.8}$$

which has the same structure as a Korteweg–de Vries (KdV) soliton [13].

Equations (2.5)–(2.7) give us the relation among the velocity, amplitude, and width of the soliton. By using these relations, we can express the velocity and width of

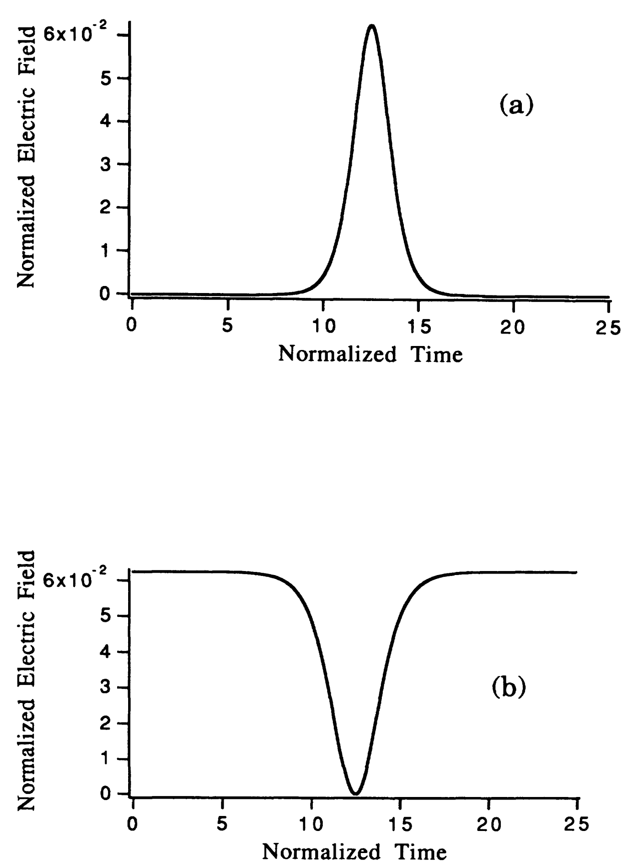

FIG. 1. Single bright (a) and dark (b) solitons with normalized field and time: $e_0=-6\epsilon(0)/d$, $\tau=[S/\epsilon(0)\omega_{\mathrm{TO}}^2]^{1/2}$. Note that the duration of the dark soliton is larger than that of the bright soliton.

the soliton by the amplitude of the soliton:

$$v=\frac{c'}{\left[1+\frac{2dE_0}{3\epsilon(0)}\right]^{1/2}}>c' ,$$

$$\Delta=\left[\frac{6S}{\omega_{\rm TO}^2(-dE_0)}\right]^{1/2}\frac{c'}{\left[1+\frac{2dE_0}{3\epsilon(0)}\right]^{1/2}} . \qquad (2.9)$$

v is velocity of the soliton, Δ is the width of the soliton in space. The pulse duration in time is given by

$$\tau=\frac{\Delta}{v}=\left[\frac{6S}{\omega_{\rm TO}^2(-dE_0)}\right]^{1/2} . \qquad (2.10)$$

Figure 2 shows the amplitude dependence of the speed and the duration of this electromagnetic soliton. From (2.9) and (2.10), we draw the following conclusions.

(i) For the electromagnetic soliton, there is only one independent parameter. For example, if the amplitude of the soliton is given, the speed (not the direction), the width, and the shape of the soliton can be determined. The larger the field E_0, the shorter the soliton pulse and the faster the soliton propagates.

(ii) $dE_0<0$ must be satisfied to get the finite soliton solution, so this soliton is negatively polarized and the soliton travels faster than the speed of the low-frequency electromagnetic wave in the medium. On the other hand, the speed of the soliton should be slower than the speed of light in a vacuum. This set a limit for the amplitude of the soliton:

$$E<E_m=\frac{3(\epsilon-1)}{2|d|} .$$

Actually when E_0 is close to E_M, the condition for our model $\omega \ll \omega_0$ is no longer valid according to (2.10).

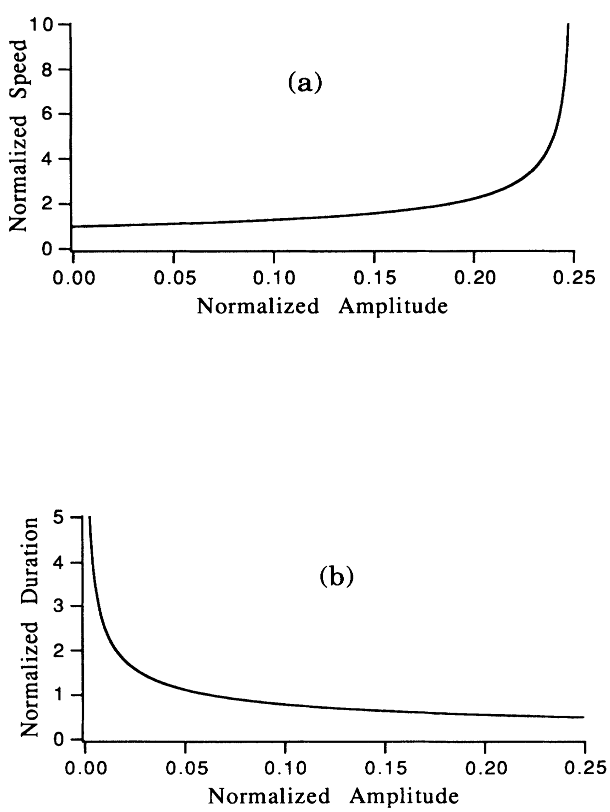

FIG. 2. The dependence of (a) the normalized speed (v/c'), (b) the normalized time duration τ on the field amplitude of the single bright soliton.

B. Colliding solitons

One of the most important features of solitons is that the solitons preserve their shapes and velocities after collisions. In Appendix A, we construct N-soliton solutions for Eq. (2.1) and give the inverse scattering transformation for the initial-condition problem. By solving the initial-condition problem, one can find the asymptotic behavior of the initial pulse—the initial pulse evolves to several solitons. In the following we study the colliding behavior of solitons by using the N-soliton solution. To simplify, we choose $N=2$. We can see that the two-soliton collision gives almost all features of soliton collisions.

From Appendix A, let $N=2$. We find the two-soliton solution for Eq. (2.1):

$$E=e_0 u , \qquad (2.11)$$

where

$$u=\frac{\partial^2}{\partial x^2}\ln f ,$$

$$f=1+e^{\eta_1}+e^{\eta_2}+Ae^{\eta_1+\eta_2} ,$$

$$\eta_i=a_i\frac{t}{\tau}-\omega_i\frac{x}{L}+\delta_i ,$$

$$\omega_i^2=a_i^2-a_i^4 ,$$

$$A=1+\frac{12a_1^2a_2^2}{(a_1+a_2)^2-(a_1+a_2)^4-(\omega_1+\omega_2)^2} , \qquad (i=1,2)$$

where

$$e_0=-\frac{6\epsilon(0)}{d},\quad \tau\left[\frac{S}{\epsilon(0)\omega_{\rm TO}^2}\right]^{1/2},\quad L=c'\tau .$$

After a simple calculation, we find the asymptotic behavior of the solution (2.11):

$$u=\begin{cases}\frac{a_1^2}{4}\mathrm{sech}^2(\eta_1+\Delta_1^-)+\frac{a_2^2}{4}\mathrm{sech}^2(\eta_2+\Delta_2^-), & t\to-\infty\\ \frac{a_1^2}{4}\mathrm{sech}^2(\eta_1+\Delta_1^+)+\frac{a_2^2}{4}\mathrm{sech}^2(\eta_2+\Delta_2^+), & t\to+\infty .\end{cases}$$

It is easy to see this solution corresponds to the case of two solitary pulses colliding. After the two pulses pass through each other, the pulses keep the same shape and velocity except for a phase shift. And $a_iL/\omega_i\tau$ gives the

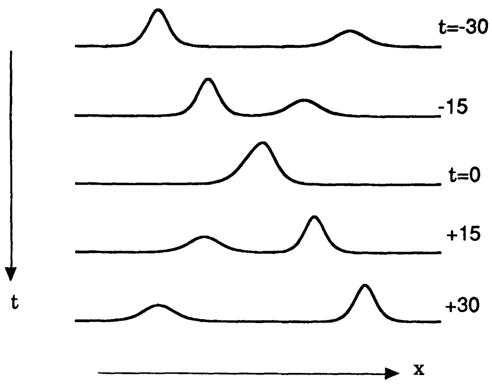

FIG. 3. The collision of the two solitons traveling in opposite directions ($a_1=0.5$, $a_2=0.3$).

velocity of the soliton i. Usually, we assume $a_i>0$ and the sign of ω_i determines the direction of the soliton propagation. In Figs. 3 and 4, we show the two-soliton collision in the case of the same and opposite propagation directions. When $\eta_1\sim 0$, $\eta_2\sim\infty$, we find E obtained from (2.11) is the same as solution (2.4). So the electromagnetic pulse governed by (2.4) is an electromagnetic soliton.

An important conclusion from solution (2.11) is that the two solitons experience a phase shift after the collision. Suppose two solitons collide at time $t\sim 0$, then $t=-\infty$ corresponds to the time long before the collision and $t=+\infty$ corresponds to the time long after the collision.

For soliton 1,

$$\Delta_1^+-\Delta_1^-=\ln A \ .$$

For soliton 2,

$$\Delta_2^+-\Delta_2^-=-\ln A \ .$$

The total shift is conserved during the collision, i.e., $\Delta_1+\Delta_2=\text{const}$. We also can see that the amplitude of soliton i is $a_i^2/4$. From $\omega_i^2=a_i^2-a_i^4$, it is easy to see $a_i<1$ which is the limit of the amplitude of the soliton. This limit of the amplitude prevents infinite propagation speed.

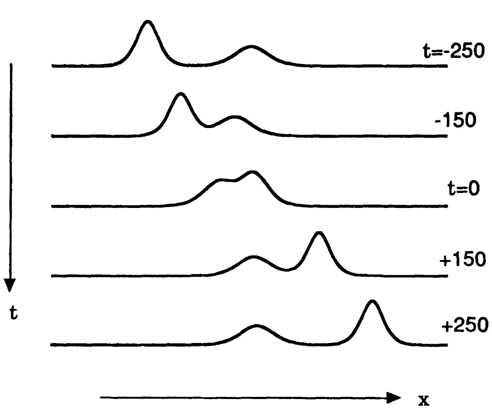

FIG. 4. The collision of the two solitons traveling in the same direction ($a_1=0.5$, $a_2=0.3$).

In order to have finite soliton solutions and finite phase shifts, $A>0$ must be satisfied. It is interesting to note that $A<0$ gives rational solutions but $A=0$ gives a finite solution which is the one-soliton solution.

C. Dark-electric soliton

In glass fibers, the optical pulse is governed by the nonlinear Schrödinger equation which has the regular-soliton (bright-soliton) solution and a dark-soliton solution [1,2]. Recently the dark soliton was generated and studied in optical fibers [14,15]. Similarly, it is interesting to study the dark-soliton solution of Eq. (2.1). For a regular soliton, the boundary condition is

$$E_\xi(\pm\infty)=0,\quad E(\pm\infty)=0\ , \tag{2.12}$$

and E reaches its maximum value at the center of the soliton. Since the intensity $I\propto E^2$, the background of this soliton is "dark" (no field) and the center is "bright" (high field). Another family of solutions can be defined as a soliton solution with the boundary condition

$$E_\xi(\pm\infty)=0,\quad E(\pm\infty)=E_{\text{dc}}\neq 0\ , \tag{2.13}$$

where E_{dc} is a constant. We further assume that the field is zero at the center of the soliton: $E(0)=0$. Compared with the bright soliton, this soliton has "bright" (nonzero) background and "dark" (zero) center, so we call it a dark soliton. It is easy to verify the existence of dark solitons from Eq. (2.1).

Let $E_d=E+E_{\text{dc}}$ represent the traveling-wave solution of (2.1) and put E_d into Eq. (2.1), then we get

$$\frac{\partial^2E}{\partial x^2}=\left[\frac{1}{c'^2}+\frac{2dE_{\text{dc}}}{c'^2}\right]\frac{\partial^2E}{\partial t^2}-\frac{S}{\omega_{\text{TO}}^2c^2}\frac{\partial^4E}{\partial t^4}+\frac{d}{c^2}\frac{\partial^2}{\partial t^2}(E^2)\ , \tag{2.14}$$

which is still a well-posed Boussinesq equation with the boundary condition (2.12). By using the same technique in Sec. II A, we find the traveling-wave solution for (2.14) satisfies the following equation:

$$E_\xi(\alpha E_{\text{dc}}+\alpha E-u)+E_{\xi\xi\xi}=0$$

with boundary conditions

$$E_\xi(\pm\infty)=0,\quad E(\pm\infty)=0\ .$$

By defining $u'=u-\alpha E_{\text{dc}}$, we get

$$E_\xi(\alpha E-u')+E_{\xi\xi\xi}=0\ ,$$

with the same boundary condition as the regular soliton. Its solution is

$$E=\frac{3u'}{\alpha}\text{sech}^2\left[\frac{\sqrt{u'}}{2}\xi\right]\ ,$$

which leads to

$$E_d=E_{\text{dc}}+\frac{3u'}{\alpha}\text{sech}^2\left[\frac{\sqrt{u'}}{2}\xi\right]\ . \tag{2.15}$$

If we want the center of the soliton to have zero field, i.e., $E_d(0)=0$, then (2.15) becomes

$$E_d=\frac{3u'}{2\alpha}\left[1-\operatorname{sech}^2\left[\frac{\sqrt{u'}}{2}\xi\right]\right] . \tag{2.16}$$

We call this solution a single dark soliton and Fig. 1(b) schematically shows this dark single soliton with the normalized time and amplitude. It has the following properties.

(i) Pulse width of the dark soliton.

$$\Delta_d=\frac{2}{\sqrt{u'}}=\left[-\frac{8}{u}\right]^{1/2}=\frac{2v^2(2S)^{1/2}}{\left[1-\frac{v^2}{c'^2}\right]^{1/2}c\omega_{\text{TO}}} . \tag{2.17}$$

(ii) Velocity of the dark soliton.

Since Δ_d should be a real number, $v<c'$ must be satisfied according to Eq. (2.16). This means the dark soliton will move slower than the regular soliton.

(iii) Amplitude of the dark soliton. At $\xi=\pm\infty$,

$$E_{d0}=E_{\text{dc}}=\frac{3u}{2\alpha}=\frac{3\left[1-\frac{v^2}{c'^2}\right]c^2}{4v^2d} . \tag{2.18}$$

At $\xi=0$, $E_d=0$. This is the dark point. For $v<c'$, $dE_{d0}>0$, which means the dark soliton is positively polarized (assuming $d>0$). Now

$$E_d=E_{d0}\left[1-\operatorname{sech}^2\left[\frac{1}{\Delta_d}(x-vt)\right]\right] .$$

Similar to the regular-soliton case, we can express the velocity and width of the dark soliton as a function of the soliton amplitude:

$$v=\frac{c'}{\left[1+\frac{4dE_{d0}}{3\epsilon(0)}\right]^{1/2}}<c' ,$$
$$\Delta_d=\left[\frac{12S}{\omega_{\text{TO}}^2(dE_{d0})}\right]^{1/2}\frac{c'}{\left[1+\frac{4dE_{d0}}{3\epsilon(0)}\right]^{1/2}} . \tag{2.19}$$

And the pulse duration in time is

$$\tau=\frac{\Delta}{v}=\left[\frac{12S}{\omega_{\text{TO}}^2(dE_0)}\right]^{1/2} . \tag{2.20}$$

It is interesting to compare the dark soliton with the regular bright soliton for the sample amplitude, i.e., $E_0=E_{d0}$. By comparing (2.19) and (2.20) with (2.9) and (2.10), we can draw the following conclusions.

(i) The dark soliton propagates with a subphotonic velocity $v<c'$, but the bright soliton has a superphotonic velocity $v>c'$. So the dark soliton moves slower than the bright soliton.

(ii) For the same amplitude, the dark soliton is $\sqrt{2}$ times wider (in time) than the bright soliton.

(iii) The dark soliton is positively polarized, but the bright soliton is negatively polarized.

III. STRONG DISSIPATION: SHOCK WAVES

If the dissipation of the medium is very strong, we have to keep the damping term in Eq. (1.4). There, the third term on the right is the damping term, which is an additional term when comparing to Eq. (2.1). For a localized traveling wave, $E_t/E_{tt}\sim\tau$, where τ is the duration of the electromagnetic field. So, if $\tau\gg 1/g$, we can drop the dispersive term E_{tt} and Eq. (1.4) reduces to

$$\frac{\partial^2E}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\epsilon(0)E-\frac{Sg}{\omega_{\text{TO}}^2}\frac{\partial E}{\partial t}+d(E^2)\right] . \tag{3.1}$$

By assuming a traveling-wave solution $E=E(\xi)=E(x-vt)$, the equation (3.1) becomes

$$E_{\xi\xi}=\frac{v^2}{c^2}\left[\epsilon(0)E-\frac{SgvE_\xi}{\omega_{\text{TO}}^2}+dE^2\right]_{\xi\xi} . \tag{3.2}$$

By assuming

$$\alpha=-\frac{2\omega_{\text{TO}}^2d}{vS} , \quad \beta=1-\frac{v^2}{c'^2} , \quad u=\frac{\beta c^2\omega_{\text{TO}}^2}{v^3S} ,$$

and the localized condition

$$E_\xi(\pm\infty)=0, \quad E(+\infty)=0 , \tag{3.3}$$

Eq. (3.2) leads to

$$E_\xi+\frac{\alpha}{2g}E^2+\frac{u}{g}E=0 . \tag{3.4}$$

The solution for (3.4) is

$$E=-\frac{u}{\alpha}\left[1-\tanh\left[\frac{u}{2g}\right]\xi\right] . \tag{3.5}$$

This has a Taylor shock-wave profile. $E(+\infty)=0$ requires $u/2g>0$, which means $v<c'$. And it also gives $E(-\infty)=-2u/\alpha$. We can obtain the following properties of the shock wave.

(i) Width of the shock wave.

$$\Delta=\frac{2g}{u}=\frac{2gSv^3}{\left[1-\frac{v^2}{c'^2}\right]c^2\omega_{\text{TO}}^2} .$$

(ii) Amplitude of the shock wave.

$$E_0=-\frac{2u}{\alpha}=\frac{\left[1-\frac{v^2}{c'^2}\right]c^2}{v^2d} .$$

We can express the velocity and the width of the shock wave in terms of the amplitude of the shock wave:

$$v=\frac{c}{\sqrt{\epsilon(0)+dE_0}} , \quad \Delta=\frac{2cgS}{\omega_{\text{TO}}^2(dE_0)\sqrt{\epsilon(0)+dE_0}} ,$$
$$\tau=\frac{\Delta}{v}=\frac{2gS}{\omega_{\text{TO}}^2(dE_0)} . \tag{3.6}$$

The shock wave can be rewritten in the following form:

$$E=\frac{E_0}{2}\left[1-\tanh\frac{1}{\Delta}(x-vt)\right] .$$

This shock wave travels with a subphotonic velocity slower than the speed of light in the medium and positive polarity ($dE_0>0$).

Another type of shock-wave solution can be found by assuming the boundary condition

$$E_\xi(\pm\infty)=0,\quad E(-\infty)=0 , \tag{3.7}$$

instead of (3.3). The shock-wave solution with this boundary condition is

$$E=\frac{u}{\alpha}\left[1+\tanh\left[\frac{u}{2g}\right]\xi\right] , \tag{3.8}$$

where

$$\alpha=-\frac{2\omega_{\rm TO}^2 d}{vS},\quad \beta=\frac{v^2}{c'^2}-1>0,\quad u=\frac{\beta c^2\omega_{\rm TO}^2}{v^3 S} . \tag{3.9}$$

This shock wave is negative polarized ($dE_0<0$) since the amplitude is given by

$$E_0=E(+\infty)=\frac{2u}{\alpha} .$$

And the relations among the pulse width, propagation velocity (superphotonic), and the amplitude are

$$v=\frac{c}{\sqrt{\epsilon(0)+dE_0}}>c',\quad \Delta=\frac{2cgS}{\omega_{\rm TO}^2(-dE_0)[\epsilon(0)+dE_0]} ,$$
$$\tau=\frac{\Delta}{v}=\frac{2gS}{\omega_{\rm TO}^2(-dE_0)} . \tag{3.10}$$

It is interesting to solve the initial-condition problem for Eq. (3.1). Under the slowly varying approximation, Eq. (3.1) can be reduced to Burgers's equation (see Appendix B) which can be solved analytically for the initial-condition problem. To simplify, we study the evolution of a shock wave whose initial waveform is a Taylor profile with a wider pulse duration than (3.5). We find that the shock wave will be sharpened and the sharpening effect will be stronger if the amplitude of the shock wave is larger.

IV. SOME EXPERIMENTAL CONSIDERATIONS FOR ELECTROMAGNETIC SOLITARY WAVES IN NONLINEAR CRYSTALS ($LiTaO_3$)

From the discussion in Secs. II and III, the media with second-order nonlinearity can be used for the propagation of electromagnetic solitons (if the loss of the media is small) or electromagnetic shock waves (if the loss of the media is large). However, it is easier to obtain a strong nonlinear effect if we choose the media with large nonlinearity due to the difficulty of obtaining a long propagation distance in the experiment and generating an ultrahigh electromagnetic field in a very short time period. For $LiTaO_3$, the second-order nonlinear coefficient d in the microwave region is quite large compared to other crystals, which is the main reason to choose it as a first experiment candidate. Before we put numbers of $LiTaO_3$ into the solutions which we derived in the previous sections, it is important to discuss some general features of the nonlinear dispersive wave equation. The nonlinear term in the wave equation will sharpen the pulse, and the dispersive term will broaden it. When the nonlinear and dispersive effects are balanced, the pulse will keep its shape, which is the property of the solitary wave.

If we consider only the nonlinear term, Eq. (2.1) becomes

$$\frac{\partial^2 E}{\partial x^2}=\frac{1}{c'^2}\frac{\partial^2}{\partial t^2}\left[E+\frac{d}{\epsilon(0)}(E^2)\right] .$$

We can rewrite the above equation as

$$\frac{\partial^2 E}{\partial x^2}=\frac{\partial^2}{\partial t^2}\left[\frac{E}{c^2(E)}\right] ,$$

where

$$c^2(E)=\frac{c'^2}{1+\dfrac{dE}{\epsilon(0)}} .$$

If $dE \ll \epsilon(0)$, then

$$c(E)\cong c'\left[1-\frac{dE}{2\epsilon(0)}\right] . \tag{4.1}$$

We could regard $c(E)$ as the field-dependent velocity of propagation. If we have an electromagnetic pulse, the center part of the pulse with $E_0\gg 0$ will move faster (because $dE_0<0$) than the wings due to the nonlinearity (4.1). This nonlinearity becomes important when the center of pulse catches up with the front edge of the pulse. More precisely, the center moves a longer distance Δ than the edge, where Δ is the width of the pulse. Suppose this process takes a time T, then we have

$$\Delta=[c(E_0)-c(0)]T ,$$

$$T=\frac{\Delta}{c(E_0)-E(0)}=\frac{\Delta}{\dfrac{-dE_0}{2\epsilon(0)}c'} .$$

At the same time, the pulse travels a distance L, and $T\sim L/c(0)=L/c'$. Now we have

$$\frac{\Delta}{\dfrac{-dE_0}{2\epsilon(0)}c'}=\frac{L}{c'}\quad\text{or}\quad L=\frac{\Delta}{\dfrac{-dE_0}{2\epsilon(0)}} . \tag{4.2}$$

When dE_0 is small, $L\sim(dE_0)^{-3/2}$ or $L\sim(\Delta)^3$ because Δ depends on dE_0 [see Eq. (2.9)]. The physical meaning of L is that the pulse experiences a strong nonlinearity after it travels a distance L. When the electromagnetic field is small, L will be very long. In an experiment, L is limited by the size of the sample and the dissipation distance of the material, so it is desirable to use an electromagnetic field as high as possible for the experiment. For most materials, the dissipation length is much shorter than L. For this strong dissipation situation, we will expect a shock wave. But if the dissipation distance is much longer than the nonlinear distance L, we should obtain the soliton.

We use the following parameters for $LiTaO_3$ [16,17]:

$$d=-3.2\times10^{-8}\ \text{m/V}\quad (d_{333})\ ,$$

$$\epsilon(0)=43,\quad \epsilon(\infty)=20,\quad S=23\ ,$$

$$\omega_{TO}=4.0\times10^{13}\ \text{s}^{-1}\ ,$$

$$g=8.16\times10^{12}\ \text{s}^{-1}\ .$$

Let us consider an electromagnetic pulse of 1 ps duration, i.e., $\tau=10^{-12}$ s. With (2.10)

$$\tau=\frac{\Delta}{v}=\left[\frac{6S}{\omega_{TO}^2(-dE_0)}\right]^{1/2}=10^{-12}$$

we find the electric field to be $E_0=27$ kV/cm. From (4.2), the nonlinear characteristic length is

$$L=\frac{\Delta}{\frac{-dE_0}{2\epsilon(0)}}=4.3\times10^{-2}\ \text{m}=4.3\ \text{cm}\ .$$

For a 1-ps pulse, the dissipation length $L_\alpha=1$ mm [17], which is 43 times shorter than nonlinear distance L. So there is no significant nonlinear effect but very strong damping for a 1-ps pulse in a $LiTaO_3$ crystal. Because the soliton exists when there is no significant damping but strong nonlinearity, a 1-ps soliton cannot exist in $LiTaO_3$.

To understand the interplay of the nonlinearity, the dissipation, and the solitary-wave amplitude in a nonlinear crystal, we introduce a parameter L_α/L, which is the ratio of the damping length to the nonlinear characteristic distance of the material. When $L_\alpha/L\gg1$, the soliton solution is a good approximation due to the weak damping; when $L_\alpha/L<1$, the soliton is impossible to propagate while the shock wave is possible due to strong damping. Figure 5 shows L_α/L as a function of the amplitude of the solitary wave for some nonlinear media. From this figure, we found the basic limitation to the observation of an electromagnetic soliton is the huge dissipation and weak nonlinearity in the $LiTaO_3$ crystal. If we can find some material with two orders of magnitude larger nonlinearity or two orders of magnitude less loss than $LiTaO_3$, the electromagnetic soliton should be observed in experiment. On the other hand, we know that the shock-wave solution can be found for the material with large loss, which fits the case of the $LiTaO_3$ crystal. If we choose

$$\tau=1\ \text{ps}=10^{-12}\ \text{s}\ ,$$

$$g=8.16\times10^{12}\ (\text{s}^{-1}),\quad 1/g=1.23\times10^{-13}\ ,$$

we find $\tau\gg1/g$ which is the shock-wave condition in Sec. III. The shock-wave amplitude is given by Eq. (3.6):

$$E_0=\frac{2gS}{\omega_{TO}^2(d\tau)}=64\ \text{kV/cm}\quad \text{for } LiTaO_3\ .$$

For this field, the nonlinear distance L is about 1.8 cm.

The $\tau=10$ ps case also satisfies the shock-wave condition $\tau\gg1/g$ for $LiTaO_3$. This requires $E_0=6.4$ kV/cm, but the nonlinear effect can be seen after 18 cm propagation which is difficult in experiments. In Fig. 6, we show the nonlinear sharpening effect for a 80-kV/cm shock wave with 5 ps initial rise time after 2 cm traveling in $LiTaO_3$. The nonlinear sharpening effect is more obvious

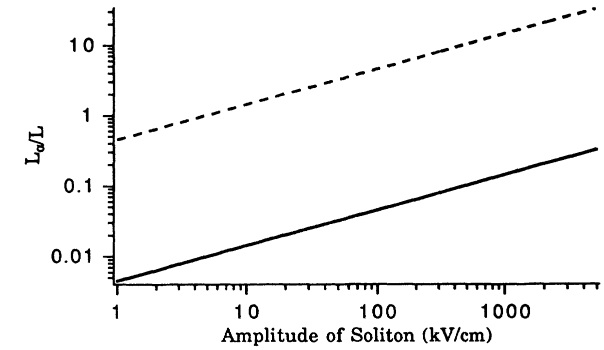

FIG. 5. The ratio L_α/L as a function of the solitary-wave amplitude. When $L_\alpha/L\gg1$, it is possible to observe the electromagnetic soliton; when $L_\alpha/L\ll1$, it is possible to observe the electromagnetic shock wave. The solid line is for the $LiTaO_3$ crystal and the dashed line is for a material with two orders of magnitude larger nonlinearity or two orders of magnitude less loss than $LiTaO_3$.

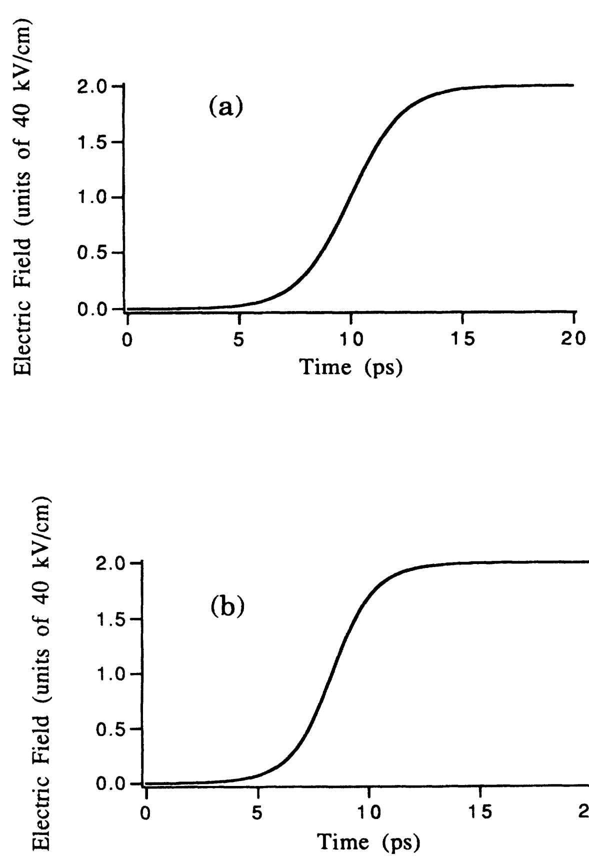

FIG. 6. The sharpening effect for the 80-kV/cm shock wave in $LiTaO_3$. (a) The initial shock wave with 5 ps duration at $x=0$. (b) The final shock wave at $x=20$ mm is sharpened to 3 ps.

for a higher field shock wave.

Another important consideration for the observation of these solitary waves is that the pulse shapes of electromagnetic transients generated experimentally do not match the soliton or shock wave perfectly. However, it can be demonstrated by solving the initial problem (see Appendixes A and B) that the initial pulse which does not match the solitary-wave shape will change its shape and finally match the solitary wave if the pulse propagates a long enough distance. As to electromagnetic amplitudes of electromagnetic transients, over 100 kV/cm electromagnetic pulses with a few picoseconds transient time have been achieved [9] due to the rapid progress in switching technique. So the basic requirements for the generation of an electromagnetic solitary wave are available now and the experimental demonstration of the electromagnetic solitary waves should be in the near future.

APPENDIX A: N-SOLITON SOLUTION OF EQ. (2.1) AND THE INVERSE SCATTERING TRANSFORMATION

We have Eq. (2.1) for an electromagnetic field in nonlinear dispersive material without dissipation:

$$\frac{\partial^2 E}{\partial x^2}=\frac{1}{c'^2}\frac{\partial^2 E}{\partial t^2}-\frac{S}{\omega_{\mathrm{TO}}^2 c^2}\frac{\partial^4 E}{\partial t^4}+\frac{d}{c^2}\frac{\partial^2}{\partial t^2}(E^2)\ .$$

This equation can be rewritten as

$$E_{tt}-\alpha E_{tttt}-\beta(E^2)_{tt}=c'^2 E_{xx}\ , \tag{A1}$$

where

$$\alpha=\frac{S}{\epsilon(0)\omega_{\mathrm{TO}}^2}>0,\quad \beta=-\frac{d}{\epsilon(0)}>0\ .$$

For convenience, we want to normalize E, x, and t so that Eq. (A1) becomes a dimensionless equation:

$$E\to u=\frac{E}{e_0},\quad x\to t'=\frac{t}{\tau},\quad t\to x'=\frac{x}{L}\ ,$$

where

$$e_0=\frac{6}{\beta}=-\frac{6\epsilon(0)}{d},\quad \tau=\sqrt{\alpha},\quad L=c'\tau\ .$$

Now (A1) becomes

$$u_{x'x'}-u_{x'x'x'x'}-6(u^2)_{x'x'}=u_{t't'}\ .$$

To simplify the notation, we drop the prime in the above equation. Finally we reach the normalized equation for Eq. (2.1):

$$u_{xx}-u_{xxxx}-6(u^2)_{xx}=u_{tt}\ . \tag{A2}$$

Equation (A2) is almost the same as a Boussinesq equation (2.2) except for the different sign for the second and third terms of the left side. The linear dispersion relation of (A2) is $\omega^2=k^2+k^4$ where k and ω are real and so (A2) is called a well-posed Boussinesq equation.

On the other hand, the Boussinesq equation (2.2)

$$u_{xx}+u_{xxxx}+6(u^2)_{xx}=u_{tt}$$

has the linear dispersion relation $\omega^2=k^2-k^4$. When $k<1$, $\mathrm{Im}(\omega)>0$, which gives instability and (2.2) is called an ill-posed Boussinesq equation. For most physical situations, we obtain (2.2) instead of (A2) even though it is an ill-posed equation; usually they are prohibited by the asymptotic derivation of the long-wavelength limit. And now (A2) is a well-posed equation, so we do not have to worry about the long-wavelength limit.

A very important technique to analyze Eq. (A2) is the inverse scattering transformation (IST) which is a nonlinear Fourier analysis to solve the initial-condition problem for some nonlinear equation. The IST technique was first developed by Gardner, Greene, Kruskal, and Miura [18] to solve the initial-value problem for the KdV equation. This method was soon expressed in general form by Lax [19]. Zakharov, Ablowitz, and Haberman extended IST theory to multidimensions and found the IST for the ill-posed Boussinesq equation [20]. One can easily find that the IST for the well-posed Boussinesq equation is the same as the ill-posed one if we choose $\beta^2=-1$ here instead of $\beta^2=+1$ in the paper [20].

IST is a very powerful mathematical method to analyze nonlinear differential equations. But in the practical situation of studying Eq. (A2), IST is too complicated a method to find some important features of the solution (like the soliton). Actually there is a class of special solutions for these nonlinear differential equations: the N-soliton solution, which gives a convenient way to study the soliton properties. For the Boussinesq equation, the N-soliton solution was found by Hirota [21]. By a similar method, we find the N-soliton solution for this well-posed Boussinesq equation (A2):

$$u(x,t)=\frac{\partial^2}{\partial x^2}\ln f(x,t)\ , \tag{A3}$$

where

$$f(x,t)=\sum_{\mu=0,1}\exp\left[\sum_{\substack{i,j\\ i=1\\ i<j}}^{N}\varphi(i,j)\mu_i\mu_j+\sum_{i=1}^{N}\mu_i\eta_i\right]\ ,$$

$$\eta_i=P_i x-\epsilon_i\Omega_i t-\eta_i^0,\quad \epsilon_i=\pm 1$$

$$\Omega_i=P_i(1-P_i^2)^{1/2}\ ,$$

$$\exp[\varphi(i,j)]=1+\frac{12P_i^2P_j^2}{(P_i+P_j)^2-(P_i+P_j)^4-(\Omega_i\epsilon_i+\Omega_j\epsilon_j)^2}=\frac{(\epsilon_i v_i-\epsilon_j v_j)^2-3(P_i-P_j)^2}{(\epsilon_i v_i-\epsilon_j v_j)^2-3(P_i+P_j)^2}\ ,$$

$$v_i=(1-P_i^2)^{1/2}\ .$$

Here P_i and η_i are the real constants relating to the amplitude and phase of the ith soliton, respectively. $\sum_{\mu=0,1}$ implies the summation over all possible combinations of $\mu_1=0,1;\ \mu_2=0,1;\ \ldots;\ \mu_N=0,1$. The second summation implies the summation over all possible pairs chosen from N elements.

The proof that (A3) is the solution of (A2) is similar to the proof in Hirota's paper [21]. It is easily seen that $u(x,t)$ defined by (A3) is a solution of (A2) provided $f(x,t)$ satisfied the following equation:

$$\left[\left[\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right]^2-\left[\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right]^2+\left[\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right]^4\right]f(x,t)f(x',t')\Bigg|_{x=x',t=t'}=0\ . \tag{A2'}$$

Substituting $f(x,t)$ in (A3) into (A2′), we get

$$\sum_{\mu=0,1}\sum_{\nu=0,1}\left[\left[\sum_{i=1}^{N}\epsilon_i\Omega_i(\mu_i-\nu_i)\right]^2-\left[\sum_{i=1}^{N}P_i(\mu_i-\nu_i\right]^2+\left[\sum_{i=1}^{N})P_i(\mu_i-\nu_i)^4\right]\right]$$
$$\times\exp\left[\sum_{\substack{i,j\\ i<j\\ i=1}}^{N}\varphi(i,j)(\mu_i\mu_j+\nu_i\nu_j)+\sum_{i=1}^{N}(\mu_i+\nu_i)\eta_i\right]=0\ .$$

Following Hirota's method [21], this identity can be proved by mathematical induction. So (A3) is a solution of Eq. (A2). It is easy to show $u(x,t)$ splits into N solitons in the limit $|t|\to\infty$, i.e., $u(x,t)\to(P_i/2)^2\mathrm{sech}^2[1/2\ (\eta_i-\eta_i^0)]$.

Because (A2) has the N-soliton solution, it should have infinite numbers of conservation constants. We find some conservation constants by breaking (A2) into two coupling equations by introducing a new function $w(x,t)$:

$$\frac{\partial u}{\partial t}=-\frac{\partial w}{\partial x}\ , \tag{A4}$$

$$\frac{\partial w}{\partial t}=-\frac{\partial}{\partial x}(u-6u^2-u_{xx})\ . \tag{A5}$$

When $|x|,|t|\to\infty$, $u(x,t),w(x,t)$ and their derivatives are zero. From (A4), $\int_{-\infty}^{+\infty}(\partial u/\partial t)dx=-w|_{-\infty}^{+\infty}=0$, we get one constant $I_1=\int_{-\infty}^{+\infty}u\,dx$.

In the same way, we can find more constants. Here is the list of some conservation constants:

$$I_1=\int_{-\infty}^{+\infty}u\,dx\ ,$$
$$I_2=\int_{-\infty}^{+\infty}w\,dx\ ,$$
$$I_3=\int_{-\infty}^{+\infty}uw\,dx\ ,$$
$$I_4=\int_{-\infty}^{+\infty}\left[\frac{u^2}{2}+\frac{w^2}{2}-2u^3-\frac{u_x^2}{2}\right]dx\ ,$$
$$D_1=\int_{-\infty}^{+\infty}w\,dt\ ,$$
$$D_2=\int_{-\infty}^{+\infty}(u-6u^2-u_{xx})dt\ ,$$
$$D_3=\int_{-\infty}^{+\infty}\left[\frac{u^2}{2}+\frac{w^2}{2}+\frac{u_x^2}{2}-4u^3-uu_{xx}\right]dt\ ,$$
$$D_4=\int_{-\infty}^{+\infty}(-wu+6wu^2+wu_{xx}-w_xu_x)dt\ .$$

Finally we want to mention that under the weak nonlinearity and long-wave length condition, Eq. (A2) can be reduced to the KdV equation by the reductive perturbation method. Importantly, we can apply a lot of well-known properties of the KdV soliton to our electromagnetic soliton under this limit.

If we introduce a small parameter ϵ and choose the new "space" and "time" variables

$$\xi=\epsilon^p(x-vt),\quad \tau=\tfrac{1}{2}\epsilon^q t\ , \tag{A6}$$

we can expand u by the power of ϵ:

$$u=\epsilon u^{(1)}+\epsilon^2u^{(2)}+\cdots\ . \tag{A7}$$

By taking $p=\frac{1}{2}$, $q=\frac{3}{2}$ (A2) is reduced to

$$\pm u_{\xi\tau}^{(1)}+6(u^{(1)})^2_{\xi\xi}+u^{(1)}_{\xi\xi\xi\xi}=0\ .$$

If we take an integral about ξ, we get the exact KdV equations:

$$\pm u_\tau^{(1)}+12u^{(1)}u_\xi^{(1)^2}+u^{(1)}_{\xi\xi\xi}=0\ , \tag{A8}$$

where the + and − sign correspond to two KdV equations which can propagate in "right" and "left" directions.

Here, we only consider the $u^{(1)}$ term, which means there is only a weak nonlinear effect. Because $p<q$, the "time" will change slower than "space" from (A6). But τ corresponds to the space scale x in (2.1), so we have a slow changing in real space which means the long-wavelength situation.

APPENDIX B: INITIAL-CONDITION PROBLEM FOR SHOCK-WAVE EQUATION (3.1)

When the dispersion of the medium is much smaller than the dissipation, the equation for the polariton is given by (3.1):

$$\frac{\partial^2E}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\epsilon(0)E-\frac{Sg}{\omega_{\mathrm{TO}}^2}\frac{\partial E}{\partial t}+d(E^2)\right]\ .$$

To simplify (3.1), we normalize the equation by rescaling the variables:

$$\tau=\frac{t}{T},\quad z=\frac{x}{L},\quad u=\frac{E}{E_0}\ ,$$

where

$$T=\frac{Sg}{\epsilon(0)\omega_{\mathrm{TO}}^2},\quad L=Tc',\quad E_0=-\frac{\epsilon(0)}{d}\ . \tag{B1}$$

Now (3.1) becomes

$$u_{\tau\tau}-u_{\tau\tau\tau}-(u^2)_{\tau\tau}=u_{zz}\ . \tag{B2}$$

If the solution of (B2) is a slowly changing function, we have $u(z,\tau)=u(\epsilon z,\tau-z)$ where ϵ is a small parameter. Introducing $\zeta=\tau-z$, ignoring the ϵ^2 term, and taking an integral about ζ, (B2) becomes

$$2u_z-u_{\zeta\zeta}-(u^2)_\zeta=0\ , \tag{B3}$$

where we assume the boundary condition $u=$const when $\zeta=\pm\infty$. (B3) is almost the same as Burgers's equation [22] which is a famous nonlinear diffusion or heat equation except that the "diffusion constant" is negative. By the Cole-Hopf transformation (B3) can be linearized as

$$f_z=\tfrac{1}{2}f_{\zeta\zeta}\ . \tag{B4}$$

where $u=(\partial/\partial\zeta)\ln f(z,\zeta)$ is the Cole-Hopf transformation. The initial-value problem of (B4) is easy to solve. If $u_0(\tau)=u(0,\zeta)$ is known, u at any time and position can be expressed as

$$u=\frac{\partial}{\partial\zeta}\ln f(z,\zeta)\ ,$$

$$f(z,\zeta)=\frac{1}{(2\pi z)^{1/2}}\int_{-\infty}^{+\infty}f(0,\tau)\exp\left[-\frac{(\zeta-\tau)^2}{2z}\right]d\tau\ ,$$

$$f(0,\tau)=\exp\left[\int^{\tau}u(0,\tau')d\tau'\right]\ .$$

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