Čerenkov emission from an axial-wiggler-magnetoactive plasma

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The electrodynamics of a relativistic electron beam passing through a nonrelativistic plasma embedded in an axial-wiggler magnetic field has been worked out in the framework of nonequilibrium statistical mechanics as developed by Prigogine and co-workers. The emerging radiation has a synergic frequency that is a combination of the original frequency, the plasma frequency, and the cyclotron frequencies due to the axial and wiggler magnetic fields. An *exact* summation of the relevant diagrams in the Dyson series corresponding to the self-consistent-field approximation has brought out non-Markovian nonlinear aspects of optics. The change in polarization has been evaluated due to the presence of magnetic fields. The axial and wiggler fields remove the degeneracy in the electronic states, and the summation over the Landau levels discriminates between the spin-up and spin-down species. The measurement of the energy loss enables one to make an estimate of the population of these species.

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I. INTRODUCTION

We propose to investigate the electrodynamics of a relativistic electron beam passing through a nonrelativistic plasma embedding axial (Ω_0) and wiggler (Ω) magnetic fields specified by their cyclotron frequencies in the framework of nonequilibrium statistical mechanics as developed by Prigogine and co-workers [1]. The linear dispersion relation in this problem has been worked out because of its relevance to free-electron lasers and is given in an excellent review by Roberson and Sprangle [2]. The nonlinear aspect has recently been investigated by Pratap and Sen [3] in the absence of an axial field starting from the Vlasov set of equations in a nonperturbative manner, to study the nonlinear saturation properties. However, it is by now well known that nonlinear dynamics results in the phenomenon of synergism [4–6]. It was shown that a quantum-mechanical treatment of this problem in both tapered [7] and helical [8] magnetic fields introduce only small corrections over the classical results. However, in a Dirac theory of an electron passing through an axial-wiggler field [9] the axial component would introduce Landau splitting of energy levels and the wiggler field would remove degeneracy [see Eq. (29)].

The most significant results of the present analysis are the following: (a) An axial magnetic field introduces the Landau energy-level quantization; it is well known that the results are different for the spin-up and spin-down species. The summation over the Landau levels, however, results in different density profiles; this enables one to determine experimentally the species ratios in the incident beam. (b) This difference, however, does not manifest itself in the classical nonrelativistic plasma which constitutes the medium. In this case the density profile has a single maximum as a function of $\beta\hbar\Omega_0/2$, β being the Boltzmann factor $(k_B T)^{-1}$. For higher values of this parameter the number density decreases and thereby restricts the domain. (c) The nonlinear interactions in the system produce synergism in the frequency and for certain specific values of the axial and wiggler synchrotron frequencies, the radiation passes through the system unimpaired, or "ducts" through the system. (d) The optics of the system is contained in an expression for the refractive index (40) in terms of effective radiation frequency (36). This is dependent on the plasma density through χ as given by (A19) as also through $\bar{v}_\lambda^2$ as defined in the Hamiltonian (7). Also there is a change in polarization. One can define an effective polarization as $\sqrt{\chi}\mathbf{e}_\lambda$. (e) We shall also define an expression for the Čerenkov cone angle (32′) which is a function of the synchrotron frequencies Ω_0 and Ω, and also the plasma frequency $\omega_{\rm pl}$ through $\bar{v}_\lambda$ and σ. The axis of the cone is tangential to the effective electron trajectory and as the trajectory wobbles, so does the axis and thereby the cone.

The paper is organized in the following manner. Section II describes the Hamiltonian of the system consisting of a relativistic test particle interacting with a nonrelativistic plasma in an axial-wiggler magnetic field and ambient radiation field. Sec. III gives the Liouville equation in the natural coordinates of the system together with the formal solution of the Liouville equation in the resolvant formalism. Section IV gives a discussion of the operators which are noncommuting and hence a Baker-Hausdorff expansion is effected. Sec. V gives the one-particle distribution function (OPDF) together with the collective modes given by the response function [9], the explicit evaluation of which is given in the Appendix. An evaluation of this requires an initial state for the test particle which is constructed out of wave functions corresponding to the unperturbed part of the relativistic Hamiltonian [8] which is given in Sec. VI. Using the OPDF, the energy loss suffered by the particle is evaluated in Sec. VII. In this section we have also evaluated the Čerenkov condition and the modification effected due to the two magnetic fields. An expression for the refractive index has been derived and this and Sec. VIII explains the optics of the system. Section VIII discusses the change in polarization due to the two magnetic fields. The linearized results obtained earlier [10,11] have been derived as special cases from the present formalism in Sec. IX. The paper closes with Sec. X giving the main conclusions.

II. HAMILTONIAN OF THE SYSTEM

We propose to solve the Liouville equation defined in a $6N$-dimensional phase space formally, and find the one-particle distribution function by integrating over all the variables except that of the test particle. The subset of infinite diagrams selected from the Dyson series on the basis of the interaction time scale $[(m/ce^2)^{1/2}$, m and e being the electronic mass and charge and c the concentration in the thermodynamic limit $N\to\infty;\ v\to\infty,\ N/v=c$, a constant] is then summed up to obtain the OPDF, due to self-consistent-field approximation or the ring approximation. We now average the test particle Hamiltonian with this OPDF; differentiating with respect to time, we get the rate of energy loss and evaluate power loss per unit time and per unit length. We cast this into the usual Čerenkov form as given by Frank and Tamm [12].

The system under consideration consists of a relativistic test particle (T), a system of field particles denoted by l, and a radiation field (λ) embedded in a space containing an axial magnetic field (denoted by the cyclotron frequency Ω_0) and wiggler field (Ω). Thus the total Hamiltonian is

$$\mathcal{H}=\mathcal{H}_T+\sum_l \mathcal{H}_l+\sum_\lambda \mathcal{H}_\lambda \ . \tag{1}$$

$\mathcal{H}_T$ and $\mathcal{H}_l$ are functions of radiation field components through the vector potential, hence this is not a strictly separable Hamiltonian. Interactions in the system are taken into account through the virtual-photon creation-annihilation processes.

The test particle is relativistic and the Hamiltonian is written as

$$\mathcal{H}_T=mc^2(1+\mathbf{u}_T^2)^{1/2} \ , \tag{2}$$

where $\mathbf{u}_T$ is defined as

$$mc\mathbf{u}_T=\tilde{\mathbf{P}}_T+\frac{m\Omega_o}{2}(\mathbf{j}x-\mathbf{i}y)+\frac{m\Omega}{k_w}(\mathbf{i}\cos k_w z+\mathbf{j}\sin k_w z)-\frac{e}{c}\mathbf{A} \ , \tag{3}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in a Cartesian coordinate system, and $k_w=2\pi/\lambda_w$ is the wave number corresponding to the wiggler field. In the above $\tilde{\mathbf{P}}_T$ is the canonical momentum, the second term is the vector potential for the axial field, the third is that due to the wiggler field, and the last term is the interaction vector potential $\mathbf{A}$ which is a function of $\mathbf{q}_T$ the position vector of the test particle as well as J_λ and w_λ, the radiation field variables in action-angle representation. We consider the transverse component of the radiation field so that $\mathbf{e}_\lambda\cdot\mathbf{K}_\lambda=0$, $\mathbf{e}_\lambda$ being the polarization vector and $\mathbf{K}_\lambda$ the propagation vector.

The nonrelativistic medium particles Hamiltonian is given by

$$\mathcal{H}_l=\frac{1}{2m_l}\left[\mathbf{P}_l-\frac{e_l}{c}\mathbf{A}_l\right]^2 \ , \tag{4}$$

where

$$\mathbf{P}_l=\tilde{\mathbf{P}}_l+\frac{m_l\Omega_0}{2}(\mathbf{j}x_l-\mathbf{i}y_l)+\frac{m_l\Omega}{k_w}(\mathbf{i}\cos k_w z+\mathbf{j}\sin k_w z) \ . \tag{5}$$

It should be noted that the interaction vector potential is not included in the definition of $\mathbf{P}_l$ in (5).

The radiation field is given by the Hamiltonian of a system of harmonic oscillators in action-angle representation as

$$\mathcal{H}_\lambda=\sum_\lambda \nu_\lambda J_\lambda \ . \tag{6}$$

If we expand (4), the quadratic term in the vector potential would change the radiation frequency ν_λ to $\bar{\nu}_\lambda$ in random-phase approximation [6] where $\bar{\nu}_\lambda^2=\nu_\lambda^2+\omega_{\rm pl}^2$. Hence we write the Hamiltonian of the system as

$$\mathcal{H}_T=mc^2(1+u^2)^{1/2} \ ,$$
$$\mathcal{H}_l=\frac{\mathbf{P}_l^2}{2m_l}-\frac{e_l}{m_l c}(\mathbf{P}_l\cdot\mathbf{A}_l) \ , \tag{7}$$
$$\mathcal{H}_\lambda=\sum_\lambda \bar{\nu}_\lambda J_\lambda$$

and we shall use the above Hamiltonian in writing the Liouville equation.

III. LIOUVILLE EQUATION

We define a Liouville density ρ in the $6N$-dimensional phase space as

$$\rho=\rho(\mathbf{q}_T,\mathbf{u}_T;\mathbf{P}_l,\mathbf{q}_l;J_\lambda,\omega_\lambda;t) \tag{8}$$

and this density satisfies the Liouville equation as

$$\frac{\partial\rho}{\partial t}+\dot{\mathbf{u}}_T\cdot\frac{\partial\rho}{\partial\mathbf{u}_T}+\dot{\mathbf{q}}_T\cdot\frac{\partial\rho}{\partial\mathbf{q}_T}+\dot{\mathbf{P}}_l\cdot\frac{\partial\rho}{\partial\mathbf{P}_l}+\dot{\mathbf{q}}_l\cdot\frac{\partial\rho}{\partial\mathbf{q}_l}+\dot{J}_\lambda\frac{\partial\rho}{\partial J_\lambda}+\dot{\omega}_\lambda\frac{\partial\rho}{\partial\omega_\lambda}=0 \ . \tag{9}$$

The time derivatives appearing in (9) are obtained from Hamilton's equations and we write the Liouville equation as

$$\frac{\partial\rho}{\partial t}+\mathcal{L}\rho=e(\delta L)\rho \ , \tag{10}$$

where

$$\mathcal{L}=\mathcal{L}_T+\mathcal{L}_l+\mathcal{L}_\lambda \tag{11}$$

with

$$\mathcal{L}_T=c\boldsymbol{\beta}\cdot\frac{\partial}{\partial \mathbf{q}_T}+\frac{\bar{\Omega}}{\gamma}\boldsymbol{\epsilon}_3\times\boldsymbol{\beta}\cdot\frac{\partial}{\partial\boldsymbol{\beta}}\ ,$$
$$\mathcal{L}_l=\frac{\mathbf{P}_l}{m_l}\cdot\frac{\partial}{\partial \mathbf{q}_L}+\bar{\Omega}\boldsymbol{\epsilon}_3\times\mathbf{P}_l\cdot\frac{\partial}{\partial \mathbf{P}_l}\ , \tag{12}$$
$$\mathcal{L}_\lambda=\bar{v}_\lambda\frac{\partial}{\partial\omega_\lambda}\ .$$

These equations are written in coordinates natural to the present system defined as

$$\boldsymbol{\epsilon}_1=\frac{\Omega}{\bar{\Omega}}\hat{\mathbf{e}}_3+\frac{\Omega_0}{\bar{\Omega}}\hat{\mathbf{e}}_1=\frac{\Omega_0}{\bar{\Omega}}(\mathbf{i}\cos k_w z+\mathbf{j}\sin k_w z)+\frac{\Omega}{\bar{\Omega}}\mathbf{k}\ ,$$
$$\boldsymbol{\epsilon}_2=\hat{\mathbf{e}}_2=-\mathbf{i}\sin k_w z+\mathbf{j}\cos k_w z\ , \tag{13}$$
$$\boldsymbol{\epsilon}_3=\frac{\Omega_0}{\bar{\Omega}}\hat{\mathbf{e}}_3-\frac{\Omega}{\bar{\Omega}}\hat{\mathbf{e}}_1=\frac{\Omega_0}{\bar{\Omega}}\mathbf{k}-\frac{\Omega}{\bar{\Omega}}(\mathbf{i}\cos k_w z+\mathbf{j}\sin k_w z)\ ,$$

where $\bar{\Omega}=(\Omega_0^2+\Omega^2)^{1/2}$. It may be seen that by setting $\Omega_0=0$ and $\Omega/\bar{\Omega}=1$, the above reduces to the usual wiggler coordinates relabeled. One also realizes the fact that the Jacobian of transformation from the Cartesian as well as the wiggler coordinates to the present one is unity.

The operators

$$\delta L=\mathcal{A}_T+\mathcal{A}_l+\mathcal{B}_T+\mathcal{B}_l \tag{14}$$

are given by

$$\mathcal{A}_T=\left[\frac{1}{m_T}\left[\frac{8}{v\bar{v}_\lambda}\right]^{1/2}a_\lambda\right]\frac{\partial}{\partial\mathbf{u}_T}\cdot\left[-\frac{\bar{v}_\lambda}{c}\mathbf{e}_\lambda(\sqrt{J_\lambda}\sin\omega_\lambda)\cos(\mathbf{K}_\lambda\cdot\mathbf{q}_T)+\boldsymbol{\beta}\times(\mathbf{K}_\lambda\times\mathbf{e}_\lambda)(\sqrt{J_\lambda}\cos\omega_\lambda)\sin(\mathbf{K}_\lambda\cdot\mathbf{q}_T)\right]\ ,$$
$$\mathcal{A}_l=\left[\frac{1}{m_l}\left[\frac{8}{v\bar{v}_\lambda}\right]^{1/2}a_\lambda\right]\sqrt{J_\lambda}\cos\omega_\lambda)\left[\frac{\partial}{\partial\mathbf{q}_l}\cdot\mathbf{e}_\lambda+m\bar{\Omega}\frac{\partial}{\partial\mathbf{P}_l}\cdot(\boldsymbol{\epsilon}_3\times\mathbf{e}_\lambda)\right]\cos(\mathbf{K}_\lambda\cdot\mathbf{q}_l)\ ,$$
$$\mathcal{B}_T=\left[\frac{8c^2}{v\bar{v}_\lambda}\right]^{1/2}a_\lambda\left[(\boldsymbol{\beta}\cdot\mathbf{e}_\lambda)\cos(\mathbf{K}_\lambda\cdot\mathbf{q}_T)(\sqrt{J_\lambda}\sin\omega_\lambda)\frac{\partial}{\partial J_\lambda}\right]\ , \tag{15}$$
$$\mathcal{B}_l=\frac{1}{m_l}\left[\frac{8}{v\bar{v}_\lambda}\right]^{1/2}a_\lambda\left[(\mathbf{P}_l\cdot\mathbf{e}_\lambda)\cos(\mathbf{K}_\lambda\cdot\mathbf{q}_l)(\sqrt{J_\lambda}\sin\omega_\lambda)\frac{\partial}{\partial J_\lambda}\right]\ .$$

In writing (15), we have used the vector potential

$$\mathbf{A}_\lambda=\left[\frac{8c^2}{v\bar{v}_\lambda}\right]^{1/2}a_\lambda\mathbf{e}_\lambda[(\sqrt{J_\lambda}\cos\omega_\lambda)\cos(\mathbf{K}_\lambda\cdot\mathbf{q})+(\sqrt{J_{-\lambda}}\cos\omega_{-\lambda})\sin(\mathbf{K}_\lambda\cdot\mathbf{q})]\ . \tag{16}$$

In the above, we have used the notation $\mathbf{u}_T/(1+\mathbf{u}_T^2)^{1/2}=\boldsymbol{\beta}$; this gives $(1+\mathbf{u}_T^2)^{1/2}=(1-\boldsymbol{\beta}^2)^{-1/2}=\gamma$ as the Lorentz factor. a_λ is the magnitude of the vector potential and $\mathbf{e}_\lambda$ is the unite polarization vector. a_λ is dimensionless and denotes the interaction strength. In the above operators, we have taken only the λ part. The $-\lambda$ part is obtained by replacing $\cos(\mathbf{K}_\lambda\cdot\mathbf{q})$ by $\sin(\mathbf{K}_\lambda\cdot\mathbf{q})$ and $\sin(\mathbf{K}_\lambda\cdot\mathbf{q})$ by $-\cos(\mathbf{K}_\lambda\cdot\mathbf{q})$. Also we have retained only one term in the Poisson bracket in the operators $\mathcal{B}$, since we assume the initial state to be angle independent.

We can now formally integrate Eq. (10) and write the solution as

$$\rho(t)=e^{-\mathcal{L}t}\rho(0)+e\int_0^t d\tau\, e^{-\mathcal{L}(t-\tau)}(\delta L)\rho(\tau)\ , \tag{17}$$

where $\rho(0)$ is the initial state of the system (at $t=0$). Equation (17) consists of two parts, the first one being the transport of the initial state to time t without any interaction or the free flow term. The second term however gives all the correlations in the system and thereby gives all the dynamics, since (δL) contains operators which annihilate and create correlations in the system.

If we iterate the above equation (17), we get the Dyson series

$$\rho(t)=\sum_{n=0}^{\infty}e^n\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n e^{-\mathcal{L}(t-t_1)}(\delta L)e^{-\mathcal{L}(t_1-t_2)}(\delta L)\cdots e^{-\mathcal{L}(t_{n-1}-t_n)}(\delta L)e^{-\mathcal{L}t_n}\rho(0)\ . \tag{18}$$

Equation (18) contains all the information in the system. $n=0$ gives the first term in (17), $n=2$ gives the interaction of the particle with the field, $n=4$ gives the interaction of the test particle with the field and a field particle, and so on, all in terms of the initial state $\rho(0)$. This constitutes the series having terms with coefficients (e^2c/m) where c is the concentration as defined before. The above series (18) can be written in the resolvant form as

$$\rho(t)=\frac{1}{2\pi}\sum_{n=0}^{\infty}e^n\int dz\, e^{-izt}R^0(z)[(\delta L)R^0(z)]^n\rho(0)\ , \tag{19}$$

where we have defined a Laplace transform as

$$\mathcal{A}(z)=\int e^{izt}\mathcal{A}(t)dt \tag{20}$$

and the corresponding inverse. The diagrams such as $\langle 0|\delta L|0\rangle$, $\langle 0|\delta L|n\rangle\langle n|\delta L|0\rangle\cdots$ with the initial and final states the same form the ring approximation or the self-consistent series, which when summed up gives the solution in self-consistent-field approximation. In the above $R^0(z)$ is the Laplace transform of the propagator $e^{-\mathcal{L}(t_i-t_{i+1})}$ and these operators appear in a convolution form, hence the advantage. This, incidentally, gives the subdynamics which go over to the kinetic regime in the asymptotic limit in time. One can obtain the OPDF by integrating (18) or (19) over all the particle variables except that of the test particle after selecting the diagrams. Each term can be seen to create and annihilate correlations with one, two, three, etc., particles and hence Eq. (18) gives the exact correlation evolution in the system.

IV. OPERATORS

The operators given in (12) and (15) have peculiar properties. It may be seen that the two members in $\mathcal{L}_T$ and $\mathcal{L}_l$ do not give a C number on commutation. Hence $e^{-\mathcal{L}_T\tau}$ and $e^{-\mathcal{L}_l\tau}$ have to be subjected to a Baker-Hausdorff expansion [13]. We then get

$$e^{-\mathcal{L}_T\tau}=\exp\left[-\frac{\alpha\Omega_0}{\omega}(\beta_1\sin\omega\tau+\beta_2\overline{1-\cos\omega\tau})\frac{\partial}{\partial\xi}+\frac{\alpha\Omega_0}{\omega}(\beta_1\overline{1-\cos\omega\tau}-\beta_2\sin\omega\tau)\frac{\partial}{\partial\eta}-\alpha\Omega_0\beta_3\tau\frac{\partial}{\partial\zeta}\right]e^{-\omega\tau(\partial/\partial\theta)}, \tag{21}$$

where

$$\alpha=\left[\frac{mc^2}{\hbar\Omega_0}\right]^{1/2},\quad \omega=\overline{\Omega}/\gamma\ ,\quad \beta_1=\beta_\rho\cos\theta\ ,\quad \beta_2=\beta_\rho\sin\theta$$

and

$$e^{-\mathcal{L}_l\tau}=\exp\left[-\frac{\Omega_0}{\overline{\Omega}}(P_1\sin\overline{\Omega}\tau+P_2\overline{1-\cos\overline{\Omega}\tau})\frac{\partial}{\partial\xi}+\frac{\Omega_0}{\overline{\Omega}}(P_1\overline{1-\cos\overline{\Omega}\tau}-P_2\sin\overline{\Omega}\tau)\frac{\partial}{\partial\eta}-P_3\Omega_0\tau\frac{\partial}{\partial\zeta}\right]e^{-\overline{\Omega}\tau(\partial/\partial\theta)} \tag{22}$$

($P_1=P_\rho\cos\theta$, $P_2=P_\rho\sin\theta$). In writing (21) and (22), the variables ξ, η, and ζ are coordinates in the $(\epsilon_1,\epsilon_2,\epsilon_3)$ system; they are made dimensionless by measuring space in units of Landau length $(\hbar/m\Omega_0)^{1/2}$ and momentum $(\hbar m\Omega_0)^{1/2}$. These will act as shift operators and propagate the system from t_n to t_{n-1} and so on. Also, these operators will operate on terms immediately to the right of this, and *not* on all terms on the right of the operator.

As regards the operators in δL, $\mathcal{A}_T$ will always be on the extreme left and $\mathcal{B}_T$ on the extreme right. The $\mathcal{B}$ vertex is a correlation creation vertex and $\mathcal{A}$ annihilates the correlation. The properties of these operators are discussed in Pratap [12].

V. OPDF

One can evaluate the relevant terms in the series having the coefficient e^2c/m, and on summing the series, the OPDF can be written as [12] [see Eq. (3.7)]

$$\rho(t)=e^2\int_0^t dt_1\int_0^{t_1}dt_2 e^{-\mathcal{L}(t-t_1)}\mathcal{A}_T e^{-\mathcal{L}(t_1-t_2)}\left[\int dz\, e^{-iz(t_1-t_2)}u(z)\frac{1}{1-\Delta(z)}\right]\mathcal{B}_T e^{-\mathcal{L}_T t_2}\rho_T(0)\ , \tag{23}$$

where $\Delta(z)$ is the response function due to the interaction with the medium. The explicit evaluation of $\Delta(z)$ is relegated to the Appendix. Substituting the expressions for $\Delta(z)$ from (A19) and evaluating the inverse Laplace transform, we have

$$\begin{aligned}\rho(t)=e^2\int_0^t dt_1\int_0^{t_1}dt_2 e^{-\mathcal{L}(t-t_1)}\mathcal{A}_T e^{-\mathcal{L}(t_1-t_2)}\mathcal{B}_T e^{-\mathcal{L}_T t_2}\rho_T(0)\\ \times\left[\frac{1}{2}\left[1+\frac{\overline{\Omega}^2-\overline{v}_\lambda^2}{\sigma}\right]\cos\left[\frac{\overline{v}_\lambda^2+\overline{\Omega}^2-\sigma}{2}\right]^{1/2}(t_1-t_2)\right.\\ \left.+\frac{1}{2}\left[1-\frac{\overline{\Omega}^2-\overline{v}_\lambda^2}{\sigma}\right]\cos\left[\frac{\overline{v}_\lambda^2+\overline{\Omega}^2+\sigma}{2}\right](t_1-t_2)\right],\end{aligned} \tag{24}$$

where

$$\sigma^2=4\chi+(\overline{v}_\lambda^2-\overline{\Omega}^2)^2\ . \tag{25}$$

After operating with the propagators, and adding the $-\lambda$ part, we have

$$\rho(t)=\frac{e^2a_\lambda^2}{2m\pi^2}\int d\mathbf{K}_\lambda\int_0^t dt_1\int_0^{t_1}dt_2\frac{\partial}{\partial\mathbf{u}}\cdot\mathbf{e}_\lambda(\mathbf{e}_{\lambda'}\cdot\overline{\boldsymbol{\beta}})\cos(l\xi+\phi)e^{-\mathcal{L}t_2}\rho_T(0)$$
$$\times\left[\left[1+\frac{\overline{\Omega}^2-\overline{\nu}_\lambda^2}{\sigma}\right]\cos\left[\frac{\overline{\nu}_\lambda^2+\overline{\Omega}^2-\sigma}{2}\right]^{1/2}(t_1-t_2)+\left[1-\frac{\overline{\Omega}^2-\overline{\nu}_\lambda^2}{\sigma}\right]\cos\left[\frac{\overline{\nu}_\lambda^2+\overline{\Omega}^2+\sigma}{2}\right]^{1/2}(t_1-t_2)\right] \quad (26)$$

with

$$\overline{\boldsymbol{\beta}}=\boldsymbol{\beta}\cos[\omega(t_1-t_2)]+\boldsymbol{\beta}\times\boldsymbol{\epsilon}_3\sin[\omega(t_1-t_2)]+\boldsymbol{\epsilon}_3(\boldsymbol{\beta}\cdot\boldsymbol{\epsilon}_3)[1-\cos(\omega\overline{t_1-t_2})]\ ,$$
$$\phi=\frac{\alpha\Omega_0}{\omega}\{\boldsymbol{\epsilon}_3\cdot\boldsymbol{\beta}\times\mathbf{K}_\lambda[1-\cos(\omega\overline{t_1-t_2})]-(\mathbf{K}_\lambda\cdot\boldsymbol{\beta})\sin(\omega\overline{t_1-t_2})+(\boldsymbol{\epsilon}_3\cdot\mathbf{K}_\lambda)(\boldsymbol{\epsilon}_3\cdot\boldsymbol{\beta})[\sin(\omega\overline{t_1-t_2})-\omega\overline{t_1-t_2}]\}\ . \quad (27)$$

In writing Eq. (26), we have used the relation $(8\pi^3/v)\sum_{\mathbf{K}_\lambda}\rightarrow\int d\mathbf{K}_\lambda$. Also $\mathbf{K}_{\lambda'}=\mathbf{K}_\lambda+\boldsymbol{\epsilon}_1 l$ and

$$\mathbf{e}_{\lambda'}=\mathbf{e}_\lambda-\boldsymbol{\epsilon}_3\frac{(\mathbf{e}_\lambda\cdot\boldsymbol{\epsilon}_1)}{(\mathbf{K}_\lambda\cdot\boldsymbol{\epsilon}_3)}l$$

which ensures $\mathbf{e}_{\lambda'}\cdot\mathbf{K}_{\lambda'}=0$. This completes the evaluation of OPDF and it may be seen that it is a function of ξ as well as β. Hence in using (26) to evaluate averages, we have to integrate over $\mathbf{q}_T$ and β. We also have to define an initial state of the relativistic test-particle beam.

VI. INITIAL STATE

The test-particle beam consists of relativistic electrons which are mutually noninteracting and pass through vacuum in the presence of the axial-wiggler field. At $t=0$, the beam enters into the plasma column. Hence the initial state is characterized by the wave functions due to a noninteracting relativistic Hamiltonian. We shall take this as the Dirac Hamiltonian [8] and construct the density matrix with these wave functions ψ weighted with a Fermi function. Thus

$$\rho_T(0)=\sum_{n,m}\left\langle\psi_m\left|\frac{1}{e^{\beta(\mathcal{H}_0-\mu)}+1}\right|\psi_n\right\rangle$$
$$=\sum_n e^{-(\beta_2+\xi)^2}\frac{H_n^2(\beta_2+\xi)}{2^n n!\sqrt{\pi}}\delta(\beta_1)\delta(\beta_2)\delta(\beta_3-\beta_0)\delta(\eta)\delta(\zeta)(e^{\beta(E_n-\mu_0)}+1)^{-1}\ , \quad (28)$$

where H_n are Hermite functions and these are normalized. The eigenvalues are given by

$$E_n=mc^2\left[1+\frac{\hbar\Omega_0}{mc^2}[(2N+p_3^2)^{1/2}+k_w/2]^2\right]^{1/2} \quad (29)$$

with $N=n+1$ for spin-up particles and n for spin down, n being the Landau level number. It should be pointed out that n occurs due to the presence of the axial component, while k_w is due to the wiggler magnetic field. The initial state consists of the dependence of ξ through the Hermite functions; since we are working in the Landau gauge, only ξ dependence needs to be specified. Hence for η we have a δ function and the initial point at which the beam is lauched is taken as zero, hence $\delta(\zeta)$. With this initial state, we shall evaluate the energy loss suffered by the test particle as it travels through the column.

VII. ENERGY LOSS

We shall evaluate the average energy loss by averaging the test-particle Hamiltonian with the OPDF as given by (26), together with the initial state given by (28). We then differentiate this with respect to time and obtain $d\overline{E}/dt$ and subsequently $d\overline{E}/dl$ as $dl=c\,dt$. This gives the power loss per unit length as obtained by Frank and Tamm. It may, however, be mentioned that in the two terms in (26) we have a condition $\overline{\nu}_\lambda^2+\overline{\Omega}^2>\sigma$ otherwise the argument of the cosine term would become imaginary and we would get absorption. We hence retain the term satisfying this condition and have

$$\frac{d\bar{E}}{dl}=\frac{ce^2a_\lambda^2}{2\pi^2}\left[\left[\coth\frac{\beta\hbar\Omega_0}{2}\mp 1\right]e^{-(l^2/4)\coth(\beta\hbar\Omega_0/2)}\right]$$
$$\times\int d\mathbf{K}_\lambda\int d\xi\int_0^t dt_2\beta_0\left[\left[1-\frac{K_{3\lambda}^2}{K_\lambda^2}\right]-\frac{lK_{1\lambda}}{K_\lambda^2}\right]\cos(\alpha K_{3\lambda}\beta_0\Omega_0\overline{t-t_2})$$
$$\times\frac{1}{2}\left[1+\frac{\bar{v}_\lambda^2-\bar{\Omega}^2}{\sigma}\right]\cos\left[\frac{\bar{v}_\lambda^2+\bar{\Omega}^2+\sigma}{2}\right]^{1/2}(t-t_2)\delta(\xi-\beta_0\Omega_0t_2)\ . \tag{30}$$

In writing (30), we have substituted for $\bar{\beta}$ from (27) and $\mathbf{e}_{\lambda'}$, as in (41), integrated over ξ, η, and β, and summed over the Landau levels, using the generating function for the Laguerre polynomials [14]. The minus sign above is for spin-up particles and the plus sign is for spin down. It may be seen that in the spin-up case the function has a peak and goes to zero as a function of $\beta\hbar\Omega_0/2$, while for the spin-down case the saturation is at a higher level. Hence for smaller value of $\beta\hbar\Omega_0/2$ the two curves are distinct and there is a measurable difference, while for larger values the difference is constant as can be seen from the figure. As $l^2/4$ is increased, the spin-up peak shifts to the right. This gives the relative population of spin-up to spin-down particles in the launching beam and could be used to measure the spin ratios.

We shall now integrate with respect to t_2. As the range of t_2 is between 0 and t, ξ has the range from 0 to $\beta_0\Omega_0 t$. We then combine the two cosine terms, integrate t_2 and obtain

$$\frac{d\bar{E}}{dl}=\frac{ce^2a_\lambda^2}{2\pi^2\Omega_0}\left[\left[\coth\frac{\beta\hbar\Omega_0}{2}\mp 1\right]e^{-(l^2/4)\coth(\beta\hbar\Omega_0/2)}\right]$$
$$\times\int d\mathbf{K}_\lambda\int d\xi(1-\mu^2)\left[1+\frac{\bar{v}_\lambda^2-\bar{\Omega}^2}{\sigma}\right]\left\{\cos\left[\alpha K_\lambda\beta_0\Omega_0\mu+\left[\frac{\bar{v}_\lambda^2+\bar{\Omega}^2+\sigma}{2}\right]^{1/2}\right]\left[t-\frac{\xi}{\beta_0\Omega_0}\right]\right.$$
$$\left.+\cos\left[\alpha K_\lambda\beta_0\Omega_0\mu-\left[\frac{\bar{v}_\lambda^2+\bar{\Omega}^2+\sigma}{2}\right]^{1/2}\right]\left[t-\frac{\xi}{\beta_0\Omega_0}\right]\right\}, \tag{31}$$

where μ is the cosine of the polar angle that K_λ makes. We integrate now with respect to ξ and take the asymptotic limit in time. We then have the usual resonance condition, viz.

$$\frac{d\bar{E}}{dl}=\frac{ce^2a_\lambda^2}{8\pi^2}\left[\left[\coth\frac{\beta\hbar\Omega_0}{2}\mp 1\right]e^{-(l^2/4)\coth(\beta\hbar\Omega_0/2)}\right]$$
$$\times\int K_\lambda^2 dK_\lambda d\mu\, d\phi\,\beta_0(1-\mu^2)\left[1+\frac{\bar{v}_\lambda^2-\bar{\Omega}^2}{\sigma}\right]$$
$$\times\left\{\delta\left[\alpha\Omega_0K_\lambda\beta_0\mu+\left[\frac{\bar{v}_\lambda^2+\bar{\Omega}^2+\sigma}{2}\right]^{1/2}\right]+\delta\left[\alpha\Omega_0K_\lambda\beta_0\mu-\left[\frac{\bar{v}_\lambda^2+\bar{\Omega}^2+\sigma}{2}\right]^{1/2}\right]\right\}. \tag{32}$$

In Eq. (32), K_λ is dimensionless ($=\bar{v}_\lambda/\alpha\Omega_0$) and $\mu=\cos\theta$. Equation (32) gives the usual Čerenkov condition, viz.

$$\mu=\pm\left[\frac{\bar{v}_\lambda^2+\bar{\Omega}^2+\sigma}{2\beta_0^2\bar{v}_\lambda^2}\right]^{1/2}. \tag{32'}$$

Integrating with respect to ϕ and μ and writing $cK_\lambda=\bar{v}_\lambda$, we have

$$\frac{d\bar{E}}{dl}=\frac{e^2\Gamma}{c^2}\int\bar{v}_\lambda d\bar{v}_\lambda\left[1+\frac{\bar{v}_\lambda^2-\bar{\Omega}^2}{\sigma}\right]\left[1-\frac{\bar{v}_\lambda^2+\bar{\Omega}^2+\sigma}{2\beta_0^2\bar{v}_\lambda^2}\right], \tag{33}$$

where

$$\Gamma=\frac{a_\lambda^2}{4\pi}\left[\coth\frac{\beta\hbar\Omega_0}{2}\mp 1\right]\exp\left[-\frac{l^2}{4}\coth\frac{\beta\hbar\Omega_0}{2}\right]. \tag{34}$$

One can easily see from the definition of σ (25) that

$$2\bar{v}_\lambda d\bar{v}_\lambda\left[1+\frac{\bar{v}_\lambda^2-\bar{\Omega}^2}{\sigma}\right]=d(\bar{v}_\lambda^2+\bar{\Omega}^2+\sigma)=d\tilde{\omega}^2\ , \tag{35}$$

say, where

$$\tilde{\omega}^2=(\bar{v}_\lambda^2+\bar{\Omega}^2+\sigma)\ . \tag{36}$$

We then have

$$\frac{d\bar{E}}{dl}=\frac{e^2\Gamma}{c^2}\int\tilde{\omega}\,d\tilde{\omega}\left[1-\frac{\tilde{\omega}^2}{2\beta_0^2\bar{v}_\lambda^2}\right]. \tag{37}$$

If we now define

$$2\bar{v}_\lambda^2=\varpi^2 N^2(\varpi) \tag{38}$$

we then have the Frank and Tamm relation

$$\frac{d\bar{E}}{dl}=\frac{e^2\Gamma}{c^2}\int \varpi\, d\varpi \left[1-\frac{1}{\beta_0^2 N^2}\right] , \tag{39}$$

where N the refractive index is given by

$$N^2=1+\left[\frac{2\chi}{\bar{\Omega}^2}\right]\frac{1}{\varpi^2}+\left[\frac{2\chi}{\bar{\Omega}^2}\right]\frac{1}{2\bar{\Omega}^2-\varpi^2} \tag{40}$$

in which χ is given by (A19). The refractive index (40) consists of three terms, the first being that of vacuum. The second is due to the photon field, and the third is the shifted frequency because of the magnetic field appearing in $\bar{\Omega}$. It may, however, be noted that the photon frequency is the effective one as defined in (36); this is density dependent through $\bar{v}_\lambda^2$ as well as χ since the plasma frequency appears in both these factors.

VIII. POLARIZATION

The passage of radiation through matter introduces a change in polarization. This consists of change due to the magnetic fields (Ω_0 and Ω), and is given by

$$\mathbf{e}_{\lambda'}=\mathbf{e}_\lambda-\boldsymbol{\epsilon}_3\frac{\mathbf{e}_\lambda\cdot\boldsymbol{\epsilon}_1}{\mathbf{K}_\lambda\cdot\boldsymbol{\epsilon}_3}l . \tag{41}$$

Thus the change in polarization due to the magnetic fields is given by

$$\begin{aligned}\delta\mathbf{e}_\lambda&=-\boldsymbol{\epsilon}_3\frac{\mathbf{e}_\lambda\cdot\boldsymbol{\epsilon}_1}{\mathbf{K}_\lambda\cdot\boldsymbol{\epsilon}_3}l\\ &=\left[\left[\frac{\Omega}{\bar{\Omega}}\right](\mathbf{i}\cos kz+\mathbf{j}\sin kz)-\mathbf{k}\left[\frac{\Omega_0}{\bar{\Omega}}\right]\right]F ,\end{aligned} \tag{42}$$

where

$$F=-\left|\frac{l}{K_\lambda}\right|\left|\left[\frac{\Omega_0\sin\theta_e\cos(kz-\phi_e)+\Omega\cos\theta_e}{\Omega\sin\theta_k\cos(kz-\phi_k)-\Omega_0\cos\theta_k}\right]\right| \tag{43}$$

we can make use of the relation that $\mathbf{e}_\lambda\cdot\mathbf{K}_\lambda=0$ or $\cos\theta_e\cos\theta_k+\sin\theta_e\sin\theta_k\cos(\phi_e-\phi_k)=0$ and simplify the above expression. However from Eq. (42), it is evident that the polarization vector rotates as we advance in the z direction with a periodicity of the wiggler field. One can also define an effective polarization vector $\sqrt{\chi}\mathbf{e}_\lambda$ from (A6) and the presence of χ in this effective polarization shows the effect of the Čerenkov radiation through the characteristic plasma frequency.

IX. LINEARIZED SYSTEM

In this section we propose to derive the earlier results [10] as obtained in the case of an electron beam passing through a wiggler field in vacuum and in the presence of radiation as a special case of the present formalism. It should be realized that the results can be obtained by taking carefully the limit as the two magnetic fields (axial and wiggler) combine in a peculiar fashion. We now set $\Omega_0=0$ and $\bar{\Omega}=\Omega$ in (13). This reduces the coordinate system to

$$\boldsymbol{\epsilon}_1=\mathbf{k},\quad \boldsymbol{\epsilon}_2=\hat{\mathbf{e}}_2,\quad \boldsymbol{\epsilon}_3=-\hat{\mathbf{e}}_1 . \tag{44}$$

Again $\mathbf{q}_l$ and $\mathbf{P}_l$ are identically zero and expression (3) becomes

$$mc\mathbf{u}_T=mc\gamma\boldsymbol{\beta}=\mathbf{P}_T+\frac{m_T\Omega}{k_w}\hat{\mathbf{e}}_1-\frac{e_T}{c}\mathbf{A}_T^\lambda , \tag{45}$$

where now $\mathbf{u}_T=\gamma\boldsymbol{\beta}$. One can readily obtain the time derivatives as

$$\begin{aligned}\dot{\mathbf{q}}_T&=c\boldsymbol{\beta} ,\\ \dot{\mathbf{P}}_T&=-m\Omega c(\boldsymbol{\beta}\cdot\hat{\mathbf{e}}_2)\mathbf{k}+e[(\boldsymbol{\beta}\cdot\nabla)\mathbf{A}+\boldsymbol{\beta}\times\nabla\times\mathbf{A}] .\end{aligned} \tag{46}$$

In the absence of a radiation field ($\mathbf{A}_\lambda$), one can easily see that $\dot{P}_x=\dot{P}_y=0$ which gives P_x and P_y as constants in time which can be taken as zero without loss of generality. This however is not quite true in the presence of the interaction vector potential $\mathbf{A}_\lambda$ which is necessary to take into account the interaction of the beam with radiation. Nevertheless in this approximation from Eq. (45) we can write

$$\boldsymbol{\beta}=\frac{K}{\gamma}\hat{\mathbf{e}}_1+\mathbf{k}\beta_0 , \tag{47}$$

where $K=\Omega/ck_w$. This makes $\dot{P}_3=0$ as $\hat{\mathbf{e}}_1$ and $\mathbf{k}$ are orthogonal to $\hat{\mathbf{e}}_2$ thereby giving $P_3=mc\gamma\beta_0$. This makes the corresponding operators defined in (12) to be

$$\mathcal{L}_T=c\boldsymbol{\beta}\cdot\frac{\partial}{\partial\mathbf{q}} . \tag{48}$$

Since β_1 and β_2 are functions of z we expand the above operator in a Baker-Hausdorff scheme and get

$$e^{-\mathcal{L}\tau}=\exp\left[\frac{K}{\gamma k_w\beta_0}\right]\left[[\sin k_w z-\sin(k_w z-ck_w\beta_0\tau)]\frac{\partial}{\partial x}+[\cos(k_w z-ck_w\beta_0\tau)-\cos k_w z]\frac{\partial}{\partial y}\right]e^{-c\beta_0\tau\partial/\partial z} \tag{49}$$

with $\boldsymbol{\beta}$ as defined in (47). One can solve for β_0 from the definition of $\gamma=(1-\beta^2)^{-1/2}$ as

$$\beta_0=\left[1-\frac{K^2+1}{\gamma^2}\right]^{1/2} . \tag{50}$$

We shall now take the $n=2$ term in the iterated series (18), since this corresponds to the term which characterizes the interaction of the test particle with the radiation field. Using this as a distribution function we evaluate the energy loss and obtain the power emitted as

$$\frac{d^2E}{d\Omega\,dt}=\frac{e^2a_\lambda^2}{\pi^2c}\int \nu_\lambda^2 d\nu_\lambda \int_0^t dt_2 \int d\beta\gamma^5\beta^2\sin^2\vartheta$$

$$\times\cos\left[2K_\lambda\sin\theta\sin\left[\frac{ck_w\beta_0\overline{t-t_2}}{2}\right]\cos\left[k_w z+\frac{ck_w\beta_0\overline{t-t_2}}{2}+\phi\right]-ck_{3\lambda}\beta_0\overline{t-t_2}\right]$$

$$\times\cos\nu_\lambda\overline{t-t_2}\,\delta(\beta_1)\delta(\beta_2)\,\delta(\beta_3-\beta_0)\;, \tag{51}$$

where ϑ is the angle between $\mathbf{K}_\lambda$ and $\boldsymbol{\beta}$, θ, and ϕ are the polar angles of the propagation vector $\mathbf{K}_\lambda$ and the initial state has been taken as a δ function with a velocity component in the axial direction. Effecting the $\boldsymbol{\beta}$ integration and expanding the cosine term in terms of Bessel functions, the term independent of the azimuthal angle ϕ is retained and Eq. (51) takes the form

$$\frac{d^2E}{d\Omega\,dt}=\frac{e^2a_\lambda^2}{\pi^2c}\int \nu_\lambda^2 d\nu_\lambda \int_0^t dt_2\gamma^5\beta_0^2\sin^2\vartheta J_0\left[2K_\lambda\sin\theta\sin\frac{ck_w\beta_0\overline{t-t_2}}{2}\right]\cos(cK_{3\lambda}\beta_0\overline{t-t_2})\cos[\nu_\lambda(t-t_2)]\;. \tag{52}$$

We can write an expansion for $J_0(z\sin\alpha)$ as

$$J_0(z\sin\alpha)=J_0^2\left[\frac{z}{2}\right]+2\sum_{n=1}^{\infty}J_n^2\left[\frac{z}{2}\right]\cos(2n\alpha)\;. \tag{53}$$

Using this, Eq. (52) can be written as

$$\frac{d^2E}{d\Omega\,dt}=\frac{e^2a_\lambda^2}{\pi^2c}\int \nu_\lambda^2 d\nu_\lambda \int_0^t dt_2\gamma^5\beta_0^2\sin^2\vartheta\left[J_0^2(K_\lambda\sin\theta)+2\sum_{n=1}^{\infty}J_n^2(K_\lambda\sin\theta)\cos(nck_w\beta_0\overline{t-t_2})\right]$$

$$\times\cos(cK_{3\lambda}\beta_0\overline{t-t_2})\cos(\nu_\lambda\overline{t-t_2})\;. \tag{54}$$

Combining the cosine terms in $(t-t_2)$, integrating over t_2, and taking the asymptotic limit in time, we obtain the resonance condition as

$$\delta(\nu_\lambda-\nu_\lambda\beta_0\cos\theta-nck_w\beta_0)\;. \tag{55}$$

For the case $n=1$ and in the direction $\theta=0$, we have

$$\nu_\lambda=\frac{ck_w\beta_0}{1-\beta_0}=\frac{2ck_w\gamma^2}{1+K^2}\;. \tag{56}$$

In writing (56) we have used (50) and expand β_0^{-1} in $ck_w/(\beta_0^{-1}-1)$ binomially. This is the resonance relation obtained by Murphy and Pellegrini [10].

X. CONCLUSION

It is well known that for a relativistic electron in a magnetic field, the Dirac equation has different Landau level index in its eigenvalues for the spin-up $[2(n+1)]$ and spin-down $(2n)$ cases. However when we do the summation over Landau levels [14] [see Eqs. (B6) and (B7)] we get different factors as in (30) which is written separately in (34), the negative sign being for the spin up and the positive for spin down. These functions are plotted in Figs. 1 and 2 as a function of $\beta\hbar\Omega_0/2$ and for two different values of $l^2/4$ ($=0.25$ and 1 so that $l=1$ and 2). The spin-up particle distribution has a single maximum and attains a low saturation value for higher values of the parameter, while in the other case the curve is similar to a sigmoidal function. The two distributions show different features at lower values of the argument, while for higher values the difference also saturates. It may however be noted that the curve flattens as l is increased, so much so that the difference (b)−(a) in Figs. 1 and 2 also flattens, thereby showing that the quantum effects become pronounced only for smaller values of $l^2/4$ and the argument $\beta\hbar\Omega_0/2$. These plots were however done for finite temperature when the Fermi function is replaced by a Boltzmann distribution.

In the case of the nonrelativistic plasma component

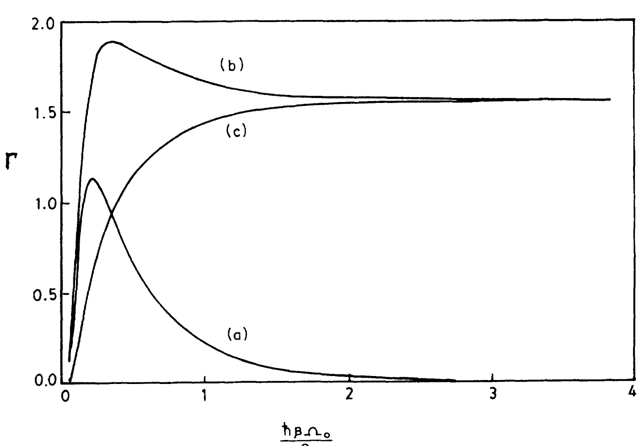

FIG. 1. The distribution Γ of spin-up (a) and spin-down (b) particles after interaction with the medium and fields as a function of $\hbar\Omega_0\beta/2$ in arbitrary units. (c) is a plot of (b)−(a). These plots are $l=1$.

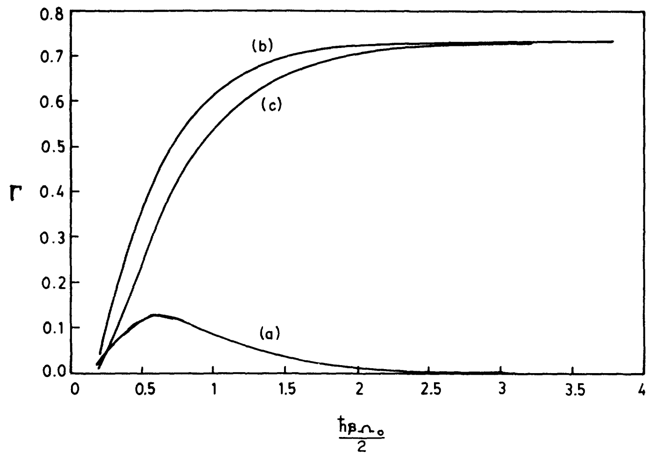

FIG. 2. The same set of curves as in Fig (1) for $l=2$.

which constitutes the medium, this function [see (A13)] appears with a single peak as plotted in Fig. 3. Here again the peak is pronounced for smaller values of $l^2/4$ and is shifted to the right and flattened for higher values of $l^2/4$ and $\beta\hbar\Omega_0/2$. Again for smaller values of $l^2/4$ the significant domain of $\beta\hbar\Omega_0/2$ is small; this gets extended for higher values of the parameter. Thus the response function has a multiplicative factor which changes significantly as the parameter takes higher values. The summation over the Landau levels is very crucial and is a significant step in a statistical theory of electrons in a magnetic field, since that introduces collective effects due to the statistics of a large number of energy levels.

The response function given in (A16) is the most significant result of this paper, as can be seen from Eq. (23). $\Delta(z)$ is the modification of the frequency $\bar{\nu}_\lambda$ appearing in $u(z)$ and the fact that this is a function of z implies that the non-Markovian dynamics considered here has given rise to a time-dependent response function. To obtain the kinetic regime, one takes the asymptotic limit in time (which is the same as $z\rightarrow 0$) in (A16) or (A17) resulting in (A18). This is necessary to wash out the memory effects introduced by the propagator which appears in a convolution form. This is discussed at length by Balescu [1]. Even in this limit, the synergism is quite obvious in the way the various frequencies $\omega_{\rm pl}^2$, Ω_0, Ω, and ν_λ appear in (A19). While the Landau level summation restricts the domain of $\beta\hbar\Omega_0/2$ to a narrow region, the above function would completely vanish if either $lP_0\Omega_0/\bar{\Omega}$ or $lP_0(1+\Omega_0/\bar{\Omega})$ is a zero of J_0. This implies that the frequencies Ω_0 and $\bar{\Omega}$ satisfying the above condition are incommensurate. In this case, one can see that the radiation frequency would reduce to $\bar{\nu}_\lambda$ or $\bar{\Omega}$ from (25) and (36). We thus have, for this particular choice of magnetic fields, the fact that the medium behaves like a vacuum and that the light passes through the medium unimpaired or is "ducted" through the medium. If however they are the $m^{\rm th}$ and $n^{\rm th}$ zeros of J_0, then $lP_0=x_n-x_m$, and the ratio of the Wiggler to Axial cyclotron frequency $\Omega/\Omega_0=[\alpha(\alpha-2)]^{1/2}$ α being x_n/x_m, which is irrational. It may be pointed out that l and P_0 are the dimensionless wave number and the axial momentum component, and that the former is the outcome of the conservation law.

The optics of this medium is obtained through the refractive index as defined in (40). This function consists of three terms, the first being that of vacuum. The second comes from the effective radiation with frequency $\tilde{\omega}$ as given in (36). The third term is this frequency shifted by $2\bar{\Omega}^2$ due to the combined magnetic fields in the system. The last two terms are weighted by a factor $2\chi/\bar{\Omega}^2$; this introduces the medium effect, as χ (A19) is a function of the plasma frequency. The particular choice of the synchrotron frequencies which make $\chi\equiv 0$ as mentioned earlier reduces the refractive index to unity and the phenomena of ducting takes place. In this case $\tilde{\omega}$ as defined (36) becomes $\bar{\nu}_\lambda$ or $\bar{\Omega}$ and thereby modifies the cosine terms appearing in (24). In this case the emerging radiation would either be the radiation existing in the medium or the pure synchrotron radiation with effective frequency $\bar{\Omega}$. For all other frequencies except this class the refractive index is a function of density through plasma frequency as well as $\hbar\Omega_0/k_BT$ as seen in (A19). The polarization of the incident beam also changes as given in (41). One can easily see that this change in polarization has three components in the original Cartesian system and in the cylindrical coordinate system: $(\delta e)_\rho=(\Omega F/\bar{\Omega})\cos(kz-\theta)$, $(\delta e)_\theta=(\Omega F/\bar{\Omega})\sin(kz-\theta)$ and $(\delta e)_z=-(\Omega_0/\bar{\Omega})F$, F being defined in (43). The change is due to the two magnetic fields Ω_0 and Ω as well as changes introduced by the Wiggler field geometry which manifests itself in the definition of F. One can also define an effective polarization vector $\sqrt{\chi}\mathbf{e}_\lambda$ in (A6). Evidently this polarization would be a function of density and other parameters of the problem through χ.

Equation (32′) gives the measure of the Čerenkov cone and is given by $\cos\theta=\pm(\tilde{\omega}/\bar{\nu}_\lambda\beta_0\sqrt{2})$, θ being the generating angle of the cone and β_0 the Lorentz factor. This again is an explicit function of density as well as the synchrotron frequencies. This may be compared with the similar expression obtained for an unmagnetized plasma [12] [see Eq. (4.18)], in which interaction has been be-

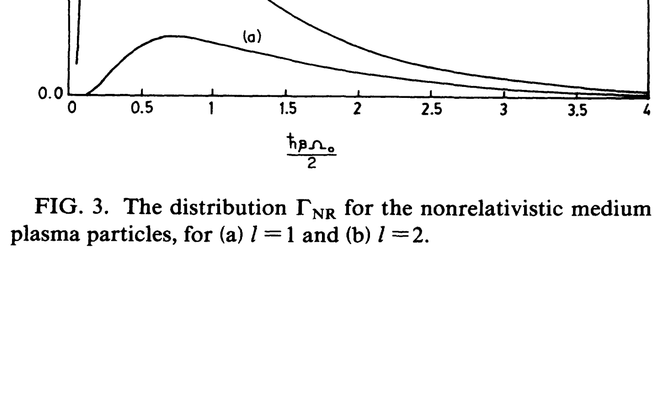

FIG. 3. The distribution $\Gamma_{\rm NR}$ for the nonrelativistic medium plasma particles, for (a) $l=1$ and (b) $l=2$.

tween the relativistic electron and the plasma. In this case however the plasma particles behaved like harmonic oscillators with frequency ν_α which in the present case is replaced by the effective synchrotron frequency $\bar{\Omega}$ due to magnetic fields. Nevertheless in the present case, for the particular choice of $\Omega_0/\bar{\Omega}$ which makes $\chi=0$, $\tilde{\omega}$ takes the value $\bar{\nu}_\lambda\sqrt{2}$ since we are considering only the positive σ due to the reality condition. This gives $\cos\theta=\beta_0^{-1}$ and may be interpreted as "Čerenkov-like" radiation due to the interaction between the synchrotron radiation and the ambient radiation in the system. The role of the usual medium is now played by the photon gas of frequency ν_λ. It may be realized that the reality condition, viz. $\bar{\nu}_\lambda^2+\bar{\Omega}^2>\sigma$ gives the condition $\bar{\Omega}^2>0$ and hence this condition exists only when the effective magnetic field is nonzero. Again remembering that β_0 is the component of β in the ϵ_3 direction which in Cartesian coordinates would read $(\Omega_0/\bar{\Omega})\beta_3-(\Omega/\bar{\Omega})\beta_\rho\cos(kz-\theta)$ where β_ρ, θ, and β_3 are the cylindrical polar coordinates, we get the Čerenkov cone twisting and wobbling around its axis as it goes in the positive z direction.

We have finally obtained the previously derived results as a special case of the present formalism. It may however be noted that taking limits in the present results in a straightforward manner would not reproduce the earlier results, since the natural coordinates in the present system with the axial-wiggler combination is distinctly different from that in the pure wiggler case. If we set $\Omega_0=0$, the new system reduces to the old one with a relabeling. On the other hand if we set $\Omega=0$, and $k_w=0$, the present coordinate system reduces to the usual Cartesian one. In the new system, the operators attain a particularly simple form in which we can perform a Baker-Hausdorff expansion very elegantly. Further, the limitations in the linear results have been brought to light and these results are strictly not applicable if one takes interaction in a consistent manner. This is seen in (46) wherein $\dot{\mathbf{P}}\equiv 0$ cannot be taken for all times. Interactions generate nonzero force components and these play a significant role in the dynamics of correlations.

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APPENDIX

Evaluation of response function

The first term in the series is the interaction of the test particle with the electromagnetic field. The matrix element is

$$\rho^{(1)}(t)=e^2\int_0^t dt_1\int_0^{t_1} dt_2 e^{-\mathcal{L}(t-t_1)}\mathcal{A}_T \times e^{-\mathcal{L}(t_1-t_2)}\mathcal{B}_T e^{-\mathcal{L}t_2}\rho_T(0)\ . \quad \text{(A1)}$$

In evaluating the energy, we have to integrate with respect to $\mathbf{u}_T$ and $\mathbf{q}_T$ and the first propagator would give unity. Substituting the operators in the above, and remembering that $\boldsymbol{\beta}\cdot\boldsymbol{\beta}\times(\mathbf{K}_\lambda\times\mathbf{e}_\lambda)\equiv 0$, we have

$$\rho^{(1)}(t)=\left[\frac{8\pi e^2}{V}\right]\int_0^t dt_1\int_0^{t_1} dt_2\frac{\partial}{\partial\mathbf{u}}\cdot[\mathbf{e}_\lambda(\mathbf{e}_{\lambda'}\cdot\bar{\boldsymbol{\beta}})\cos(\mathbf{K}_\lambda\cdot\mathbf{q}_T)\cos(\mathbf{K}_{\lambda'}\cdot\mathbf{q}_T+\phi)]e^{-\mathcal{L}t_2}\rho_T(0)\oint dz\, e^{-iz(t_1-t_2)}u(z)\ , \quad \text{(A2)}$$

where

$$u(z)=iz/(z^2-\bar{\nu}_\lambda^2) \quad \text{(A3)}$$

and $\bar{\beta}$ and ϕ are defined in Eq. (27). This has been obtained after the following operations: (a) shift operated on the photon variables and integrated over J_λ, and ω_λ and (b) acted the shift operator on $\boldsymbol{\beta}$ and $\mathbf{q}_T$.

The second term in the series is the one at which the test particle emits a virtual photon at t_4, absorbed by the field particles at t_3, which in turn emits the same at t_2 and finally the test particle absorbs at t_1 and proceeds to t. We consider this as a completion of interaction of the test particle with the field particle. Analytically one can write this as

$$\rho^{(2)}(t)=e^4\int_0^t dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3\int_0^{t_3} dt_4[e^{-\mathcal{L}(t-t_1)}\mathcal{A}_T e^{-\mathcal{L}(t_1-t_2)}\mathcal{B}_l e^{-\mathcal{L}(t_2-t_3)}\mathcal{A}'_l e^{-\mathcal{L}(t_3-t_4)}\mathcal{B}'_T e^{-Lt_4}\rho_T(0)]\ . \quad \text{(A4)}$$

The progress of interaction is to be followed from right to left. The $\mathcal{B}$ vertex creates the correlation by emitting the virtual photon and $\mathcal{A}$ vertex absorbs the photon annihilating the correlation.

We write the operators in full in (A4) and integrate over the photon variables. We then get

$$\rho^{(2)}(t)=\left[\frac{8\pi e^2}{mV}\right]\int_0^t dt_1\int_0^{t_1} dt_2\frac{\partial}{\partial u}\cdot[\mathbf{e}_\lambda(\mathbf{e}_{\lambda'}\cdot\bar{\boldsymbol{\beta}})\cos(\mathbf{K}_\lambda\cdot\mathbf{q}_T)\cos(\mathbf{K}_{\lambda'}\cdot\mathbf{q}_T+\phi)]e^{-\mathcal{L}t_2}\rho_T(0)\oint dz\, e^{-iz(t_1-t_2)}u(z)\Delta(z)\ , \quad \text{(A5)}$$

where $\Delta(z)$ is the response function in resolvant space and the Laplace transform of R given by

$$R=\sum_l \left[\frac{8\pi e^2}{mV\bar{\nu}_{\lambda'}}\right]\int_0^{t_2}dt_3\int_0^{t_3}dt_4\sin(\nu_{\lambda'}\overline{t_3-t_4})$$
$$\times\int d\mathbf{P}_l d\mathbf{q}_l[(\mathbf{e}_\lambda\cdot\mathbf{P}_l)(\mathbf{e}_{\lambda'}\cdot\mathbf{K}_\lambda)\sin(\overline{\mathbf{K}_{\lambda'}-\mathbf{K}_\lambda}\cdot\mathbf{q}_l+\phi)$$
$$+m\bar{\Omega}(\boldsymbol{\epsilon}_3\times\mathbf{e}_{\lambda'})\cdot\mathbf{e}_\lambda\cos(\overline{\mathbf{K}_{\lambda'}-\mathbf{K}_\lambda}\cdot\mathbf{q}_l+\phi)]e^{-\mathcal{L}t_3}\rho_l(0)\ , \tag{A6}$$

where

$$\phi=\frac{\Omega_0}{\bar{\Omega}}\{-(\mathbf{K}_\lambda\cdot\mathbf{P}_l)\sin(\bar{\Omega}\overline{t_2-t_3}$$
$$+\boldsymbol{\epsilon}_3\cdot\mathbf{P}_l\times\mathbf{K}_\lambda[1-\cos(\bar{\Omega}\overline{t_2-t_3})]$$
$$+K_3P_3[\sin(\bar{\Omega}\overline{t_2-t_3})-\bar{\Omega}\overline{t_2-t_3}]\}\ .$$

In obtaining (A6) we have integrated partially with respect to $\mathbf{q}_l$ and $\mathbf{P}_l$. We have also added the $-\lambda$ part. We now have to integrate the above with respect to $\mathbf{P}_l$ and $\mathbf{q}_l$ and for this we have to prescribe an initial state.

We shall construct the intitial state of the plasma as before by taking the density matrix with the eigenfunctions of the unperturbed part of the Hamiltonian $\mathcal{H}_l$ from (7). The Hamiltonian is expressed in $(\boldsymbol{\epsilon}_1,\boldsymbol{\epsilon}_2,\boldsymbol{\epsilon}_3)$ coordinate system.

The eigenfunction of the Schrödinger equation is

$$\psi(\xi,\eta,\zeta)=\sum_n e^{-(p_2+\xi)^2/2}\frac{H_n(p_2+\xi)}{(2^n n!\sqrt{\pi})^{1/2}}$$
$$\times\exp\left[i\left[-\frac{\Omega}{\bar{\Omega}\,\bar{k}_w}\xi+p_2\eta+\frac{\Omega^2}{\Omega_0\bar{\Omega}}\zeta\right]\right] \tag{A7}$$

and the corresponding eigenvalues are

$$E_n=(n+\tfrac{1}{2})\hbar\Omega_0\ . \tag{A8}$$

The initial state is therefore written as

$$\rho_l(0)=\sum_n e^{-(p_2+\xi)^2}\frac{H_n^2(p_2+\xi)}{2^n n!\sqrt{\pi}}$$
$$\times\delta(P-P_0)\delta(P_3-P_3^0)[1+\exp\beta(E_n-\mu_0)]^{-1}\ , \tag{A9}$$

where in the momentum space, it is angle dependent since there are magnetic fields. There are nonzero components in the cylindrical polar coordinates P and P_3. With this initial state we shall evaluate the response function (A6).

We substitute (A9) in (A6) and effect the operations with the operator. The δ functions are angle independent and hence the operator acts only on p_2 and ξ. We perform the ξ, η, and ζ integrations and get a conservation law as $\mathbf{K}_{\lambda'}=\mathbf{K}_\lambda+\boldsymbol{\epsilon}_1 l$ and the corresponding polarization vector ensuring $\mathbf{K}_{\lambda'}\cdot\mathbf{e}_{\lambda'}=0$ as

$$\mathbf{e}_{\lambda'}=\mathbf{e}_\lambda-\boldsymbol{\epsilon}_3\frac{\mathbf{e}_\lambda\cdot\boldsymbol{\epsilon}_1}{\mathbf{K}_\lambda\cdot\boldsymbol{\epsilon}_3}l \tag{A10}$$

and this gives

$$\mathbf{e}_{\lambda'}\cdot\mathbf{K}_\lambda=-\mathbf{e}_{1\lambda}l\ ,$$
$$\boldsymbol{\epsilon}_3\times\mathbf{e}_{\lambda'}=\boldsymbol{\epsilon}_3\times\mathbf{e}_\lambda\ , \tag{A11}$$
$$(\boldsymbol{\epsilon}_3\times\mathbf{e}_{\lambda'})\cdot\mathbf{e}_\lambda\equiv 0\ .$$

We now effect the summation over the Landau levels and obtain (A6) as

$$R=\left[\frac{2\omega_{\rm pl}^2}{\bar{\nu}_{\lambda'}}l\Gamma_{\rm NR}\sin[\bar{\nu}_{\lambda'}(t_3-t_4)]\right]$$
$$\times\int d\mathbf{P}_l\left[(P_{1l}-\frac{K_{1\lambda}(\mathbf{K}_\lambda\cdot\mathbf{P}_l)}{K_\lambda^2}\right]$$
$$\times\sin(\bar{\phi})\delta(P-P_0)\delta(P_3-P_3^0)\ , \tag{A12}$$

where (NR stands for nonrelativistic)

$$\Gamma_{\rm NR}=\frac{\exp\left[-\frac{l^2}{4}\coth\frac{\beta\hbar\Omega_0}{2}\right]}{\sinh\frac{\beta\hbar\Omega_0}{2}}\ , \tag{A13}$$

$$\bar{\phi}=\frac{\Omega_0}{\bar{\Omega}}(P_1K_{2\lambda}-P_2K_{1\lambda}+lP_2)-\Omega_0K_{3\lambda}P_3(t_2-t_3)$$
$$-\frac{\Omega_0}{\bar{\Omega}}[(K_{1\lambda}P_1+K_{2\lambda}P_2)\sin\bar{\Omega}(t_2-t_3)+(P_1K_{2\lambda}-P_2K_{1\lambda})\cos\bar{\Omega}(t_2-t_3)]$$
$$+l\left[1+\frac{\Omega_0}{\bar{\Omega}}\right](P_1\sin\bar{\Omega}t_3-P_2\cos\bar{\Omega}t_3)$$
$$=-2\frac{\Omega_0}{\bar{\Omega}}PK_\lambda\sin\theta_k\sin\frac{\bar{\Omega}t_{23}}{2}\cos(\phi_k-\phi_p+\bar{\Omega}t_{23})-K_{3\lambda}P_3\Omega_0t_{23}\cos\theta_k+lPA\,\sin(\theta_p+\alpha)\ , \tag{A14}$$

wherein

$$t_{23}=t_2-t_3,\quad A\cos\alpha=\frac{\Omega_0}{\bar{\Omega}}-\left[1+\frac{\Omega_0}{\bar{\Omega}}\right]\cos\bar{\Omega}t_3,\quad A\sin\alpha=\left[1+\frac{\Omega_0}{\bar{\Omega}}\right]\sin\bar{\Omega}t_3\ ,$$
$$A^2=\left[\frac{\Omega_0}{\bar{\Omega}}\right]^2+\left[1+\frac{\Omega_0}{\bar{\Omega}}\right]^2-2\left[\frac{\Omega_0}{\bar{\Omega}}\right]\left[1+\frac{\Omega_0}{\bar{\Omega}}\right]\cos\bar{\Omega}t_3\ . \tag{A15}$$

In writing (A14) we have used cylindrical polar coordinates for **P** and spherical polar coordinates for $\mathbf{K}_\lambda$. We now substitute (A14) in (A12) and consider the isotropic part in the integrand, i.e., terms independent of the propagation vector angles. We now perform the integration over P, θ_p, and P_3 and we write R in Laplace space as

$$\Delta(z)=\frac{\chi(z)}{(z^2-\bar{v}_\lambda^2)(z^2-\bar{\Omega}^2)} \tag{A16}$$

where

$$\chi(z)=\left[2\pi\omega_{\rm pl}^2\left[\frac{\Omega_0}{\bar{\Omega}}\right]\left[1+\frac{\Omega_0}{\bar{\Omega}}\right]P_0(P_0-2)\frac{\exp\left[-\frac{l^2}{4}\coth\frac{\beta\hbar\Omega_0}{2}\right]}{\sinh\frac{\beta\hbar\Omega_0}{2}}\right]$$
$$\times\left[\sum_n \mathcal{E}_n J_n[(\Omega_0/\bar{\Omega})P_0 l]J_n[\overline{1+(\Omega_0/\bar{\Omega})P_0 l}]\bar{\Omega}\bar{v}_\lambda\right]$$
$$\times\left[\frac{(n+1)(3z^2+(\overline{n+1}\,\bar{\Omega})^2-\bar{v}_\lambda^2)(z^2-\bar{v}_\lambda^2)(z^2-\bar{\Omega}^2)}{[z^2-(\overline{n+1}\bar{\Omega})^2][z^2-(\overline{n+1}\bar{\Omega}-\bar{v}_\lambda)^2][z^2-(\overline{n+1}\bar{\Omega}+\bar{v}_\lambda)^2]}\right.$$
$$\left.-\frac{(n-1)[3z^2+(\overline{n-1}\bar{\Omega})^2-\bar{v}_\lambda^2](z^2-\bar{v}_\lambda^2)(z^2-\bar{\Omega}^2)}{[z^2-(\overline{n-1}\bar{\Omega})^2][z^2-(\overline{n-1}\bar{\Omega}+\bar{v}_\lambda)^2][z^2-(\overline{n-1}\bar{\Omega}-\bar{v}_\lambda)^2]}\right] \tag{A17}$$

with $\mathcal{E}_n=1$ for $n=0$ and 2 for $n\neq 0$; $n=1,2,3,\ldots$.

The dominant term in (A17) is when $n=0$ and we then have

$$\chi(z)=\left[4\pi\omega_{\rm pl}^2\left[\frac{\Omega_0}{\bar{\Omega}}\right]\left[1+\frac{\Omega_0}{\bar{\Omega}}\right]P_0(P_0-2)\frac{\exp\left[-\frac{l^2}{4}\coth\frac{\beta\hbar\Omega_0}{2}\right]}{\sinh\frac{\beta\hbar\Omega_o}{2}}\right]$$
$$\times\left[\frac{\bar{\Omega}\bar{v}_\lambda(3z^2+\bar{\Omega}^2-\bar{v}_\lambda^2)(z^2-\bar{v}_\lambda^2)}{[z^2-(\bar{\Omega}+\bar{v}_\lambda)^2][z^2-(\bar{\Omega}-\bar{v}_\lambda)^2]}\right]J_0\left[lP_0\frac{\Omega_0}{\bar{\Omega}}\right]J_0[lP_0\overline{1+(\Omega_0/\bar{\Omega})}]\ . \tag{A18}$$

In evaluating these integrals, we had obtained $J_o(lP_0A)$, A being given by (A15). We expand this using the addition theorem in Bessel function.

We shall now go over to the kinetic regime by taking the asymptotic limit in time, i.e., $z\to 0$ in (A18) and write (A16) as

$$\Delta(z)=\frac{\chi(0)}{(z^2-\bar{v}_\lambda^2)(z^2-\bar{\Omega}^2)}\ , \tag{A19}$$

where now

$$\chi(0)=-\left[4\pi\omega_{\rm pl}^2\left[\frac{\Omega_0}{\bar{\Omega}}\right]\left[1+\frac{\Omega_0}{\bar{\Omega}}\right]\frac{(\bar{\Omega}\bar{v}_\lambda)^2}{\bar{\Omega}^2-\bar{v}_\lambda^2}\left[\frac{\bar{v}_\lambda}{\bar{\Omega}}\right]P_0(P_0-2)\frac{\exp\left[-\frac{l^2}{4}\coth\frac{\beta\hbar\Omega_0}{2}\right]}{\sinh\frac{\beta\hbar\Omega_0}{2}}\right]J_0\left[lP_0\frac{\Omega_0}{\bar{\Omega}}\right]J_0[lP_0\overline{1+(\Omega_0/\bar{\Omega})}]\ . \tag{A20}$$

The appearance of Bessel functions in (A19) give $\chi(0)$ as a multivalued function of l. In particular $\chi(0)$ can be zero if $lP_0(\Omega_0/\overline{\Omega})$ or $[lP_0\overline{1+(\Omega_0/\overline{\Omega})}]$ or both are zeros of J_0, and this makes the response function zero or the medium acts as a vacuum for the radiation. Again if they are mth and nth zeros of J_0, we then have

$$\frac{1+\frac{\Omega_0}{\overline{\Omega}}}{\frac{\Omega_0}{\overline{\Omega}}}=\frac{x_n}{x_m} \tag{A21}$$

or the ratio of the cyclotron frequencies would be

$$\frac{\Omega}{\Omega_0}=[\alpha(\alpha-2)]^{1/2} , \tag{A22}$$

α being x_n/x_m. Again this would give a measure of lP_0, viz.

$$lP_0=x_n-x_m . \tag{A23}$$

l and P_0 are still arbitrary. If we choose $|l|=|k_w|$, then $P_0=(x_n-x_m)/|k_w|$. We then get a quantization condition for P_0 as x_n-x_m are fixed quantities for the different sets of n and m.

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