Retarded radiation field and spontaneous emission for the hydrogen atom as an emitting antenna

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We have evaluated explicitly from the current of the electron in a nonstationary superposition state the near and far potentials and fields, then the radiated flux, and with it, the rate of spontaneous emission with retardation. The Schrödinger current gives a power which is four times smaller than the observed rate of spontaneous emission.

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I. INTRODUCTION

The radiative-transition process of an electron between two atomic electronic levels can be described using various theories. The two main families of approaches are the following.

(i) The perturbative QED model [1–3] where the field is quantized. The state of the atom is represented by a product wave function $|a,\alpha\rangle$ of the electron and photons. The total Hamiltonian

$$H=H_{\text{atom}}+\sum aa^{\dagger}\hbar\omega+\sum g(a\sigma^{+}+a^{\dagger}\sigma^{-}) \tag{1}$$

includes the field Hamiltonian and an interaction term.

(ii) The "semiclassical model" [4–6] where the electromagnetic field is not quantized.

Here we will use the self-energy quantum electrodynamics, which we view as a complete self-consistent formalism, to compute the potentials and fields emitted by the electron in transition. A simple calculation of these fields has been made previously [4]. In this paper, a different method will be used which is more systematic, and the potentials will be evaluated without approximations. Notably, the dipole approximation will not be used, which means that the retardation exponentials will be treated fully.

In this paper, as a prototype, only the $2P\rightarrow 1S$ partial transition in the hydrogen atom will be considered. The electron is assumed to have the wave function

$$\psi=\frac{1}{\sqrt{2}}[\psi_{1s}(\mathbf{r},t)+\psi_{2p}(\mathbf{r},t)] =\frac{1}{\sqrt{2}}[e^{-i(E_1/h)t}\psi_{1s}(\mathbf{r})+e^{-i(E_2/h)t}\psi_{2p}(\mathbf{r})] . \tag{2}$$

In self-energy quantum electrodynamics, the wave function is determined self-consistently together with all other observable energy shifts and spontaneous emission. It seems appropriate, however, for the calculation of a partial transition rate between two levels in the lowest order of iteration to assume the above state whose current contains a single frequency $E_2-E_1=\omega_{12}\hbar$. This wave function is a solution of the Schrödinger equation without self-energy term. In addition, the starting $2P$ orbital and the final $1S$ orbital are weighted by equal coefficients.

Although this study is restricted to the wave function (2), one might consider next to extend it to a more appropriate wave function [5,7,8] of the form

$$\psi=\alpha(t)e^{-i(E_1/h)t}\psi_{1s}(\mathbf{r})+\sqrt{1-\alpha^2(t)}e^{-i(E_2/h)t}\psi_{2p}(\mathbf{r}) , \quad \text{with } \alpha(0)=1 \text{ and } \alpha(t_{\text{final}})=0 , \tag{3}$$

which could satisfy the Schrödinger equation with a self-energy term. In (3), $\alpha(t)$ varies slowly with respect to the exponentials; therefore, the wave function (2) can be interpreted as an approximation of (3) $[\alpha(t)=1/\sqrt{2}]$.

In the state ψ, we will compute successively (i) the charge and charge current densities ρ and $\mathbf{J}$, and their Fourier transforms, (ii) the scalar and vector potentials ϕ and $\mathbf{A}$, using a Fourier transform $\mathcal{F}$ in space and time $(n=4)$, followed by the reciprocal transform $\bar{\mathcal{F}}$:

$$A\equiv(\phi,c\,\mathbf{A})=\frac{(2\pi)^{n/2}}{\epsilon_0}\bar{\mathcal{F}}(\mathcal{F}G\times\mathcal{F}J) , \tag{4}$$

i.e.,

$$A(x)=\frac{(2\pi)^{n/2}}{\epsilon_0}4\pi^2\int d^4k\, e^{ikx}\mathcal{F}G\mathcal{F}J ,$$

where

$$(\mathcal{F}f)(k)=\frac{1}{4\pi^2}\int d^4x\, e^{-ikx}f(x)$$

with $J\equiv(\rho,\mathbf{J}/c)$; G is the Green's function

$$G=(4\pi r)^{-1}\delta\left[t-\frac{r}{c}\right], \tag{4'}$$

(iii) the electromagnetic fields **E** and **B** given by the Maxwell equations, and (iv) the Poynting vector **S** and with it the power radiated by the electron, and the spontaneous-emission lifetime.

II. THE DENSITY OF CHARGE AND ITS FOURIER TRANSFORM

(i) The charge density in the state ψ is defined by

$$\rho=e\psi^*\psi \quad (e<0)\ . \tag{5}$$

Substitution of (2) in (5) gives

$$\rho=e\left[\tfrac{1}{2}(|\psi_1|^2+|\psi_2|^2)+\cos\left[\frac{E_1-E_2}{\hbar}t\right]\psi_1\psi_2\right]. \tag{6}$$

In (6), the hydrogenic wave functions are

$$\psi_1=\psi_{1s}(\mathbf{r})=\frac{1}{\sqrt{\pi a^3}}e^{-r/a}, \tag{7}$$

$$\psi_2=\psi_{2p_z}(\mathbf{r})=\frac{1}{4\sqrt{2\pi a^3}}\frac{r}{a}e^{-r/2a}\cos\theta', \tag{8}$$

where $r\equiv|\mathbf{r}|$, $\theta'\equiv\angle(\mathbf{e}_3,\mathbf{r})$; $\mathbf{e}_3$ is the axis of the $2p$ orbital, and a is the Bohr radius. Now we compute the Fourier transforms of the three terms in (6). The spacetime metric is $(+---)$.

(ii) The Fourier transform of the charge density $\rho_1=(e/2)|\psi_1|^2$.

This part of ρ is time independent. Its space Fourier transform is

$$(\mathcal{F}\rho_1)(\mathbf{k})\equiv\frac{e}{2}\frac{1}{(2\pi)^{3/2}}\int d^3\mathbf{r}\, e^{+i\mathbf{k}\cdot\mathbf{r}}\psi_1^2(\mathbf{r})\ . \tag{9}$$

In dimension 3, the Fourier transform of a radial function $f(r)$ is the radial function [9]

$$(\mathcal{F}f)(k)=\frac{1}{(2\pi)^{3/2}}\int_0^\infty dr\, r\sin(kr)f(r) \qquad (k\equiv|\mathbf{k}|)\ . \tag{10}$$

Equation (10) applied to $f=\rho_1$ gives

$$\mathcal{F}\rho_1=\frac{e}{2}\frac{1}{(2\pi)^{3/2}}\frac{8h}{a^3(h^2+k^2)^2}\qquad\left[h\equiv\frac{2}{a}\right]. \tag{11}$$

(iii) The Fourier transform of $\rho_2=(e/2)|\psi_2|^2$.

Let us compute $\mathcal{F}\rho_2$ in spherical coordinates (r,θ,φ) of **k**, with the yz plane containing the axis $\mathbf{e}_3$ of the $2p$ orbital:

$$\mathcal{F}\rho_2=\frac{e}{2}\frac{1}{32\pi a^5}\frac{1}{(2\pi)^{3/2}}\times\int_0^\infty dr\, r^4e^{-r/a}\int_0^\pi d\theta\,\sin\theta e^{ikr\cos\theta}\int_0^{2\pi}d\varphi\cos^2\theta' \tag{12}$$

$[\theta'\equiv\angle(\mathbf{e}_3,\mathbf{r})]$. We have

$$\cos\theta'=\cos\beta\cos\theta+\sin\theta\sin\varphi\sin\beta\ , \tag{13}$$

with $\beta\equiv\angle(\mathbf{k},\mathbf{e}_3)$ and $\theta\equiv\angle(\mathbf{k},\mathbf{r})$.

In (12), we find

$$\int d\theta\, d\varphi=\frac{4\pi}{k}\left[\cos^2\beta\frac{\sin(kr)}{r}+(1-3\cos^2\beta)\frac{1}{kr^2}\left[-\cos(kr)+\frac{\sin(kr)}{kr}\right]\right]. \tag{14}$$

Integration over r then gives

$$\mathcal{F}\rho_2=\frac{e}{2}\frac{1}{(2\pi)^{3/2}a^6(h^2+k^2)^4}(-6k^2\cos^2\beta+h^2+k^2)\ , \tag{15}$$

where $h\equiv a^{-1}$.

(iv) The Fourier transform of the crossed density of charge.

The Fourier transform of the crossed term in (6),

$$\rho_{12}\equiv e\cos(\omega t)\psi_1\psi_2\ ,\quad \omega\equiv\frac{E_2-E_1}{\hbar}\ , \tag{16}$$

is

$$(\mathcal{F}\rho_{12})(K)=\frac{1}{4\pi^2}\left[\int dy_0 e^{-ik_0y_0}\cos\left[\frac{\omega}{c}y_0\right]\right]I_{21}(\mathbf{k})\ , \tag{17}$$

with $K\equiv(k_0,\mathbf{k})$ and

$$I_{nm}(\mathbf{k})\equiv\int d^3\mathbf{y}\, e^{i\mathbf{k}\cdot\mathbf{y}}\psi_n(\mathbf{y})\psi_m(\mathbf{y})\ . \tag{18}$$

In (17) we have

$$\int dy_0=\pi\sum_{+,-}\delta\left[k_0\pm\frac{\omega}{c}\right], \tag{19}$$

and

$$I_{21}=i\frac{6\sqrt{2}}{a^5}\cos\beta\frac{k}{(h^2+k^2)^3}\qquad\left[h\equiv\frac{3}{2a}\ ,\ k\equiv|\mathbf{k}|\right]. \tag{20}$$

III. THE CURRENT DENSITY AND ITS FOURIER TRANSFORM

(i) The charge current density is defined by [Ref. [4] Eq. (7.3); Ref. [10]]

$$\mathbf{J}=e\,\mathrm{Re}\left[\psi^*\frac{\hbar}{im}\nabla\psi\right]=\frac{e\hbar}{2im}(\psi^*\nabla\psi-\psi\nabla\psi^*)\ . \tag{21}$$

Since ψ is defined as (2), we have

$$\psi^*\nabla\psi=\tfrac{1}{2}(\psi_2^*\nabla\psi_2+\psi^*\nabla\psi_1+e^{i(E_2-E_1/h)t}\psi_2^*\nabla\psi_1 + e^{-i[(E_2-E_1)/h]t}\psi_1^*\nabla\psi_2) . \quad (22)$$

Since the space orbitals ψ_1 and ψ_2 are real, $\mathbf{J}$ reduces to its crossed term

$$\mathbf{J}=\frac{e\hbar}{2m}\sin\left[\frac{E_2-E_1}{\hbar}t\right](\psi_2\nabla\psi_1-\psi_1\nabla\psi_2) . \quad (23)$$

(ii) The Fourier transform of $\mathbf{J}$ is

$$(\mathcal{F}\mathbf{J})(K)=\frac{e\hbar}{2m}\frac{1}{4\pi^2}\left[\int dy_0 e^{-ik_0y_0}\sin\left[\frac{\omega}{c}y_0\right]\right] \times\int d^3\mathbf{y}\, e^{i\mathbf{k}\cdot\mathbf{y}}(\psi_2\nabla\psi_1-\psi_1\nabla\psi_2)(\mathbf{y}) . \quad (24)$$

In (24) we have

$$\int dy_0=-i\pi\left[\delta\left[k_0-\frac{\omega}{c}\right]-\delta\left[k_0+\frac{\omega}{c}\right]\right] \quad (25)$$

and

$$\int d^3\mathbf{y}=\mathbf{T}_{21}(\mathbf{k})-\mathbf{T}_{12}(\mathbf{k}) , \quad (26)$$

with the definition

$$\mathbf{T}_{nm}(\mathbf{k})\equiv\int d^3\mathbf{y}\, e^{i\mathbf{k}\cdot\mathbf{y}}\psi_n^*(\mathbf{y})\nabla\psi_m(\mathbf{y}) . \quad (27)$$

By virtue of the identity

$$\mathbf{T}_{nm}(\mathbf{k})=-\mathbf{T}_{mn}(\mathbf{k})-i\mathbf{k}I_{nm}(\mathbf{k}) \quad (28)$$

[where I_{nm} is still defined by (18)], we have in (24)

$$\int d^3\mathbf{y}=2\mathbf{T}_{21}(\mathbf{k})+i\mathbf{k}I_{21}(\mathbf{k}) . \quad (29)$$

The component of $\mathbf{T}_{21}$ orthogonal to $\mathbf{k}$ has been calculated previously [15]

$$\mathbf{T}_{21}^{\perp}=-\mathbf{j}\frac{\sqrt{2}\sin\beta}{a^5(h^2+k^2)^2} \quad (30)$$

[$h\equiv3/2a$; $\beta\equiv\angle(\mathbf{k},\mathbf{e}_3)$], where $\mathbf{j}$ is the unit vector orthogonal to $\mathbf{k}$ and contained in the half plane with side $\mathbf{k}$ and containing $\mathbf{e}_3$.

The component of $\mathbf{T}_{21}$ on $\mathbf{e}_3$ can be calculated using spherical coordinates of $\mathbf{k}$, and is

$$\mathbf{T}_{21}^{e_3}=\mathbf{e}_3\frac{\sqrt{2}}{a^5(h^2+k^2)^2}\left[\frac{4k^2}{h^2+k^2}\cos^2\beta-1\right] . \quad (31)$$

$\mathbf{T}_{21}$ is then easily deduced from (30) and (31):

$$\mathbf{T}_{21}=\frac{\sqrt{2}}{a^5(h^2+k^2)^2}\left[\mathbf{w}\cos\beta\frac{4k^2}{h^2+k^2}-\mathbf{e}_3\right] , \quad (32)$$

where $\mathbf{w}\equiv\mathbf{k}/|\mathbf{k}|$. The explicit value of $\mathcal{F}\mathbf{J}$ is obtained when (25), (29), (32), and (20) are substituted into (24).

IV. THE SCALAR ELECTROMAGNETIC POTENTIAL ϕ

The potential A is defined by

$$A=J\circ G ,$$

i.e.,

$$A(x)=\int d^4y\, J(y)G(x-y) , \quad (33)$$

where G is the retarded Green's function (4′). In order to simplify the calculations, we shall use instead of (4′) the half-sum

$$G=(4\pi r)^{-1}\frac{1}{2}\left[\delta\left[t-\frac{r}{c}\right]+\delta\left[t+\frac{r}{c}\right]\right] \quad (34)$$

of the retarded and advanced Green's functions. However, it must be emphasized that (34) is only a mathematical artifice, the function having physical interest being (4′). At the end of the calculations, only the retarded terms will be selected (and multiplied by 2) in the expressions for the fields $\mathbf{E}$ and $\mathbf{B}$. The convolution theorem applied to (33) gives (4). The Fourier transform of (34) is

$$\mathcal{F}G(K)=-\frac{1}{4\pi^2}\mathrm{P}\frac{1}{k_0^2-k^2} \quad (35)$$

[$K\equiv(k_0,\mathbf{k})$, $k\equiv|\mathbf{k}|$, P means principal value].

(i) The scalar potential ϕ_1 due to the charge density $\rho_1=(e/2)|\psi_1|^2$. Because ρ_1 is time independent, Eq. (4) can be applied with space Fourier transforms only. Therefore, we set $n=3$ in (4), and (35) is replaced by

$$(\mathcal{F}G)(\mathbf{k})=\frac{1}{2\pi)^{3/2}}\frac{1}{k^2} . \quad (35')$$

Substituting (11) and (35′) in (4), we have

$$\phi_1(\mathbf{r})=\frac{e}{\epsilon_0a^3\pi^2}\int_0^\infty dk\frac{h}{(h^2+k^2)^2}\int_0^\pi d\theta\sin\theta\, e^{-ikr\cos\theta}$$
$$=-\frac{1}{h}\int_0^\infty dk\frac{\sin(kr)k}{(h^2+k^2)^2} +\frac{1}{h^3}\left[\int_0^\infty dk\frac{\sin(kr)}{k}-\int_0^\infty dk\frac{k\sin(kr)}{h^2+k^2}\right] . \quad (36)$$

The integrals in (36) are known [11], hence,

$$\phi_1(\mathbf{r})=\phi_1(r)=\frac{e}{4\pi\epsilon_0r}\frac{1}{2}\left[1-e^{-2r/a}\left[1+\frac{r}{a}\right]\right] . \quad (37)$$

It is interesting to compare ϕ_1 with the potential ϕ_1' due only to the part of the distribution ρ_1 which is inside the sphere $(\mathbf{0},r)$, namely,

$$\phi_1'=\frac{e}{4\pi\epsilon_0r}\frac{1}{2}\left[1-e^{-2r/a}\left[\frac{2r^2}{a^2}+2\frac{r}{a}+1\right]\right] . \quad (38)$$

(ii) The scalar potential ϕ_2 due to the charge density $\rho_2=(e/2)|\psi_2|^2$. Let us insert (15) and (35′) into (4) (with $n=3$). This leads to

$$\phi_2(\mathbf{r})=\frac{e}{2\epsilon_0(2\pi)^3a^6}\int_0^\infty dk\left[-\frac{6k^2}{(h^2+k^2)^4}\int_0^\pi d\theta\sin\theta\, e^{-ikr\cos\theta}\int_0^{2\pi}d\varphi\cos^2\beta+\frac{2\pi}{(h^2+k^2)^3}\int_0^\pi d\theta\sin\theta\, e^{-ikr\cos\theta}\right], \tag{39}$$

with $\theta\equiv\angle(\mathbf{k},\mathbf{r})$, $\beta\equiv\angle(\mathbf{k},\mathbf{e}_3)$. Integration of (39) gives (Appendix A)

$$\phi_2(\mathbf{r})=\frac{3ea^2}{4\pi\epsilon_0 r^3}\left[\cos^2\gamma\left[3-\frac{e^{-rh}}{48}P\right]-1+\frac{r^2h^2}{6}+e^{-rh}Q\right], \tag{40}$$

where $h\equiv a^{-1}$, $\gamma\equiv\angle(\mathbf{e}_3,\mathbf{r})$, and P and Q are the polynomials

$$P\equiv r^5h^5+6r^4h^4+75r^3h^3+3\times 48rh+3\times 48\ ,$$
$$Q\equiv\frac{r^3h^3}{24}+\frac{r^2h^2}{3}+rh+1\ . \tag{41}$$

At large distances r, e^{-rh} is small, and therefore

$$\phi_2\simeq\frac{e}{4\pi\epsilon_0}\frac{1}{2}\left[\frac{1}{r}+\frac{6a^2}{r^3}(3\cos^2\gamma-1)\right]. \tag{42}$$

(iii) The scalar potential ϕ_{12} due to the crossed charge density ρ_{12}. Let us insert (17), (19), (20), and (35) into (4). We obtain

$$\begin{aligned}\phi_{12}(x_0,\mathbf{x})&=-\frac{1}{8\pi^3\epsilon_0}\int d^3\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{x}}I_{21}(\mathbf{k})\\&\quad\times\int dk_0\frac{e^{ik_0x_0}}{k_0^2-k^2}\\&\quad\times\tfrac{1}{2}\sum_{+,-}\delta\left[k_0\pm\frac{\omega}{c}\right]\\&=-\frac{\cos(\omega t)}{8\pi^3\epsilon_0}\int d^3\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{I_{21}(\mathbf{k})}{\omega^2/c^2-k^2}\ .\end{aligned} \tag{43}$$

In spherical coordinates (k,θ,φ) of $\mathbf{x}$, we have

$$\phi_{12}=C_0 i\int_0^\infty dk\frac{k^3}{(\omega^2-k^2)(h^2+k^2)^3}\times\int_0^\pi d\theta\sin\theta\, e^{-ikr\cos\theta}\times\int_0^{2\pi}d\varphi\cos\beta\ , \tag{44}$$

$(h\equiv 3/2a)$ with

$$\cos\beta=\cos\theta\cos\gamma+\sin\gamma\sin\theta\sin\varphi$$

and

$$C_0\equiv-\frac{6\sqrt{2}}{2\pi^2}\frac{e}{4\pi\epsilon_0 a^5}\cos(\omega t)\ . \tag{45}$$

This leads to

$$\phi_{12}=C_0 4\pi\cos(\gamma)r^4(-C_{23}+S_{13})\ , \tag{46}$$

where C_{23} and S_{13} are defined by (B1).

Defining

$$\Omega\equiv\frac{\omega}{c}r\ ,\quad H\equiv hr\equiv\frac{3r}{2a}\ ,\quad L^{-1}\equiv\Omega^2+H^2\ , \tag{47}$$

and substituting (B4) and (B6) into (46), we obtain

$$\phi_{12}=-6\sqrt{2}\cos(\omega t)\frac{e}{4\pi\epsilon_0}\cos\gamma\frac{r^4L^3}{a^5}\times\left[-\Omega\sin\Omega-\cos\Omega+e^{-H}\left[\frac{L^{-2}}{8H}+\frac{L^{-1}}{2}+H+1\right]\right]. \tag{48}$$

If $\omega/c=3\alpha/8a$ (α is the fine-structure constant) is neglected with respect to $h\equiv 3/2a$, then we have the following approximations in (48):

$$L^3\simeq H^{-6}\ ,$$
$$e^{-H}\left[\frac{L^{-2}}{8H}+\frac{L^{-1}}{2}+H+1\right]\simeq e^{-H}\left[\frac{H^3}{8}+\frac{H^2}{2}+H+1\right]. \tag{49}$$

At large distances r $(r\gg a)$, we have $H\gg 1$. Hence,

$$\phi_{12}\simeq D\frac{\cos\gamma}{r^2}\cos(\omega t)\left[\frac{\omega r}{c}\sin\frac{\omega r}{c}+\cos\frac{\omega r}{c}\right]$$

with

$$D\equiv\frac{2^7\sqrt{2}}{3^5}\frac{ea}{4\pi\epsilon_0}\ . \tag{50}$$

We notice that (50) is a solution of the free wave equation, as it must be. When $\phi_{12}\equiv\frac{1}{2}(\phi_{\rm ret}+\phi_{\rm adv})$ is written in the form

$$\phi_{12}=\frac{D\cos\gamma}{2}\left[\frac{\omega}{cr}\left\{\sin\left[\omega\left[t+\frac{r}{c}\right]\right]-\sin\left[\omega\left[t-\frac{r}{c}\right]\right]\right\}+\frac{1}{r^2}\left\{\cos\left[\omega\left[t+\frac{r}{c}\right]\right]+\cos\left[\omega\left[t-\frac{r}{c}\right]\right]\right\}\right], \tag{51}$$

the retarded potential is easily selected (and multiplied by 2):

$$\phi_{12}=D\cos\gamma\left\{-\frac{\omega}{cr}\sin\left[\omega\left[t-\frac{r}{c}\right]\right]+\frac{1}{r^2}\cos\left[\omega\left[t-\frac{r}{c}\right]\right]\right\} . \quad (52)$$

When the retardation is neglected in either (51) or (52) (that is, $r/c \ll |t|$), we obtain the dipole-approximation results

$$\phi_{12}=-\frac{1}{4\pi\epsilon_0}\mu\cdot\nabla\left[\frac{1}{r}\right] \quad (53)$$

with

$$\mu\equiv\mathbf{e}_3\frac{2^7\sqrt{2}}{3^5}ea\cos(\omega t)=\cos(\omega t)e\langle 2p|\mathbf{x}|1s\rangle . \quad (54)$$

V. THE VECTOR ELECTROMAGNETIC POTENTIAL A

In (4), we now insert (35) and $\mathcal{FJ}$ as computed in Sec. III. This gives the half sum of the retarded and advanced potentials:

$$\mathbf{A}_{12}(x_0,\mathbf{x})\equiv\mathbf{A}=\frac{ie\hbar}{4\pi\epsilon_0 mc^2 8\pi^2}\int d^3\mathbf{k}\,e^{-i\mathbf{k}\cdot\mathbf{x}}[2\mathbf{T}_{21}(\mathbf{k})+i\mathbf{k}I_{21}(\mathbf{k})]\int dk_0\frac{e^{ik_0x_0}}{k_0^2-k^2}\left[\delta\left[k_0-\frac{\omega}{c}\right]-\delta\left[k_0+\frac{\omega}{c}\right]\right] . \quad (55)$$

In (55) we have

$$\int dk_0=\frac{2i}{\omega^2/c^2-k^2}\sin(\omega t)\quad (x_0\equiv ct) . \quad (56)$$

Hence,

$$\mathbf{A}=C(\mathbf{A}_T+\mathbf{A}_I) , \quad (57)$$

with the definitions

$$C\equiv\frac{-e\hbar}{4\pi\epsilon_0 mc^2 2\pi^2}\sin(\omega t) , \quad (58)$$

$$\mathbf{A}_T\equiv\int d^3\mathbf{k}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{\mathbf{T}_{21}(\mathbf{k})}{\omega^2/c^2-k^2} \quad (59)$$

$$\mathbf{A}_I\equiv\frac{i}{2}\int d^3\mathbf{k}\,\mathbf{k}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{I_{21}(\mathbf{k})}{\omega^2/c^2-k^2} . \quad (60)$$

Integration of (59) and (60) (Appendix C) and substitution into (57) leads to

$$\mathbf{A}=-\frac{e}{4\pi\epsilon_0 c}\sqrt{2}\alpha\sin(\omega t)\frac{r^3L^3}{a^4}\left\{\mathbf{e}_3\left[-\Omega\sin\Omega+(L^{-1}-1)\cos\Omega+e^{-H}\left[-\frac{3}{8}\frac{L^{-2}}{H}-\frac{L^{-1}}{2}+H+1\right]\right]+\mathbf{n}\cos\gamma\left[3\Omega\sin\Omega+(3-\Omega^2)\cos\Omega-e^{-H}\left[\frac{L^{-2}}{8}+\frac{HL^{-1}}{2}+\frac{L^{-1}}{2}+H^2+3H+3\right]\right]\right\} , \quad (61)$$

where $\alpha\equiv e^2/(4\pi\epsilon_0\hbar c)$ is the fine-structure constant.

Since $\omega^2\ll h^2$, we have $L^{-1}\simeq H^2$, and therefore

$$\mathbf{A}\simeq-\frac{e}{4\pi\epsilon_0 c}\frac{2^6\sqrt{2}}{3^6}\alpha\sin(\omega t)\left\{\frac{9}{4}\frac{1}{r}\cos\frac{\omega r}{c}\left[\mathbf{e}_3-\frac{\alpha^2}{16}\mathbf{n}\cos\gamma\right]+\frac{3\alpha}{8}\frac{a}{r^2}\sin\frac{\omega r}{c}(-\mathbf{e}_3+3\mathbf{n}\cos\gamma)+\frac{a^2}{r^3}\cos\frac{\omega r}{c}(-\mathbf{e}_3+3\mathbf{n}\cos\gamma)+\frac{3^4}{2^5}\frac{r}{a^2}e^{-3r/2a}\left[-\frac{\mathbf{n}}{4}\cos\gamma+O\left[\frac{r}{a}\right]\right]\right\} . \quad (62)$$

This is the half sum of the retarded and advanced potentials. Using

$$\sin(\omega t)\cos\frac{\omega r}{c}=\frac{1}{2}\left\{\sin\left[\omega\left[t+\frac{r}{c}\right]\right]+\sin\left[\omega\left[t-\frac{r}{c}\right]\right]\right\} ,$$

we obtain the far field going like r^{-1} of the retarded potential:

$$\mathbf{A}\simeq-\frac{e}{4\pi\epsilon_0 c}\frac{2^4}{3^4}\sqrt{2}\alpha\frac{1}{r}\sin\left[\omega\left[t-\frac{r}{c}\right]\right]\left[\mathbf{e}_3-\frac{\alpha^2}{16}\mathbf{n}\cos\gamma\right]. \tag{63}$$

VI. THE ELECTRIC FIELD

Let us derive separately the electric fields $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_{12}$ corresponding to the potentials ϕ_1, ϕ_2, and $(\phi_{12}, \mathbf{A}_{12})$. The relation

$$\mathbf{E}=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t} \tag{64}$$

reduces to $\mathbf{E}=-\nabla\phi$ in the first two cases.

(i) Expression (37) gives

$$\mathbf{E}_1(\mathbf{r})=\frac{1}{2}\frac{e}{4\pi\epsilon_0}\frac{\mathbf{n}}{r^2}\left[1-e^{-2r/a}\left[2\frac{r^2}{a^2}+2\frac{r}{a}+1\right]\right] \tag{65}$$

$(\mathbf{n}\equiv\mathbf{r}/r)$.

(ii) Let us use expression (42) for ϕ_2, valid for $r\gg a$:

$$\mathbf{E}_2(\mathbf{r})=\frac{e}{4\pi\epsilon_0}\frac{1}{2}\left[\frac{\mathbf{n}}{r^2}+\frac{18a^2}{r^4}[(5\cos^2\gamma-1)\mathbf{n}-2\cos\gamma\,\mathbf{e}_3]\right]+\text{exponential terms} \tag{66}$$

$[\gamma\equiv\angle(\mathbf{e}_3,\mathbf{r})]$.

(iii) Equation (64) is now applied to the crossed potentials ϕ_{12} and $\mathbf{A}_{12}$ given by (48) and (61), neglecting ω^2 with respect to h^2 $(L^{-1}\simeq H^2)$. This gives

$$\begin{aligned}\mathbf{E}_{12}=\frac{e}{4\pi\epsilon_0}\frac{2^7}{3^5}\sqrt{2}a\cos\omega t\Bigg[&\frac{\omega^2}{r}\cos\left[\frac{\omega r}{c}\right]\left[\mathbf{e}_3-\mathbf{n}\cos\gamma\left[1+\frac{\alpha^2}{16}\right]\right]\\&+\left[\frac{\omega}{r^2}\sin\frac{\omega r}{c}+\frac{1}{r^3}\cos\frac{\omega r}{c}\right](3\mathbf{n}\cos\gamma-\mathbf{e}_3)\left[1+\frac{\alpha^2}{16}\right]\\&+\frac{e^{-H}}{r^3}\Bigg\{\mathbf{e}_3\left[\frac{H^3}{8}+\frac{H^2}{2}+H+1+\frac{\alpha^2}{16}\left[-\frac{3}{8}H^3-\frac{H^2}{2}+H+1\right]\right]\\&-\mathbf{n}\cos\gamma\Bigg[\frac{H^4}{8}+\frac{H^3}{2}+\frac{3}{2}H^2+3H+3\\&+\frac{\alpha^2}{16}\left[\frac{H^4}{8}+\frac{H^3}{2}+\frac{3}{2}H^2+3H+3\right]\Bigg]\Bigg\}\Bigg].\end{aligned} \tag{67}$$

The r^{-1} part of the retarded field can be isolated in (67):

$$\mathbf{E}_{12}\simeq\frac{e}{4\pi\epsilon_0}\frac{2^7}{3^5}\sqrt{2}a\cos\left[\omega\left[t-\frac{r}{c}\right]\right]\times\frac{\omega^2}{r}\left[\mathbf{e}_3-\mathbf{n}\cos\gamma\left[1+\frac{\alpha^2}{16}\right]\right]. \tag{68}$$

VII. THE MAGNETIC FIELD

In

$$\mathbf{B}=\nabla\times\mathbf{A}, \tag{69}$$

$\mathbf{A}$ is given by (61). Still making the approximation $\omega^2\ll h^2$, we obtain

$$\mathbf{B}=-\frac{e}{4\pi\epsilon_0 c^2}\frac{2^4}{3^4}\sqrt{2}\alpha\sin(\omega t)\mathbf{i}\sin\gamma\times\left[\frac{\omega}{rc}\sin\frac{\omega r}{c}+\frac{1}{r^2}\cos\frac{\omega r}{c}-\frac{e^{-H}}{r^2}\left[\frac{H^2}{2}+H+1\right]\right] \tag{70}$$

with $\mathbf{i}\sin\gamma\equiv\mathbf{e}_3\times\mathbf{n}$. The r^{-1} part of the retarded magnetic field is therefore

$$\mathbf{B}\simeq-\frac{e}{4\pi\epsilon_0 c^2}\frac{2^4}{3^4}\sqrt{2}\alpha\mathbf{i}\sin\gamma\frac{\omega}{rc}\cos\left[\omega\left[t-\frac{r}{c}\right]\right]. \tag{71}$$

VIII. THE RADIATED POWER

The four-dimensional energy-momentum tensor T of the electromagnetic field in the radiation zone satisfies the conservation law

$$T^{\mu\nu}_{,\nu}=0 \ . \tag{72}$$

For any sphere $(\mathbf{0},r)$ centered on the atom, Eq. (72) gives

$$c\int\int_{(0,r)} d\mathbf{s}_{\text{ext}}\cdot\mathbf{S}=\frac{dU}{dt} \ , \tag{73}$$

$$c\int\int_{(0,r)} d\mathbf{s}_{\text{ext}}\cdot(-\mathbf{T})=\frac{d\mathbf{P}}{dt} \tag{74}$$

in terms of the Poynting vector $\mathbf{S}$ and of the space part $\mathbf{T}$ of the energy-momentum tensor; U and $\mathbf{P}$ are the energy and momentum of the electromagnetic field within the sphere.

The Poynting vector is given by

$$\begin{aligned}\mathbf{S}&=\epsilon_0 c(\mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_{12})\times(\mathbf{B}_1+\mathbf{B}_2+\mathbf{B}_{12})\\&=\epsilon_0 c(\mathbf{E}_1+\mathbf{E}_2+\mathbf{E}_{12})\times\mathbf{B}_{12} \ .\end{aligned} \tag{75}$$

When Eq. (73) is time averaged over one period, only the crossed term $S=\epsilon_0 c\mathbf{E}_{12}\times\mathbf{B}_{12}$ contributes to the outcoming flux because $\mathbf{E}_1$ and $\mathbf{E}_2$ are time independent and $\mathbf{E}_{12}$ and $\mathbf{B}_{12}$ are periodic.

For large values of r, the r^{-1} terms (68) and (71) of $\mathbf{E}_{12}$ and $\mathbf{B}$ dominate. Hence,

$$\mathbf{S}\simeq\mathbf{n}\frac{2}{3^6\pi}\alpha^6 mc^2\frac{c}{a}\sin^2\gamma\frac{1}{r^2}\cos^2\left[\omega\left[t-\frac{r}{c}\right]\right] \ . \tag{76}$$

The mean value of $\cos^2[\omega(t-r/c)]$ over one period is $\frac{1}{2}$, and therefore the energy radiated per unit time, hence our final result, is

$$P=\frac{2^3}{3^7}\alpha^6 mc^2\frac{c}{a} \ . \tag{77}$$

IX. CONCLUSIONS

The wave function ψ as defined by (2) represents a nonstationary state between two stationary states with probability $\frac{1}{2}$ for each sate. If we assume, following Schiff [4], that P as given by (77) is the average radiated power during the transition, then, the mean emission lifetime is

$$\tau_s=\hbar\omega/P \ , \tag{78}$$

and the Einstein coefficient would be

$$A=\tau_s^{-1}=\left[\frac{2}{3}\right]^8\alpha^4\frac{c}{a} \ . \tag{79}$$

However, this result disagrees both with experiment [12] and with Schiff's calculation [4] by a factor of 4. The difference between our result and Schiff's result is very surprising because the two calculations seem to be founded on similar hypotheses (only the method of calculation is different).

In fact, a careful examination shows that the charge and charge current densities as assumed by Schiff are twice the standard ones we used [i.e., (5) and (21)] plus a term which does not contribute to the potentials: Schiff [Ref. [4], Eqs (45.2) and (45.20)] defines the current as

$$\begin{aligned}\mathbf{J}(\mathbf{r},t)&\equiv 2\,\text{Re}[\mathbf{J}(\mathbf{r})e^{-i\omega t}]\\&=\mathbf{J}(\mathbf{r})e^{-i\omega t}+\text{c.c.}\end{aligned} \tag{80}$$

with

$$\mathbf{J}(\mathbf{r})\equiv-\frac{ie\hbar}{m}\psi_n^*(\mathbf{r})\nabla\psi_m(\mathbf{r}) \ . \tag{81}$$

Since the space orbitals $\psi_n\equiv\psi_{2p}$ and $\psi_m\equiv\psi_{1s}$ are real, Eqs. (80) and (81) give

$$\mathbf{J}=-\frac{2e\hbar}{m}\psi_n\nabla\psi_m\sin(\omega t) \ . \tag{82}$$

Let us compare (82) to (23): Eq. (28) shows that

$$\left[\int d^3\mathbf{y}\, e^{-i\mathbf{k}\cdot\mathbf{y}}(\psi_n\nabla\psi_m-\psi_m\nabla\psi_n)(\mathbf{y})\right]^{\perp}=2\left[\int d^3\mathbf{y}\, e^{-i\mathbf{k}\cdot\mathbf{y}}\psi_n\nabla\psi_m\right]^{\perp} , \tag{83}$$

where $\perp$ represents the component of the vector orthogonal to $\mathbf{k}$. Therefore, the vector potential $\mathbf{A}$ resulting from (82) is larger by a factor 2 than that coming from (23). The charge density used by Schiff is deduced from the current $\mathbf{J}$ through the equation of continuity (Ref. [4], p. 406); it is therefore the double of the standard density (5).

This factor 2 in ρ and $\mathbf{J}$ results in a factor 2 in $\mathbf{E}$ and $\mathbf{B}$, and hence in a factor 4 in $\mathbf{S}$ or P. The factor 2 in Schiff's current is not explained. We think, however, that it should not be there with the normalization used in Eq. (2).

Instead, one of the authors (A.O.B.) suggests that the discrepancy with the experimental result may be due to the fact that there are two spin states $1S_{1/2,\pm}$ and $2P_{1/2,\pm}$ each carrying a Schrödinger wave function (2), and the partial decay rates add. Thus, from an analysis of decay rates one would have perhaps guessed that the electronic states are doubled. The fully relativistic calculation in self-energy quantum electrodynamics with Dirac wave functions gives not only the correct factor 4 but also accounts for the two polarization states of the emitted photons [13,14]. In order to establish the role of the spin, we plan to study by the same method the $M1$ transition and the transitions between other states.

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APPENDIX A: THE SCALAR POTENTIAL ϕ_2 DUE TO THE CHARGE DENSITY $\rho_2=(e/2)|\psi_2|^2$

Let us split expression (39) into the form

$$\phi_2=\frac{e}{2\epsilon_0(2\pi)^3a^6}\left[{}^1\int dk+{}^2\int dk\right] \ . \tag{A1}$$

The first integral over θ in (39) is equal to

$$^{1}\int d\theta = 4\pi \left[\cos^2\gamma \left[3\frac{\cos u}{u^2} + \frac{\sin u}{u} - \frac{3\sin u}{u^3} \right] - \frac{\cos u}{u^2} + \frac{\sin u}{u^3} \right] , \tag{A2}$$

where $u \equiv kr$. The corresponding integral over k in (39) is [11]

$$^{1}\int dk = -\frac{12\pi^2}{r^3 h^8} \left[\cos^2\gamma \left[\frac{e^{-rh}}{48} P - 3 \right] + 1 - e^{-rh} R \right] , \tag{A3}$$

where P and R are the polynomials defined by (41) and (A4):

$$R \equiv \frac{r^4 h^4}{48} + \frac{7}{48} r^3 h^3 + \frac{r^2 h^2}{2} + rh + 1 . \tag{A4}$$

The remaining term in (39) is

$$^{2}\int dk = \int_0^\infty dk \frac{2\pi}{(h^2+k^2)^3} 2 \frac{\sin(kr)}{kr} = \frac{2\pi^2}{h^6 r} \left[1 - \frac{e^{-rh}}{8} (r^2 h^2 + 5rh + 8) \right] . \tag{A5}$$

Equation (40) is obtained when (A3) and (A5) are substituted into (A1).

APPENDIX B: INTEGRALS OF THE FORM

$$\left. \begin{matrix} S_{nm} \\ C_{nm} \end{matrix} \right\} = \int_0^\infty du \frac{u^n}{(\Omega^2 - u^2)(H^2 + u^2)^m} \times \left\{ \begin{matrix} \sin u \\ \cos u \end{matrix} \right. . \tag{B1}$$

When the fraction in (B1) is split into simple elements, S_{nm} or C_{nm} becomes a sum of the known integrals [11]

$$\int_0^\infty dx \frac{x^p}{(b^2 \pm x^2)^q} \times \left\{ \begin{matrix} \sin(ax) \\ \cos(ax) \end{matrix} \right. . \tag{B2}$$

We thus obtain

$$S_{12} = \frac{\pi}{2} L^2 \left[-\cos\Omega + e^{-H} \left[\frac{L^{-1}}{2H} + 1 \right] \right] , \tag{B3}$$

$$S_{13} = \frac{\pi}{2} L^3 \left[-\cos\Omega + e^{-H} \left[\frac{L^{-2}(1+H)}{8H^3} + \frac{L^{-1}}{2H} + 1 \right] \right] , \tag{B4}$$

$$S_{33} = \Omega^2 S_{13} - e^{-H} \frac{\pi}{16H^3} (1+H) , \tag{B5}$$

$$C_{23} = \frac{\pi}{2} \left[\Omega L^3 \sin + e^{-H} \left[-\frac{L}{8H^3}(3 + 3H + H^2) + \Omega^2 \frac{L^3}{H} + \Omega^2 \frac{L^2}{2H^3}(1+H) \right] \right] , \tag{B6}$$

where $L^{-1} = \Omega^2 + H^2$ as in (47).

APPENDIX C: CALCULATION OF THE VECTOR POTENTIAL TERMS $\mathbf{A}_T$ AND $\mathbf{A}_l$ DEFINED BY (59) AND (60)

(a) $\mathbf{A}_T$. The integration over $\mathbf{k}$ in (59) will be performed in spherical coordinates (r,θ,φ) of $\mathbf{x}$, in the frame $(\mathbf{i},\mathbf{j},\mathbf{n})$ defined as follows: $\mathbf{n} \equiv \mathbf{x}/|\mathbf{x}|$; $\mathbf{j}$ is the unit vector perpendicular to $\mathbf{n}$ and contained in the half plane of edge $\mathbf{n}$ and containing $\mathbf{e}_3$; $\mathbf{i} = \mathbf{j} \times \mathbf{n}$. We have

$$\mathbf{A}_T = \int_0^\infty dk \frac{k^2}{\omega^2/c^2 - k^2} \times \int_0^\pi d\theta \sin(\theta) e^{ikr\cos\theta} \int_0^{2\pi} d\varphi \, \mathbf{T}_{21}(\mathbf{k}) . \tag{C1}$$

In the frame $(\mathbf{i},\mathbf{j}',\mathbf{e}_3)$ ($\mathbf{j}' \equiv \mathbf{e}_3 \times \mathbf{i}$ is coplanar with $\mathbf{n}$, $\mathbf{e}_3$, and $\mathbf{j}$), let us consider the normalized projection $\mathbf{v}$ of $\mathbf{k}$ on the plane $(\mathbf{i},\mathbf{j}')$. In the plane $(\mathbf{e}_3,\mathbf{k})$, $\mathbf{v}$ is orthogonal to $\mathbf{e}_3$ and on the same side as $\mathbf{k}$.

Expression (32) is equivalent to

$$\mathbf{T}_{21}(\mathbf{k}) = \mathbf{e}_3 \frac{\sqrt{2}}{a^5(h^2+k^2)^2} \left[\frac{4k^2}{h^2+k^2} \cos^2\beta - 1 \right] + \mathbf{v} \frac{2\sqrt{2}k^2}{a^5(h^2+k^2)^3} \sin(2\beta) \tag{C2}$$

$[\beta \equiv \angle(\mathbf{e}_3,\mathbf{k})]$. Defining $\varphi' \equiv <(\mathbf{i},\mathbf{v})$, we have

$$\mathbf{v} = \mathbf{i} \cos\varphi' + \mathbf{j}' \sin\varphi' . \tag{C3}$$

For symmetry reasons $\mathbf{A}_T$ belongs to the $(\mathbf{x},\mathbf{e}_3)$ plane, and therefore the first term in (C3) does not contribute to (C1); in the second term of (C2), $\mathbf{j}'$ does not depend on $\mathbf{k}$. When (C2) is substituted into (C1), the following terms occur:

$$\cos^2\beta = \cos^2\theta \cos^2\gamma + \sin^2\theta \sin^2\varphi \sin^2\gamma + \cdots , \tag{C4}$$

$$\sin(2\beta)\sin\varphi' = 2(\sin^2\theta \sin^2\varphi \sin\gamma \cos\gamma - \cos^2\theta \sin\gamma \cos\gamma) + \cdots \tag{C5}$$

$[\gamma \equiv \angle(\mathbf{x},\mathbf{e}_3),\ \theta \equiv \angle(\mathbf{x},\mathbf{k})]$; the terms omitted in (C4) and (C5) do not contribute to (C1).

We find

$$\mathbf{A}_T = \frac{\sqrt{2}\pi}{a^5} \int_0^\infty dk \frac{k^4}{(\omega^2/c^2 - k^2)(h^2+k^2)^3} \times \left[4\mathbf{e}_3 \, {}^{1}\int_\theta - \mathbf{e}_3 \frac{h^2+k^2}{k^2} \, {}^{2}\int_\theta + 4\mathbf{j}' \, {}^{3}\int_\theta \right] , \tag{C6}$$

where

$$^{1}\int_\theta \equiv \cos^2(\gamma) 2\Theta_{12}(kr) + \sin^2(\gamma) \Theta_{30}(kr) , \tag{C7}$$

$$^{2}\int_{\theta}\equiv 2\Theta_{10}(kr)\ , \tag{C8}$$

$$^{3}\int_{\theta}\equiv \sin(\gamma)\cos(\gamma)\Theta_{30}(kr)-2\sin(\gamma)\cos(\gamma)\Theta_{12}(kr)\ . \tag{C9}$$

Replacing the integrals

$$\Theta_{mn}\equiv\int_0^{\pi}d\theta\,\sin^m\theta\,\cos^n\theta e^{ir\cos\theta}\ . \tag{C10}$$

by their values [15] and using Appendix B, we obtain

$$\mathbf{A}_T=2\frac{\sqrt{2}}{a^5}\pi^2r^3L^3(\mathbf{e}_3A_1+\mathbf{j}'A_2) \tag{C11}$$

with

$$A_1\equiv 4\cos^2\gamma\left[2\Omega\sin\Omega-(\Omega^2-2)\cos\Omega-e^{-H}\left[\frac{L^{-2}}{8}-\frac{L^{-2}}{8H}+\frac{L^{-1}H}{2}+H^2+2H+2\right]\right]$$
$$+4\sin^2(\gamma)\left[-\Omega\sin\Omega-\cos\Omega+e^{-H}\left[\frac{L^{-2}}{8H}+\frac{L^{-1}}{2}+H+1\right]\right]+L^{-1}\left[\cos\Omega-e^{-H}\left[\frac{L^{-1}}{2H}+1\right]\right] \tag{C12}$$

and

$$A_2\equiv 2\sin(2\gamma)\left[-3\Omega\sin\Omega+(\Omega^2-3)\cos\Omega+e^{-H}\left[\frac{H}{2}(L^{-1}+6)+\frac{L^{-2}}{8}+\frac{3}{2}L^{-1}-\Omega^2+3\right]\right]\ . \tag{C13}$$

(b) $\mathbf{A}_I$. Definition (60) is equivalent to

$$\mathbf{A}_I=-\tfrac{1}{2}\nabla_{\mathbf{x}}\int d^3\mathbf{k}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{I_{21}(\mathbf{k})}{\omega^2/c^2-k^2}\ . \tag{C14}$$

This integral is the same as in (43):

$$\mathbf{A}_I=\frac{4\pi^3\epsilon_0}{\cos(\omega t)}\nabla_{\mathbf{x}}\phi_{12}\ . \tag{C15}$$

By symmetry, the component of the gradient orthogonal to the plane $(\mathbf{e}_3,\mathbf{x})$ does not contribute to (C14):

$$\nabla=\mathbf{n}\frac{\partial}{\partial r}-\mathbf{j}\frac{1}{r}\frac{\partial}{\partial\gamma}+\cdots \tag{C16}$$

($\mathbf{n}$, $\mathbf{j}$, and γ defined as above). Replacing ϕ_{12} in (C15) by its expression (50), and using (C16), we obtain

$$\mathbf{A}_I=-6\frac{\sqrt{2}}{a^5}\pi^2r^3L^3\left\{\mathbf{n}\cos\gamma\left[2\Omega\sin\Omega+(2-\Omega^2)\cos\Omega-e^{-H}\left[\frac{L^{-2}}{8}-\frac{L^{-2}}{8H}+\frac{L^{-1}H}{2}+H^2+2H+2\right]\right]\right.$$
$$\left.+\mathbf{j}\sin\gamma\left[-\Omega\sin\Omega-\cos\Omega+e^{-H}\left[\frac{L^{-2}}{8H}+\frac{L^{-1}}{2}+H+1\right]\right]\right\}\ . \tag{C17}$$

(c) The total vector potential $\mathbf{A}$ is obtained by substituting (C11)–(C13) and (C17) into (57), and expressing vectors $\mathbf{j}$ and $\mathbf{j}'$ with respect to $\mathbf{n}$ and $\mathbf{e}_3$:

$$\mathbf{j}'\sin\gamma=\mathbf{e}_3\cos\gamma-\mathbf{n}\ , \tag{C18}$$

$$\mathbf{j}\sin\gamma=\mathbf{e}_3-\mathbf{n}\cos\gamma\ . \tag{C19}$$

This leads to expression (61).

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