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Exact study of adiabaticity

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A practical example, which admits a fully exact solution, is presented and then employed to study adiabaticity. The example consists of an oscillator whose oscillation frequency changes at a finite rate within a finite time interval. An analytically exact formula is given for the adiabatic ratio, which is a good measure of the adiabaticity. General and practical features of the adiabaticity, including the adiabatic theorem, are presented.

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We shall consider a time-dependent harmonic oscillator described by the Hamiltonian

$$H(x,p;t)=\tfrac{1}{2}[p^2+\omega^2(t)x^2]\ , \tag{1}$$

where $\omega(t)$ is a time-varying parameter. It is well known that, for a sufficiently slow change in $\omega(t)$, the action variable of system (1) is approximately constant, which we designate as an adiabatic constant. The proof for adiabatic invariance of the action integral has been given many times in the literature [1]. Unfortunately, the adiabatic theorem does not tell quantitatively how slow the change in the parameter must be for adiabaticity to hold. When the change in $\omega(t)$ is rapid, rather than nearly adiabatic, it may be possible to estimate an increase in the action integral but, so far, only approximate methods to calculate this increase have been presented [2–5] by several authors. In particular, serious work, by a number of authors, has been restricted to the following two cases:

$$\omega(t)=[\omega_\infty-(\omega_\infty-\omega_0)\exp(-\beta^2t^2)]\ , \tag{2}$$

where ω_∞ is the asymptotic limit $t\to\pm\infty$, ω_0 is the value at $t=0$; and

$$\omega(t)=\omega_0[\tfrac{1}{2}(n+1)+\tfrac{1}{2}(n-1)\tanh(\alpha t)]\ , \tag{3}$$

where n is an integer. Calculational methods in the literature require an infinite time interval for the change of parameter $\omega(t)$; the requirement cannot be met in most real cases. In addition, difficulties associated with the initial phase are unavoidable, if not insurmountable.

In order to avoid the initial phase problem, a simple technique to estimate an increment in the action integral by observing the Courant-Snyder invariant [6,7] curve has been proposed by Symon [8] and revived by Takayama [9]. This method can also be applied in a typical application where $\omega(t)$ changes at a finite varying rate and in a finite time interval. Unfortunately, the example treated in Ref. [9] did not admit exact analytic solutions because an auxiliary function describing the behavior of the Courant-Snyder invariant in the phase space cannot be written in a fully analytic form. As a consequence, the result was an approximate expression for the adiabatic ratio which will be defined later, and the validity of the expression is limited to the nearly adiabatic case. Accordingly, the question of the adiabaticity in a practical case, where the parameter $\omega(t)$ continuously changes from a constant value ω_1 to another constant value ω_2 in a finite time interval $\tau=t_2-t_1$, as shown in Fig. 1, has been left unsolved.

Recently a class of solvable Hill equations and their solutions have been found [10–12] and studies on the application [13] of those purely mathematical results to physics have been initiated. It is worth noting that the time-varying coefficient of such solvable Hill equations is directly applicable to the subject of the time variation of $\omega(t)$. This means that the auxiliary function can be obtainable in a completely analytic form through the entire time region because of a direct relationship [7] between the Hamiltonian (1) and the auxiliary function [14]. In this article we employ this mathematical result and give an exact solution for the adiabaticity problem, that is, an analytically exact expression of the adiabatic ratio. The result will be described in terms of a normalized time interval measured by the early oscillation period $2\pi/\omega_1$ and a frequency multiplication parameter $k=\omega_2/\omega_1$. As a result, a given formula may be used in a straightforward manner in many fields of physics or other sciences. The formula will confirm an empirical rule well known in accelerator physics that, in any system as described by (1), τ must be longer than $2\pi/\omega_1$ to minimize the increment of

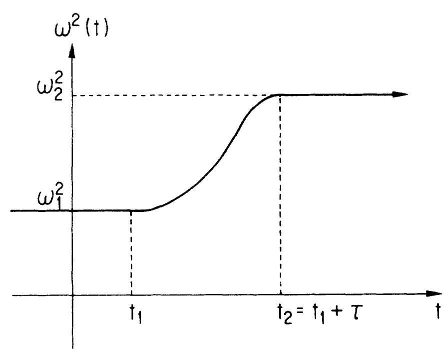

FIG. 1. Time-varying coefficient $\omega^2(t)$.

action variable to less than a few percent.

The Courant-Snyder invariant is written as

$$I(x,p;t)\equiv\frac{1}{2\beta(t)}\{x^2+[\tfrac{1}{2}\dot\beta(t)x-\beta(t)p]^2\}=I_0\ , \tag{4}$$

where $\beta(t)$ satisfies the auxiliary differential equation

$$\tfrac{1}{2}\beta\ddot\beta-\tfrac{1}{4}\dot\beta^2+\omega^2(t)\beta^2=1\ . \tag{5}$$

When $\omega(t)$ is constant, say ω_1, the invariant I is exactly identical to the action variable of the system (1) provided the initial conditions $\beta(-\infty)=1/\omega_1$ and $\dot\beta(-\infty)=0$ are chosen. The Courant-Snyder invariant curve is an ellipse in the phase space characterized by its maximal edges on the x and p axes, ξ and δ, which are functions of $\beta(t)$ and $\dot\beta(t)$, $\xi(t)=\sqrt{2I_0\beta(t)}$, $\delta(t)=\sqrt{2I_0\gamma(t)}$, where $\gamma(t)=[1+\dot\beta^2(t)/4]/\beta(t)$. Before the Hamiltonian changes, the form of the ellipse remains unchanged and its motion is simply a parallel displacement along the time axis. At $t=t_1$ where $\omega(t)$ starts to change, the ellipse begins to move, following the time evolution of $\beta(t)$. After the change in $\omega(t)$ is completed at $t=t_2$, the ellipse tumbles in the phase space because of so-called mismatching, unless $(\beta(t_2),\dot\beta(t_2))$ is equal to $(1/\omega_2,0)$, as seen in Fig. 2.

We write the phase-space area surrounded by the invariant curve which remains constant at $t<t_1$ by $S_0=2\pi I_0$. Then we designate as S the area of the outer envelope of the ellipse tumbling after $t=t_2$; S is given by $S=\pi\max\xi(t)\max\delta(t)$. Since the phase-space area between the outer and inner envelopes is swept by phase points which have the same action integral at $t\le t_1$, it is reasonable to introduce the adiabatic ratio $R=S/S_0$ or $[\max\beta(t)\max\gamma(t)]^{1/2}$ as a measure of the adiabaticity. A solution of (5) with a constant $\omega=\omega_2$ is well known,

$$\beta(t)=(A^2+B^2+1/\omega_2^2)^{1/2}+A\cos[2\omega_2(t-t_2)] \\ +B\sin[2\omega_2(t-t_2)]\ , \tag{6}$$

$$\dot\beta(t)=2\omega_2\{-A\sin[2\omega_2(t-t_2)] \\ +B\cos[2\omega_2(t-t_2)]\}\ , \tag{7}$$

where A and B satisfy the boundary conditions $\beta(t_2)=(A^2+B^2+1/\omega_2^2)^{1/2}+A$ and $\dot\beta(t_2)=2\omega_2 B$. From (6) and (7), the maximum values of $\beta(t)$ and $\gamma(t)$ are easily obtained,

$$\max\beta(t)=(A^2+B^2+1/\omega_2^2)^{1/2}+(A^2+B^2)^{1/2}\ , \tag{8a}$$

$$\max\gamma(t)=1/[(A^2+B^2+1/\omega_2^2)^{1/2}-(A^2+B^2)^{1/2}]\ . \tag{8b}$$

Substituting (8a) and (8b) into the expression for R, we have

$$R=[\omega_2^2(A^2+B^2)+1]^{1/2}+[\omega_2^2(A^2+B^2)]^{1/2}\ . \tag{9}$$

Replacement of A^2+B^2 with terms of the auxiliary function and its time derivative leads to $R=X+(X^2-1)^{1/2}$ where

$$X=\frac{\omega_2}{2}\left[\beta(t_2)+\frac{\dot\beta^2(t_2)}{4\omega_2^2\beta(t_2)}+\frac{1}{\omega_2^2\beta(t_2)}\right]\ . \tag{10}$$

Only the values of the auxiliary function and its time derivative at $t=t_2$ are required to obtain an exact expression of the adiabatic ratio.

There is an infinite number of paths connecting two fixed points $(t_1,\omega(t_1)=\omega_1)$ and $(t_2,\omega(t_2)=\omega_2)$. In the case of $\tau=t_2-t_1\ll 2\pi/\omega_1$, the adiabatic ratio should be very sensitive to a selected path. Which path minimizes the adiabatic ratio for a fixed condition? It is difficult to answer this question, but by numerically solving the auxiliary equation, we have determined the adiabatic ratio for

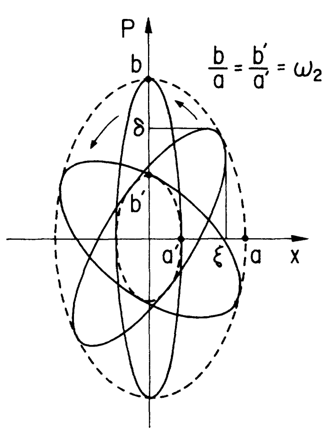

FIG. 2. Outer and inner envelopes of the tumbling Courant-Snyder ellipse.

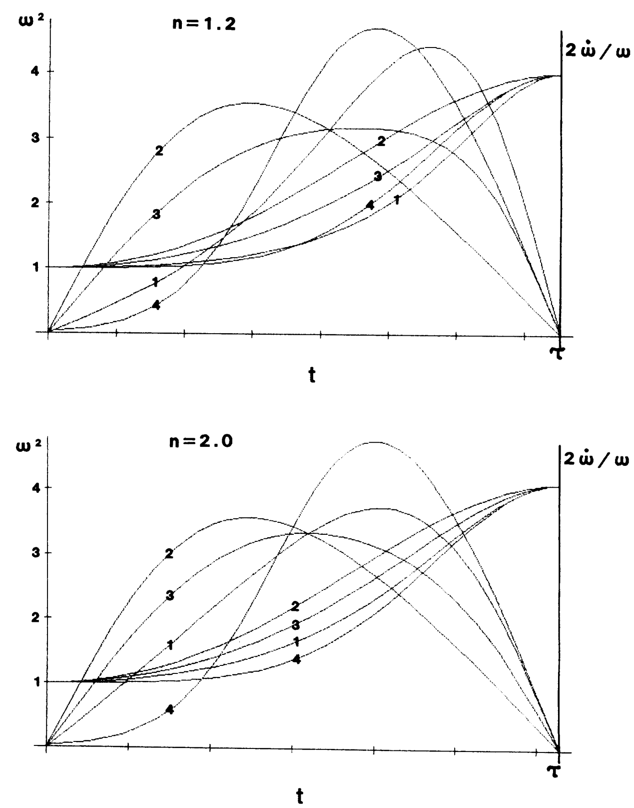

FIG. 3. Possible paths $\omega^2(t)$ and their normalized derivative $2\dot\omega(t)/\omega(t)$ for $n=2$ and 1.2 and $k=2$. Numbers denote each of the cases mentioned in the text. $2\dot\omega(t)/\omega(t)$ is shown in a relative scale.

possible paths where $\omega(t)$ is continuous on both boundaries and $\dot{\omega}(t)/\omega(t)$ has one maximum value. From this result, we observe that a relatively low $\dot{\omega}/\omega$ in the early stage $t<\tau/2$, coupled with a small maximum value of $\dot{\omega}/\omega$, through transition leads to a relatively small adiabatic ratio. This observation will be recovered later in a comparison between possible cases. When the minimal adiabatic ratio is desirable, the path should be chosen so as to satisfy these requirements.

As an example, the following model shown as case (1) in Fig. 3 is considered:

$$\omega^2(t)=\frac{\nu^2}{4}\left[1+\frac{\lambda(1-G^2)}{(1+G\cos\nu t)^2}\right](|G|<1)\ , \tag{11}$$

where π/ν is the time interval of change and

$$\lambda=\{[(2\omega_1/\nu)^2-1][(2\omega_2/\nu)^2-1]\}^{1/2}\ ,$$

$$G=\frac{\lambda+1-(2\omega_1/\nu)^2}{\lambda-1+(2\omega_1/\nu)^2}\ .$$

This model will be shown later to be close to the optimum path. Solutions of a Hill equation, $\ddot{x}+\omega^2(t)x=0$, can be shown [12] to be

$$\left.\begin{matrix}x_1\\x_2\end{matrix}\right\}=(1+G\cos\nu t)^{1/2}\times\left\{\begin{matrix}\sin\left[\dfrac{\sqrt{1+\lambda}}{2}\Theta(t)\right]\\ \cos\left[\dfrac{\sqrt{1+\lambda}}{2}\Theta(t)\right]\end{matrix}\right.\ , \tag{12}$$

where $\Theta(t)$ satisfies $\sin\Theta(t)=(1-G^2)^{1/2}\sin\nu t/(1+G\cos\nu t)$. A relationship between the above solutions and the auxiliary function is given by

$$\beta(t)=c_1x_1^2(t)+c_2x_2^2(t)+2(c_1c_2-1/W^2)^{1/2}x_1(t)x_2(t)\ , \tag{13}$$

where c_1 and c_2 are determined from the initial conditions $\beta(0)=1/\omega_1$ and $\dot{\beta}(0)=0$, $(t_1=0)$, while W is the Wronskian, $W=x_1(0)\dot{x}_2(0)-\dot{x}_1(0)x_2(0)$. After tedious manipulations, we obtain

$$\beta(\pi/\nu)=\frac{4\omega_1}{\nu^2(1+\lambda)}\sin^2\frac{\sqrt{1+\lambda}}{2}\pi+\frac{1}{\omega_1}\left[\frac{1-G}{1+G}\right]\cos^2\frac{\sqrt{1+\lambda}}{2}\pi\ , \tag{14}$$

$$\dot{\beta}(\pi/\nu)=\sqrt{1+\lambda}\left[\frac{2\omega_1}{\nu(1+\lambda)}\left[\frac{1+G}{1-G}\right]^{1/2}-\frac{\nu}{2\omega_1}\left[\frac{1-G}{1+G}\right]^{1/2}\right]\sin\sqrt{1+\lambda}\pi\ . \tag{15}$$

By introducing the relative time-interval parameter $n=2\omega_1/\nu$ and the frequency multiplication parameter $k=\omega_2/\omega_1$, we have $\lambda=[(n^2-1)(n^2k^2-1)]^{1/2}$ and $G=(\lambda+1-n^2)/(\lambda-1+n^2)$; then, we can simplify the expression for X in Eq. (10):

$$X=\frac{1}{2}\left[D_1+\frac{D_2}{4D_1}+\frac{1}{D_1}\right]\ , \tag{16}$$

where

$$D_1\equiv\omega_2\beta(\pi/\nu)=k\left[\frac{n^2}{1+\lambda}\sin^2\frac{\sqrt{1+\lambda}}{2}\pi+\frac{n^2-1}{\lambda}\cos^2\frac{\sqrt{1+\lambda}}{2}\pi\right]\ , \tag{17}$$

$$D_2\equiv\dot{\beta}^2(\pi/\nu)=(1+\lambda)\left[\frac{n^2\lambda}{(1+\lambda)^2(n^2-1)}+\frac{n^2-1}{n^2\lambda}-\frac{2}{1+\lambda}\right]\sin^2\sqrt{1+\lambda}\pi\ . \tag{18}$$

We can show the usefulness of the adiabatic ratio R by using Eq. (16) to study the fully adiabatic limit. In the limit of $n\to\infty$, $D_1=1$ and $D_2=0$ for an arbitrary k; therefore $X=1$ and $R=X+(X^2-1)^{1/2}=1$. In order to demonstrate the nature of the adiabaticity qualitatively as well as quantitatively, the adiabatic ratio for $k=2$ is plotted as a function of n in Fig. 4 [15]. As may be expected, the adiabatic ratio is shown to approach unity asymptotically as the relative time interval is increased. Up to $n=4$, which corresponds to one oscillation period before change, it diminishes rapidly down to a few percent. It is also interesting to see the adiabatic ratio as a function of k for $n=4$. From Fig. 5 one sees that the adiabatic ratio approaches a stationary value of 1.03 as k increases. From the asymptotic expression of R in the limit of $k\to\infty$, $R=D_1=n/(n^2-1)^{1/2}$, we can also derive $R=1.0328$ for $n=4$. The saturation may be understandable by noting that the system (1) experiences the time interval of change expanding as $\omega(t)$ increases. Thus the empirical rule previously stated is theoretically confirmed. Prior to this work, those characteristics of the adiabaticity, except for the adiabatic theorem itself, have been demonstrated only through computer simulations.

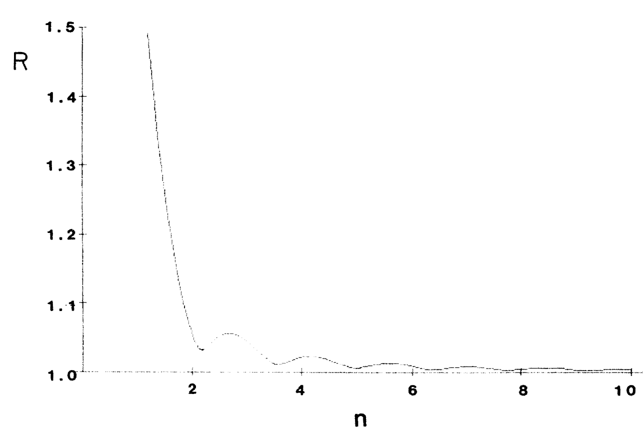

FIG. 4. The adiabatic ratio vs the normalized time interval n for $k=2$ where $n=4$ corresponds to $2\pi/\omega_1$.

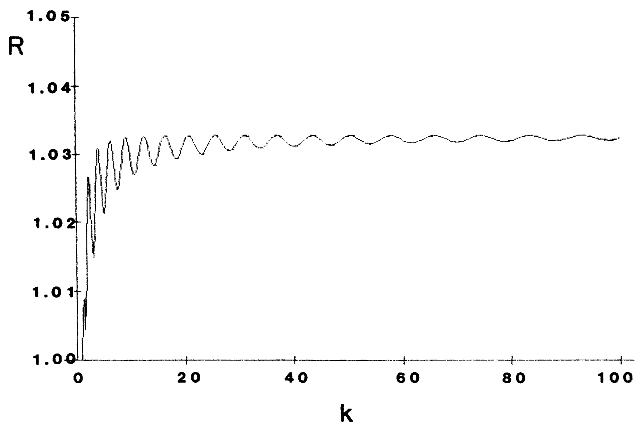

FIG. 5. The adiabatic ratio vs the frequency multiplication parameter k for $n=4$.

In order to manifest the universal validity of the model, a comparison is made with other cases of possible paths. The selected possible cases are (1) $\omega_{\mathrm{I}}^2(t)$, the present model; (2) $\omega_{\mathrm{II}}^2(t)=\{\frac{1}{2}[(1+k)+(1-k)\cos(2t/n)]\}^2$; (3) $\omega_{\mathrm{III}}^2(t)=\frac{1}{2}[\omega_{\mathrm{I}}^2(t)+\omega_{\mathrm{II}}^2(t)]$; and (4) $\omega_{\mathrm{IV}}^2(t)=(k^2-1)\exp[-0.8(2/n)^2(n/2-t)^2]+1$ [16], $0\le t\le n/2$. Figure 3 shows those paths $\omega^2(t)$ and their normalized time derivatives $2\dot{\omega}/\omega$ for $n=1.2$ and 2 and for $k=2$. The adiabatic ratio calculated numerically, except for case (1), is given for the range of $1<n\le 4$ in Fig. 6. The results support the previously stated observation with respect to the optimum path. For $n<1.8$, the present model seems to be somewhat off the optimum path as there apparently exist paths giving smaller values of the adiabatic ratio; however, the difference is less than

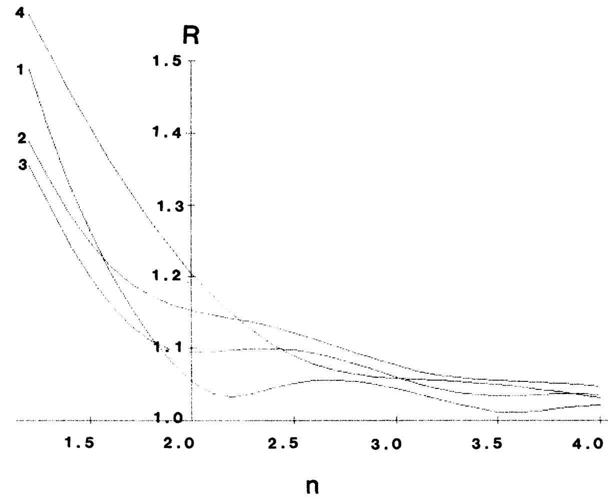

FIG. 6. The adiabatic ratios vs the normalized time interval in the range of $1<n\le 4$ for $k=2$.

20%. Although it is not shown here, we can prove in a similar manner that the model is close to the ideal path for larger k. From those comparisons, and the analytically evaluated results, we may conclude that the model is valid for quantitative estimation of the adiabaticity.

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[16] Strictly speaking, the case (4) does not satisfy the boundary conditions $\omega(t_1)=\omega_1$ and $\dot{\omega}(t_1)=0$; however, its effect is not a notable size.