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Negative-energy waves in a magnetized homogeneous plasma

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The general expression for the second-order wave energy of a Vlasov-Maxwell system derived by Morrison and Pfirsch [Phys. Rev. A **40**, 3898 (1989); Phys. Fluids B **2**, 1105 (1990)] is evaluated here for the case of electrostatic perturbations in a magnetized, homogeneous plasma. It is again shown that negative-energy waves (which could become nonlinearly unstable and cause anomalous transport) exist for *any deviation from monotonicity* and/or *any* (however small) *anisotropy* in the equilibrium distribution function of any of the particle species. The partly unexpected and particularly interesting feature of the results is that, contrary to the proof of Morrison and Pfirsch, no restricting condition has to be imposed on the perpendicular wave number $k_\perp$ of the perturbation (i.e., large $k_\perp$ is not required). Finite-gyroradius effects are therefore not expected to improve the situation. Anisotropy alone would, however, impose a restriction on k_z, the parallel wave number, relating it to the gyroradius. As far as distribution functions with $v_z(\partial f_\nu^{(0)}/\partial v_z)>0$ in some region of **v** space are concerned, however, this result agrees with a result found by Pfirsch and Morrison within the framework of drift-kinetic theory.

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I. INTRODUCTION

A general expression for the second variation of the free energy of a Vlasov-Maxwell equilibrium was previously derived by Morrison and Pfirsch [1,2], who showed that negative-energy modes exist whenever the equilibrium distribution $f_\nu^{(0)}(\mathbf{x},\mathbf{v})$ of any particle species ν satisfies the inequality

$$(\mathbf{v}\cdot\mathbf{k})\left[\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\mathbf{k}\right]>0 \tag{1}$$

for some position vector **x** and velocity **v** and for some vector **k**. Such negative-energy modes are important because they may become nonlinearly unstable [3,4] and be of relevance to anomalous transport phenomena. However, the condition for the existence of these modes may require very highly localized perturbations, i.e., very high mode numbers k. In fact, Morrison and Pfirsch made this assumption in order to prove condition (1). As far as distribution functions with $v_z(\partial f_\nu^{(0)}/\partial v_z)>0$ (v_z is the component of the velocity in the direction of the equilibrium magnetic field) in some region of **v** space are concerned, Pfirsch and Morrison [5], Eq. (144.b), obtained negative-energy perturbations within the framework of drift-kinetic theory with no conditions on the perpendicular and parallel wave numbers $k_\perp, k_z$, except $k_z\neq 0$. Since the Vlasov theory becomes inapplicable for wavelengths smaller than the Debye length, one must investigate how strongly localized the perturbations have to be. Also, if the required wavelengths are much smaller than the gyroradii, the relevance of the results is questionable, and finite-gyroradius effects would have to be taken into account. This paper treats this question for the case of a general, magnetized, homogeneous plasma; the localization needed for an inhomogeneous system is expected to be of the same order of magnitude.

In the following, the right choice of representation of the perturbations in velocity space is seen to lead to clear, simple *but partly unexpected* results, namely the fact that for the existence of negative-energy waves in the system under investigation *no restriction has to be imposed on* $k_\perp$ if a monotonicity-isotropy condition for the equilibrium distribution function $f_\nu^{(0)}$ of any particle species ν is violated. However, if only anisotropy is present, then a restriction relating the *parallel* wave number k_z to the gyroradius has to be imposed.

II. A CONVENIENT EXPRESSION FOR THE FREE ENERGY OF A GENERAL VLASOV PLASMA

Within the framework of Maxwell-Vlasov theory, Morrison and Pfirsch [1,2] derived expressions for the free energy available upon arbitrary perturbations of an arbitrary Vlasov-Maxwell equilibrium. If this free energy is denoted by $\delta^2 H$, then the expression derived in [1], Eq. (61), reads

$$\delta^2H=\sum_\nu\int d^3x\,d^3v f_\nu^{(0)}(\mathbf{x},\mathbf{v})\left\{\frac{m_\nu}{2}[(\delta\dot{\mathbf{x}}_\nu)^2-(d\delta\mathbf{x}_\nu)^2]+\frac{e_\nu}{2}\left[-2\delta\mathbf{x}_\nu\cdot\frac{\mathbf{v}\times\delta\mathbf{B}}{c}+\frac{\delta\mathbf{x}_\nu\times\mathbf{B}^{(0)}}{c}\cdot d\delta\mathbf{x}_\nu-\delta\mathbf{x}_\nu\cdot(\delta\mathbf{x}_\nu\cdot\nabla)\left[\mathbf{E}^{(0)}+\frac{\mathbf{v}\times\mathbf{B}^{(0)}}{c}\right]\right]\right\}+\frac{1}{8\pi}\int d^3x(\delta E^2+\delta B^2)\ . \tag{2}$$

Here, the species ν with equilibrium distribution function $f_\nu^{(0)}(\mathbf{x},\mathbf{v})$ consists of particles of electric charge e_ν and mass m_ν. $\mathbf{E}^{(0)}$ and $\mathbf{B}^{(0)}$ are the equilibrium electric and magnetic fields, respectively, and $\delta E^2/8\pi$ and $\delta B^2/8\pi$ are the perturbations in the electric- and magnetic-field energy densities. The operator d is defined as the equilibrium Vlasov operator, i.e.,

$$d \equiv \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}+\frac{e_\nu}{m_\nu}\left[\mathbf{E}^{(0)}+\frac{\mathbf{v}\times\mathbf{B}^{(0)}}{c}\right]\cdot\frac{\partial}{\partial \mathbf{v}} . \tag{3}$$

The particle displacement $\delta\mathbf{x}_\nu$ and the particle velocity perturbation $\delta\dot{x}_\nu$ are derived from a generating function $G_\nu(\mathbf{x},\mathbf{v})$ through the relations

$$\delta\mathbf{x}_\nu=\frac{1}{m_\nu}\frac{\partial G_\nu}{\partial \mathbf{v}} \tag{4}$$

and

$$\begin{aligned}\delta\dot{\mathbf{x}}_\nu&=-\frac{1}{m_\nu}\left[\frac{\partial G_\nu}{\partial \mathbf{x}}+\frac{e_\nu}{m_\nu c}\frac{\partial G_\nu}{\partial v_i}\left[-\frac{\partial A_i^{(0)}}{\partial \mathbf{x}}+\frac{\partial \mathbf{A}^{(0)}}{\partial x_i}\right]+d\frac{\partial G_\nu}{\partial \mathbf{v}}+\frac{e_\nu}{c}\delta\mathbf{A}\right]\\ &=-\frac{1}{m_\nu}\left[\frac{\partial G_\nu}{\partial \mathbf{x}}+\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}+d\frac{\partial G_\nu}{\partial \mathbf{v}}+\frac{e_\nu}{c}\delta\mathbf{A}\right] ,\end{aligned} \tag{5}$$

where $\mathbf{A}^{(0)}$ is the equilibrium vector potential and $\delta\mathbf{A}$ is the corresponding perturbation.

From the definition of the operator d it follows that

$$\frac{\partial}{\partial \mathbf{v}}(dG_\nu)=d\left[\frac{\partial G_\nu}{\partial \mathbf{v}}\right]+\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}+\frac{\partial G_\nu}{\partial \mathbf{x}} , \tag{6}$$

and Eq. (5) can be expressed as

$$\delta\dot{\mathbf{x}}_\nu=-\frac{1}{m_\nu}\frac{\partial}{\partial \mathbf{v}}(dG_\nu)-\frac{e_\nu}{m_\nu c}\delta\mathbf{A} . \tag{7}$$

Combining Eqs. (4) and (6) yields

$$d\,\delta\mathbf{x}_\nu=\frac{1}{m_\nu}\left[\frac{\partial}{\partial \mathbf{v}}(dG_\nu)-\frac{e_\nu}{c}\mathbf{B}^{(0)}\times\delta\mathbf{x}_\nu-\frac{\partial G_\nu}{\partial \mathbf{x}}\right] \tag{8}$$

and, therefore,

$$\begin{aligned}\frac{m_\nu}{2}[(\delta\dot{\mathbf{x}}_\nu)^2-(d\,\delta\mathbf{x}_\nu)^2]+\frac{e_\nu}{2c}\delta\mathbf{x}_\nu\times\mathbf{B}^{(0)}\cdot d\,\delta\mathbf{x}_\nu=\frac{1}{2m_\nu}&\left[2\frac{e_\nu}{c}\delta\mathbf{A}\cdot\frac{\partial}{\partial \mathbf{v}}(dG_\nu)+\frac{e_\nu^2}{c^2}(\delta\mathbf{A})^2+\frac{e_\nu}{c}\mathbf{B}^{(0)}\times\delta\mathbf{x}_\nu\cdot\frac{\partial}{\partial \mathbf{v}}(dG_\nu)\right.\\ &\left.-\frac{e_\nu}{c}\mathbf{B}^{(0)}\times\delta\mathbf{x}_\nu\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}+2\frac{\partial G_\nu}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{v}}(dG_\nu)-\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\right]^2\right] .\end{aligned} \tag{9}$$

The second-order wave energy can then be expressed as

$$\begin{aligned}\delta^2 H=\sum_\nu\int\frac{d^3x\,d^3v}{2m_\nu}f_\nu^{(0)}(\mathbf{x},\mathbf{v})&\left\{-\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\right]^2+2\frac{\partial G_\nu}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{v}}(dG_\nu)-\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}+\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{v}}dG_\nu\right.\\ &+\frac{e_\nu^2}{c^2}(\delta\mathbf{A})^2+2\frac{e_\nu}{c}\delta\mathbf{A}\cdot\frac{\partial}{\partial \mathbf{v}}(dG_\nu)+2\frac{e_\nu}{c}\mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\times\delta\mathbf{B}\\ &\left.-\frac{e_\nu}{m_\nu}\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\left[\left[\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{x}}\right]\left[\mathbf{E}^{(0)}+\frac{\mathbf{v}\times\mathbf{B}^{(0)}}{c}\right]\right]\right\}+\frac{1}{8\pi}\int d^3x(\delta E^2+\delta B^2) .\end{aligned} \tag{10}$$

The terms appearing in $\delta^2 H$ can be transformed into more convenient expressions which, with the single exception of the term quadratic in $\delta\mathbf{A}$, do not contain $f_\nu^{(0)}$ itself, but only its derivatives in **x**-**v** space. For this purpose, we define a vector $\mathbf{a}_\nu^{(0)}$ as in [1], i.e.,

$$\mathbf{a}_\nu^{(0)}\equiv\frac{e_\nu}{m_\nu}\left[\mathbf{E}^{(0)}+\frac{\mathbf{v}\times\mathbf{B}^{(0)}}{c}\right] , \tag{11}$$

and take into account the following identities:

$$2\frac{\partial G_\nu}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{v}}\left[\mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]=\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\right]^2+\frac{\partial}{\partial \mathbf{v}}\cdot\left[\left[\mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\frac{\partial G_\nu}{\partial \mathbf{x}}\right]+\frac{\partial}{\partial \mathbf{x}}\cdot\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\times\left[\mathbf{v}\times\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\right], \tag{12}$$

$$\begin{aligned}2\frac{\partial G_\nu}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{v}}\left[\mathbf{a}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]&-\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}+\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{v}}(dG_\nu)-\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\left[\left[\frac{\partial G_\nu}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{x}}\right]\mathbf{a}_\nu^{(0)}\right]\\
&=\frac{\partial}{\partial \mathbf{v}}\cdot\left[2\left[\mathbf{a}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\frac{\partial G_\nu}{\partial \mathbf{x}}-\frac{e_\nu}{m_\nu c}G_\nu\mathbf{v}\times\frac{\partial}{\partial \mathbf{v}}\left[\mathbf{B}^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]+\frac{e_\nu}{m_\nu}G_\nu\frac{\partial}{\partial \mathbf{v}}\times\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\times\mathbf{E}^{(0)}\right]\right]\\
&\quad-\frac{\partial}{\partial \mathbf{x}}\cdot\left[\left[\mathbf{a}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\frac{\partial G_\nu}{\partial \mathbf{v}}\right],\end{aligned} \tag{13}$$

$$2\frac{e_\nu}{c}\left[\delta\mathbf{A}\cdot\frac{\partial}{\partial \mathbf{v}}(dG_\nu)+\mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\times\delta\mathbf{B}\right]=2\frac{e_\nu}{c}\frac{\partial}{\partial \mathbf{v}}\cdot\left[d(G_\nu\delta\mathbf{A})-G_\nu\frac{\partial}{\partial \mathbf{x}}(\mathbf{v}\cdot\delta\mathbf{A})\right]. \tag{14}$$

These relations allow the second-order wave energy to be written as

$$\begin{aligned}\delta^2H=\sum_\nu\int\frac{d^3x\,d^3v}{2m_\nu}f_\nu^{(0)}(\mathbf{x},\mathbf{v})&\left\{\frac{\partial}{\partial \mathbf{v}}\cdot\left[\left[\mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\frac{\partial G_\nu}{\partial \mathbf{x}}+\left[\mathbf{a}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\left[\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}+2\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\right.\right.\\
&\left.-\frac{e_\nu}{m_\nu c}G_\nu\mathbf{v}\times\frac{\partial}{\partial \mathbf{v}}\left[\mathbf{B}^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]+\frac{e_\nu}{m_\nu}G_\nu\frac{\partial}{\partial \mathbf{v}}\times\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\times\mathbf{E}^{(0)}\right]\right]\\
&+\frac{\partial}{\partial \mathbf{x}}\cdot\left[\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\mathbf{v}-(dG_\nu)\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\\
&\left.+\left[\frac{e_\nu}{c}\delta\mathbf{A}\right]^2+2\frac{e_\nu}{c}\frac{\partial}{\partial \mathbf{v}}\cdot\left[d(G_\nu\delta\mathbf{A})-G_\nu\frac{\partial}{\partial \mathbf{x}}(\mathbf{v}\cdot\delta\mathbf{A}\right]\right\}+\frac{1}{8\pi}\int d^3x(\delta E^2+\delta B^2)\,.\end{aligned} \tag{15}$$

Here, all the terms in curly brackets which depend on the generating function G_ν are expressed as divergences either in $\mathbf{v}$ or in $\mathbf{x}$ space. This proves convenient for applications.

It is straightforward, but lengthy and tedious, to show that Eq. (15) is in fact the same as Eq. (10).

Through some integrations by parts and neglect of surface terms, Eq. (15) can be transformed to

$$\begin{aligned}\delta^2H=\sum_\nu\int\frac{d^3x\,d^3v}{2m_\nu}&\left\{\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\left[-\left[\mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\frac{\partial G_\nu}{\partial \mathbf{x}}-\left[\mathbf{a}_\nu^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\left[\frac{e_\nu}{m_\nu c}\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}+2\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\right.\right.\\
&\left.+\frac{e_\nu}{m_\nu c}G_\nu\mathbf{v}\times\frac{\partial}{\partial \mathbf{v}}\left[\mathbf{B}^{(0)}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]-\frac{e_\nu}{m_\nu}G_\nu\frac{\partial}{\partial \mathbf{v}}\times\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\times\mathbf{E}^{(0)}\right]\right]\\
&+\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\cdot\left[-\left[\frac{\partial G_\nu}{\partial \mathbf{x}}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\mathbf{v}+(dG_\nu)\frac{\partial G_\nu}{\partial \mathbf{v}}\right]\\
&\left.+f_\nu^{(0)}\left[\frac{e_\nu}{c}\delta\mathbf{A}\right]^2-2\frac{e_\nu}{c}\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\left[d(G_\nu\delta\mathbf{A})-G_\nu\frac{\partial}{\partial \mathbf{x}}(\mathbf{v}\cdot\delta\mathbf{A})\right]\right\}+\frac{1}{8\pi}\int d^3x(\delta E^2+\delta B^2)\,,\end{aligned} \tag{16}$$

an expression that has the same structure as Eq. (13) of Ref. [2], but with $\mathbf{x}$ and $\mathbf{v}$ as the independent variables.

III. SECOND-ORDER ELECTROSTATIC WAVE ENERGY FOR A MAGNETIZED HOMOGENEOUS PLASMA

We now consider a homogeneous equilibrium with a constant, unperturbed magnetic field and no electric field, and assume purely electrostatic perturbations, i.e., we take

$$\mathbf{B}^{(0)}=B^{(0)}\mathbf{e}_z\,,\quad \mathbf{E}^{(0)}=\mathbf{0}\,,\quad \frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}=\mathbf{0}\,, \tag{17}$$

and set

$$\delta \mathbf{A}=0 . \tag{18}$$

In this case, it is convenient to use Cartesian coordinates (x,y,z) in $\mathbf{x}$ space, and cylindrical coordinates $(v_\perp,\phi,v_z)$ in $\mathbf{v}$ space, ϕ being the angle between the projection of $\mathbf{v}$ onto the x-y plane and the (arbitrary) x axis.

With these assumptions, Vlasov's equation reduces to $\partial f_\nu^{(0)}/\partial\phi=0$. Furthermore,

$$\mathbf{a}_\nu^{(0)}\cdot\frac{\partial}{\partial \mathbf{v}}=-\omega_\nu\frac{\partial}{\partial\phi} , \tag{19}$$

where we have set

$$\omega_\nu\equiv\frac{e_\nu B^{(0)}}{m_\nu c} . \tag{20}$$

Thus, Eq. (16) becomes

$$\delta^2 H=\sum_\nu \int \frac{d^3x\, d^3v}{2m_\nu}\left[-\left[\mathbf{v}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]\left[\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]-2\omega_\nu^2\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\left[\frac{\partial G_\nu}{\partial\phi}\right]^2 +2\omega_\nu\frac{\partial G_\nu}{\partial\phi}\left[\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]+2\omega_\nu v_z\left[\frac{\partial f_\nu^{(0)}}{\partial v_z^2}-\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\right]G_\nu\frac{\partial^2 G_\nu}{\partial\phi\partial z}\right]+\frac{1}{8\pi}\int d^3x\,\delta E^2 . \tag{21}$$

Note that derivatives of G_ν in $\mathbf{v}$ space only *appear* as derivatives with respect to ϕ.

Since the equilibrium is $\mathbf{x}$ independent, an appropriate ansatz for the generating function $G_\nu(\mathbf{x},\mathbf{v})$ is

$$G_\nu(\mathbf{x},\mathbf{v})=\tfrac{1}{2}[g_\nu(\mathbf{v})e^{i\mathbf{k}\cdot\mathbf{x}}+g_\nu^*(\mathbf{v})e^{-i\mathbf{k}\cdot\mathbf{x}}] . \tag{22}$$

G_ν is obviously real, g_ν^* being the complex conjugate of g_ν. We limit ourselves here to a single $\mathbf{k}$. Any generating function G_ν could be represented as a Fourier integral over d^3k, with coefficients $g_\nu(\mathbf{v},\mathbf{k})$.

Inserting Eq. (22) in Eq. (21) and subsequent $\mathbf{x}$ integration over a periodicty volume V leads to

$$\delta^2 H=\sum_\nu \frac{V}{4m_\nu}\int d^3v\left[-(\mathbf{v}\cdot\mathbf{k})\left[\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\mathbf{k}\right]g_\nu g_\nu^* -2\omega_\nu^2\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\frac{\partial g_\nu}{\partial\phi}\frac{\partial g_\nu^*}{\partial\phi}+i\omega_\nu\left[\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\mathbf{k}\right]\left[g_\nu\frac{\partial g_\nu^*}{\partial\phi}-g_\nu^*\frac{\partial g_\nu}{\partial\phi}\right] +i\omega_\nu k_z v_z\left[\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}-\frac{\partial f_\nu^{(0)}}{\partial v_z^2}\right]\left[g_\nu\frac{\partial g_\nu^*}{\partial\phi}-g_\nu^*\frac{\partial g_\nu}{\partial\phi}\right]\right]+\frac{1}{8\pi}\int d^3x\,\delta E^2 . \tag{23}$$

The complex function $g_\nu(\mathbf{v})$ can be represented as

$$g_\nu(\mathbf{v})=\Psi_\nu(\mathbf{v})e^{i\Gamma_\nu(\mathbf{v})} , \tag{24}$$

where $\Psi_\nu(\mathbf{v})$ and $\Gamma_\nu(\mathbf{v})$ are *real* functions. Since $g_\nu(\mathbf{v})$ must be single valued, Ψ_ν and Γ_ν are subject to the periodicity conditions

$$\Psi_\nu(v_\perp,\phi+2\pi,v_z)=\Psi_\nu(v_\perp,\phi,v_z) \tag{25}$$

and

$$\Gamma_\nu(v_\perp,\phi+2\pi,v_z)=\Gamma_\nu(v_\perp,\phi,v_z)+2\pi n_\nu , \tag{26}$$

with n_ν any integer number, i.e., $n_\nu=0,\pm1,\ldots$.

Inserting Eq. (24) in Eq. (23) yields

$$\delta^2 H=\sum_\nu \frac{V}{4m_\nu}\int d^3v\left\{-(\mathbf{v}\cdot\mathbf{k})\left[\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\mathbf{k}\right]\Psi_\nu^2-2\omega_\nu^2\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\left[\left[\frac{\partial \Psi_\nu}{\partial\phi}\right]^2+\Psi_\nu^2\left[\frac{\partial \Gamma_\nu}{\partial\phi}\right]^2\right] +2\omega_\nu\left[\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\mathbf{k}\right]\Psi_\nu^2\frac{\partial\Gamma_\nu}{\partial\phi}+2\omega_\nu k_z v_z\left[\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}-\frac{\partial f_\nu^{(0)}}{\partial v_z^2}\right]\Psi_\nu^2\frac{\partial\Gamma_\nu}{\partial\phi}\right\}+\frac{1}{8\pi}\int d^3x\,\delta E^2 , \tag{27}$$

which is the general expression for the second-order energy of *electrostatic* waves of wave vector $\mathbf{k}$ in a homogeneous magnetized plasma.

Note that $\delta^2 H$ is now a functional of Ψ_ν, which appears as Ψ_ν and $\partial\Psi_\nu/\partial\phi$, and of Γ_ν, which appears *only* as $\partial\Gamma_\nu/\partial\phi$.

IV. EXTREMIZATON OF THE FREE ENERGY

It is now straightforward to minimize Eq. (27) with respect to Γ_ν. This can be done either by minimizing with respect to Γ_ν itself, with Eq. (26) taken into account as a boundary condition, or by minimizing with respect to $\partial\Gamma_\nu/\partial\phi$, but then the subsidiary condition

$$\int_0^{2\pi}\frac{\partial\Gamma_\nu}{\partial\phi}d\phi=2\pi n_\nu \tag{28}$$

would have to be introduced. We choose the first way: the Euler equation to minimize $\delta^2 H$ with respect to Γ_ν, if we write $\delta^2 H=\int d^3v\, I(\Gamma_\nu,\Gamma_{\nu,\phi})$, $\Gamma_{\nu,\phi}\equiv\partial\Gamma_\nu/\partial\phi$, is

$$\frac{\partial}{\partial\phi}\left[\frac{\partial I}{\partial\Gamma_{\nu,\phi}}\right]-\frac{\partial I}{\partial\Gamma_\nu}=0 . \tag{29}$$

Since $\partial I/\partial\Gamma_\nu=0$, Eq. (29) implies

$$\frac{\partial I}{\partial\Gamma_{\nu,\phi}}=C_\nu(v_\perp,v_z) . \tag{30}$$

Explicitly, this means that

$$-4\omega_\nu^2\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\Psi_\nu^2\frac{\partial\Gamma_\nu}{\partial\phi}+2\omega_\nu\left[\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\cdot\mathbf{k}\right]\Psi_\nu^2+2\omega_\nu k_z v_z\Psi_\nu^2\left[\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}-\frac{\partial f_\nu^{(0)}}{\partial v_z^2}\right]=C_\nu(v_\perp,v_z) . \tag{31}$$

From Eqs. (27) and (31) we then obtain

$$\delta^2 H=\sum_\nu\frac{V}{4m_\nu}\int d^3v\left[-(\mathbf{v}\cdot\mathbf{k})\left[\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\cdot\mathbf{k}\right]\Psi_\nu^2-2\omega_\nu^2\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\left[\frac{\partial\Psi_\nu}{\partial\phi}\right]^2+2\omega_\nu^2\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\Psi_\nu^2\left[\frac{\partial\Gamma_\nu}{\partial\phi}\right]^2+C_\nu\frac{\partial\Gamma_\nu}{\partial\phi}\right], \tag{32}$$

where $\partial\Gamma_\nu/\partial\phi$, as determined from Eq. (31), still has to be inserted explicitly. The electrostatic energy term has been dropped in Eq. (32) since the perturbed charge density can be made zero for the perturbations considered here. This is shown in the Appendix.

By inserting $(\partial f_\nu^{(0)}/\partial\mathbf{v})\cdot\mathbf{k}$ explicitly into Eq. (31), we see that Γ_ν can be split into a particular periodic part $\Gamma_\nu^{(1)}$ and a nonperiodic contribution $\Gamma_\nu^{(2)}$:

$$\Gamma_\nu=\Gamma_\nu^{(1)}+\Gamma_\nu^{(2)} , \tag{33}$$

where

$$\Gamma_\nu^{(1)}=\frac{v_\perp}{\omega_\nu}(k_x\sin\phi-k_y\cos\phi) , \tag{34}$$

i.e.,

$$\frac{\partial\Gamma_\nu^{(1)}}{\partial\phi}=\frac{v_\perp}{\omega_\nu}(k_x\cos\phi+k_y\sin\phi)=\frac{\mathbf{k}\cdot\mathbf{v}_\perp}{\omega_\nu} . \tag{35}$$

The term $(\mathbf{v}\cdot\mathbf{k})[(\partial f_\nu^{(0)}/\partial\mathbf{v})\cdot\mathbf{k}]$ appearing in Eq. (32) is, explicitly

$$(\mathbf{v}\cdot\mathbf{k})\left[\frac{\partial f_\nu^{(0)}}{\partial\mathbf{v}}\cdot\mathbf{k}\right]=2[v_\perp^2(\mathbf{e}_{v_\perp}\cdot\mathbf{k})^2+v_z k_z v_\perp(\mathbf{e}_{v_\perp}\cdot\mathbf{k})]\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}+2[v_z^2k_z^2+v_z k_z v_\perp(\mathbf{e}_{v_\perp}\cdot\mathbf{k})]\frac{\partial f_\nu^{(0)}}{\partial v_z^2} , \tag{36}$$

where

$$\mathbf{e}_{v_\perp}\cdot\mathbf{k}=k_x\cos\phi+k_y\sin\phi . \tag{37}$$

Inserting Eqs. (33), (35), and (36) in Eq. (32) yields

$$\delta^2 H=\sum_\nu\frac{V}{4m_\nu}\int d^3v\left[-2v_z^2k_z^2\frac{\partial f_\nu^{(0)}}{\partial v_z^2}\Psi_\nu^2-2\omega_\nu^2\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\left[\frac{\partial\Psi_\nu}{\partial\phi}\right]^2+2\omega_\nu^2\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\Psi_\nu^2\left[\frac{\partial\Gamma_\nu^{(2)}}{\partial\phi}\right]^2+C_\nu(v_\perp,v_z)\frac{\partial\Gamma_\nu^{(2)}}{\partial\phi}\right] . \tag{38}$$

Note that this expression for $\delta^2 H$ *does not contain* $\mathbf{e}_{v_\perp}\cdot\mathbf{k}$ any more, but only k_z. This means that the results will be independent of $k_\perp$, the perpendicular wave number. If we define

$$F_\nu(v_\perp,v_z)\equiv\frac{k_z v_z}{2\omega_\nu(\partial f_\nu^{(0)}/\partial v_\perp^2)}\left[\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}+\frac{\partial f_\nu^{(0)}}{\partial v_z^2}\right] , \tag{39}$$

then the function $\Gamma_\nu^{(2)}$ has to satisfy the equation

$$\left[\frac{\partial\Gamma_\nu^{(2)}}{\partial\phi}-F_\nu\right]\Psi_\nu^2=-\frac{C_\nu(v_\perp,v_z)}{4\omega_\nu^2(\partial f_\nu^{(0)}/\partial v_\perp^2)} . \tag{40}$$

The functions $C_\nu(v_\perp,v_z)$, which is constant in ϕ, and $\Gamma_\nu^{(2)}$ are determined from Eq. (40), together with the boundary condition on $\Gamma_\nu^{(2)}$, namely $\Gamma_\nu^{(2)}(\phi+2\pi)=\Gamma_\nu^{(2)}(\phi)+2\pi n_\nu$:

$$C_\nu=8\pi\omega_\nu^2\frac{1}{\int_0^{2\pi}(d\phi/\Psi_\nu^2)}(F_\nu-n_\nu)\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2} \tag{41}$$

and

$$\frac{\partial\Gamma_\nu^{(2)}}{\partial\phi}=F_\nu+2\pi(n_\nu-F_\nu)\frac{1}{\Psi_\nu^2}\frac{1}{\int_0^{2\pi}(d\phi/\Psi_\nu^2)} . \tag{42}$$

Inserting these results in Eq. (38) leads to

$$\delta^2H=\sum_\nu\frac{V}{4m_\nu}\int d^3v\,2\omega_\nu^2\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\left[\frac{v_z^2k_z^2}{4\omega_\nu^2}[1-\alpha_\nu(v_\perp,v_z)]^2\Psi_\nu^2-\left[\frac{\partial\Psi_\nu}{\partial\phi}\right]^2-\frac{2\pi}{\int_0^{2\pi}(d\phi/\Psi_\nu^2)}\left[\frac{v_zk_z}{2\omega_\nu}[1+\alpha_\nu(v_\perp,v_z)]-n_\nu\right]^2\right] , \tag{43}$$

where we have defined a *local anisotropy* parameter $\alpha_\nu(v_\perp,v_z)$:

$$\alpha_\nu(v_\perp,v_z)\equiv\frac{\partial f_\nu^{(0)}/\partial v_z^2}{\partial f_\nu^{(0)}/\partial v_\perp^2} . \tag{44}$$

We now consider this equation more closely:

A. $k_z=0$ (wave propagation perpendicular to $B^{(0)}$)

In this case we obtain

$$\delta^2H=\sum_\nu\frac{V}{4m_\nu}\int d^3v\,2\omega_\nu^2\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\times\left[-\left[\frac{\partial\Psi_\nu}{\partial\phi}\right]^2-\frac{2\pi n_\nu^2}{\int_0^{2\pi}(d\phi/\Psi_\nu^2)}\right] , \tag{45}$$

and $\delta^2H<0$ if $\partial f_\nu^{(0)}/\partial v_\perp^2>0$ for some $v_\perp,v_z$ and for any particle species ν, i.e., the presence of a local minimum in $f_\nu^{(0)}(v_\perp^2)$ guarantees $\delta^2H<0$ *for all* $k_\perp$. It suffices to localize Ψ_ν ($\partial\Psi_\nu/\partial\phi$ is then also localized) to the region in $v_\perp,v_z$ where $\partial f_\nu^{(0)}/\partial v_\perp^2>0$. Outside this region Ψ_ν vanishes and all other Ψ_μ are set equal to zero. The sign of δ^2H is then determined by the sign of the integrand in the region of localization. *There is no restriction on* $k_\perp$, contrary to the results obtained in [6]. Those results are obtained when the class of possible perturbations is restricted by a particular choice of test functions, namely $\Gamma_\nu\equiv0$ and $\partial\psi_\nu/\partial v_x\equiv0$, so that they do not correspond to the minimum δ^2H.

B. $k_z\neq0$ (either parallel or oblique wave propagation with respect to $B^{(0)}$)

If $\partial f_\nu^{(0)}/\partial v_\perp^2>0$ for some $v_\perp,v_z$, one localizes Ψ_ν around these velocities. Then if $\alpha_\nu=1$, all terms in Eq. (43) are negative; if $\alpha_\nu\neq1$, one can use n_ν to make the expression in the large square brackets negative; if $\alpha_\nu>0$, one can take $n_\nu=0$, $\partial\Psi_\nu/\partial\phi=0$; if $\alpha_\nu\leq0$, one can take $n_\nu>k_zv_z/\omega_\nu>0$ or $n_\nu<k_zv_z/\omega_\nu<0$, $\partial\Psi_\nu/\partial\phi=0$. Note that no condition is imposed on either $k_\perp$ or k_z.

If $\partial f_\nu^{(0)}/\partial v_\perp^2<0$ for some $v_\perp,v_z$, one again localizes Ψ_ν around these velocities. The case $\partial f_\nu^{(0)}/\partial v_\perp^2<0$ is the most interesting one since this condition always obtains for some $v_\perp$. The positive contribution of $(\partial\Psi_\nu/\partial\phi)^2$ can be eliminated by choosing

$$\frac{\partial\Psi_\nu}{\partial\phi}=0 . \tag{46}$$

In this case we have

$$\delta^2H=\sum_\nu\frac{V}{4m_\nu}\int d^3v\frac{k_z^2v_z^2}{2}\Psi_\nu^2\times\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}\left[(1-\alpha_\nu)^2-\left[1+\alpha_\nu-\frac{2\omega_\nu n_\nu}{k_zv_z}\right]^2\right] , \tag{47}$$

and thus, since Ψ_ν^2 is localized in $v_\perp,v_z$, the condition for $\delta^2H<0$ is

$$(1-\alpha_\nu)^2-\left[1+\alpha_\nu-\frac{2\omega_\nu n_\nu}{k_zv_z}\right]^2>0 , \tag{48}$$

which means either

$$\alpha_\nu>\frac{n_\nu\omega_\nu}{k_zv_z}>1 \tag{49}$$

(for $\alpha_\nu > 1$) or

$$\alpha_\nu < \frac{n_\nu \omega_\nu}{k_z v_z} < 1 \tag{50}$$

(for $\alpha_\nu < 1$).

The integer n_ν and the wave number k_z can be arbitrarily chosen, and it is always possible to satisfy one of the inequalities (49) or (50) for any anisotropy $\alpha_\nu \neq 1$, *without any restriction being imposed on* $k_\perp$.

If $\alpha_\nu > 1$ $(\partial f_\nu^{(0)}/\partial v_z^2 < \partial f_\nu^{(0)}/\partial v_\perp^2 < 0)$, then k_z is restricted by condition (49). If one sets $k_z v_z = (2\pi/\lambda_z) v_z = 2\pi/\tau_z$, then inequality (49) becomes

$$\alpha_\nu \frac{2\pi}{\omega_\nu} > n_\nu \tau_z > \frac{2\pi}{\omega_\nu} \ . \tag{51}$$

This means that n_ν times the time that a particle needs to travel the distance λ_z must be larger than the period of the gyromotion, but smaller than α_ν times this period.

If $1 > \alpha_\nu \geq 0$ $(\partial f_\nu^{(0)}/\partial v_\perp^2 < \partial f_\nu^{(0)}/\partial v_z^2 \leq 0)$, then k_z is restricted by condition (50) in a way similar to that in the preceding case.

If $\alpha_\nu < 0$ $(\partial f_\nu^{(0)}/\partial v_\perp^2 < 0,\ \partial f_\nu^{(0)}/\partial v_z^2 > 0)$, then choosing $n_\nu = 0$ satisfies inequality (50) *without any condition being imposed on* k_z, except $k_z \neq 0$. This is similar to the results obtained by Pfirsch and Morrison [5], Eq. (144.b), within the framework of drift-kinetic theory for equilibrium distribution functions with $v_z(\partial f_\nu^{(0)}/\partial v_z) > 0$ in some region of **v** space.

For $\alpha_\nu \equiv 1$ and $\partial f_\nu^{(0)}/\partial v_\perp^2 < 0$ everywhere, we obtain $f_\nu^{(0)} = f_\nu^{(0)}(v_\perp^2 + v_z^2)$. The equilibrium distribution is a monotonically decreasing function of the particle energy, and no negative-energy modes exist. This is consistent with the general results obtained in [7].

V. CONCLUSIONS

In the case of a magnetized, homogeneous Vlasov plasma, waves of negative energy ($\delta^2 H < 0$) exist for any deviation from monotonicity (i.e., if $\partial f_\nu^{(0)}/\partial v_\perp^2 > 0$ and/or $\partial f_\nu^{(0)}/\partial v_z^2 > 0$ for some $v_\perp, v_z$) and/or any anisotropy $\alpha_\nu(v_\perp, v_z) \neq 1$. No restricting condition is imposed on the perpendicular wave number $k_\perp$. The situation therefore cannot be expected to be alleviated by finite-gyroradius effects.

For distribution functions with both $\partial f_\nu^{(0)}/\partial v_\perp^2$ and $\partial f_\nu^{(0)}/\partial v_z^2 < 0$ everywhere, but which are anisotropic ($\alpha_\nu > 0$ and $\alpha_\nu \neq 1$ in some region of **v** space), the existence of negative-energy waves imposes a restriction on the *parallel* wave number k_z [conditions (49) or (50)]. However, if the distribution function is such that $\partial f_\nu^{(0)}/\partial v_z^2 > 0$ in some region of **v** space, then there is no restriction whatsoever on $k_\perp, k_z$, except $k_z \neq 0$. As shown by Pfirsch and Morrison [5], Eq. (144.b), this latter result is also obtained within the framework of drift-kinetic theory.

APPENDIX

Neglect of the electrostatic energy term

The contribution of the electrostatic energy term

$$\frac{1}{8\pi} \int d^3x\, \delta E^2 \tag{A1}$$

has been neglected. To justify this, let us consider the perturbed electric charge density $\delta\rho$. Generally, the charge density is

$$\rho = \sum_\nu e_\nu \int f_\nu d^3v \tag{A2}$$

and the perturbed charge density is

$$\delta\rho = \sum_\nu e_\nu \int \delta f_\nu d^3v \ . \tag{A3}$$

The perturbation in the distribution function is given by

$$\delta f_\nu = \left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}} \cdot \delta\mathbf{x}_\nu + \left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{p}_\nu}\right|_{\mathbf{x}} \cdot \delta\mathbf{p}_\nu \ , \tag{A4}$$

with $\mathbf{p}_\nu$ the canonical momentum of species ν, i.e.,

$$\mathbf{p}_\nu = m_\nu \mathbf{v} + \frac{e_\nu}{c} \mathbf{A}^{(0)}(\mathbf{x}) \ . \tag{A5}$$

It therefore follows that

$$\left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right|_{\mathbf{x}} = m_\nu \left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{p}_\nu}\right|_{\mathbf{x}} , \tag{A6}$$

$$\begin{aligned} \left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{v}} &= \left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}} + \left.\frac{\partial (p_\nu)_i}{\partial \mathbf{x}}\right|_{\mathbf{v}} \left.\frac{\partial f_\nu^{(0)}}{\partial (p_\nu)_i}\right|_{\mathbf{x}} \\ &= \left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}} + \left.\frac{e_\nu}{c} \frac{\partial A_i^{(0)}}{\partial \mathbf{x}} \frac{\partial f_\nu^{(0)}}{\partial (p_\nu)_i}\right|_{\mathbf{x}} , \end{aligned} \tag{A7}$$

$$\begin{aligned} \left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{p}} &= \left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{v}} - \left.\frac{e_\nu}{c} \frac{\partial A_i^{(0)}}{\partial \mathbf{x}} \frac{\partial f_\nu^{(0)}}{\partial (p_\nu)_i}\right|_{\mathbf{x}} \\ &= \left.\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\right|_{\mathbf{v}} - \left.\frac{e_\nu}{m_\nu c} \frac{\partial A_i^{(0)}}{\partial \mathbf{x}} \frac{\partial f_\nu^{(0)}}{\partial v_i}\right|_{\mathbf{x}} . \end{aligned} \tag{A8}$$

The perturbations $\delta\mathbf{x}_\nu$ and $\delta\mathbf{p}_\nu$ are given by

$$\begin{aligned} \delta\mathbf{x}_\nu &= \left.\frac{\partial G_\nu}{\partial \mathbf{p}_\nu}\right|_{\mathbf{x}} \\ &= \frac{1}{m_\nu} \left.\frac{\partial G_\nu}{\partial \mathbf{v}}\right|_{\mathbf{x}} , \end{aligned} \tag{A9}$$

$$\begin{aligned} \delta\mathbf{p}_\nu &= -\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{p}} \\ &= -\left.\frac{\partial G_\nu}{\partial \mathbf{x}}\right|_{\mathbf{v}} + \left.\frac{e_\nu}{m_\nu c} \frac{\partial A_i^{(0)}}{\partial \mathbf{x}} \frac{\partial G_\nu}{\partial v_i}\right|_{\mathbf{x}} . \end{aligned} \tag{A10}$$

Employing the relations above, one obtains δf_ν as a function of **x** and **v**:

$$\delta f_\nu=\frac{1}{m_\nu}\left[\frac{\partial f_\nu^{(0)}}{\partial \mathbf{x}}\cdot\frac{\partial G_\nu}{\partial \mathbf{v}}-\frac{e_\nu}{m_\nu c}\left\{\mathbf{B}^{(0)}\times\frac{\partial G_\nu}{\partial \mathbf{v}}\right\}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}-\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\frac{\partial G_\nu}{\partial \mathbf{x}}\right]. \quad \text{(A11)}$$

Specializing this expression to the equililbrium given by Eqs. (17), we obtain

$$\delta f_\nu=\frac{1}{m_\nu}\left\{2\omega_\nu\frac{\partial G_\nu}{\partial \phi}\frac{\partial f_\nu^{(0)}}{\partial v_\perp^2}-\frac{\partial G_\nu}{\partial \mathbf{x}}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\right\} \quad \text{(A12)}$$

and

$$\delta\rho=-\sum_\nu\frac{e_\nu}{m_\nu}\int d^3v\frac{\partial G_\nu}{\partial \mathbf{x}}\cdot\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}, \quad \text{(A13)}$$

where we have used the fact that G_ν is single valued, and that $f_\nu^{(0)}$ is ϕ independent.

Taking into account G_ν as given by Eqs. (22) and (24) yields

$$\delta\rho=-\sum_\nu\int d^3v\frac{i}{2}\frac{e_\nu}{2m_\nu}\left\{\frac{\partial f_\nu^{(0)}}{\partial \mathbf{v}}\cdot\mathbf{k}\right\}\Psi_\nu(\mathbf{v})\times(e^{i\Gamma_\nu(\mathbf{v})+i\mathbf{k}\cdot\mathbf{x}}-e^{-i\Gamma_\nu(\mathbf{v})-i\mathbf{k}\cdot\mathbf{x}}). \quad \text{(A14)}$$

The perturbed charge density $\delta\rho$ can be made zero since our expression for $\delta^2 H$ only contains Ψ_ν^2, $(\partial\Psi_\nu/\partial\phi)^2$, which are then chosen localized in $v_\perp$ or v_z. This distribution of *signs* in Ψ_ν and $\partial\Psi_\nu/\partial\phi$ is free. For instance, one can take Ψ_ν piecewise continuous in $v_\perp$ or v_z, with changing signs so that positive and negative contribution to $\delta\rho$ balance each other.

[1] P. J. Morrison and D. Pfirsch, Phys. Rev. A **40**, 3898 (1989).
[2] P. J. Morrison and D. Pfirsch, Phys. Fluids B **2**, 1105 (1990).
[3] M. Kotschenreuther and P. J. Morrison, Institute for Fusion Studies Report ISFR No. 280, Austin, TX, 1987.
[4] P. J. Morrison, Z. Naturforsch. A **42**, 1115 (1987).
[5] D. Pfirsch and P. J. Morrison, Phys. Fluids B **3**, 271 (1991).
[6] D. Pfirsch, Z. Naturforsch. A **43**, 533 (1988).
[7] H. Weitzner and D. Pfirsch, Phys. Rev. A **43**, 4532 (1991).