

## Statistical mechanics, Euler's equation, and Jupiter's Red Spot

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We construct a statistical-mechanical treatment of equilibrium flows in the two-dimensional Euler fluid which respects all conservation laws. The vorticity field is fundamental, and its long-range Coulomb interactions lead to an *exact* set of nonlinear mean-field equations for the equilibrium state. The equations depend on an infinite set of parameters, in one-to-one correspondence with the infinite set of conserved variables. We illustrate the equations by solving them numerically in simple cases. In more complicated cases we use Monte Carlo techniques, with the eventual aim of detailed comparison with the Red Spot dynamical simulations of Marcus: Preliminary efforts show good agreement. We review previous literature on two-dimensional (2D) Euler flow in light of our theory. In particular, we derive the Kraichnan energy-entropy theory and the Lundgren and Pointin point-vortex mean-field theory as special cases of our own. Our techniques may be generalized to a number of other Coulomb-like Hamiltonian systems with an infinite number of conservation laws, including some in higher dimensions. For example, we rederive Lynden-Bell's theory of stellar-cluster formation, as well as the Debye-Hückel theory of electrolytes. Our results may also be applicable to cylindrically bound guiding-center plasmas, which under idealized conditions provide another realization of 2D Euler flow.

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### I. INTRODUCTION

We present the statistical-mechanics theory of the ideal inviscid two-dimensional (2D) (Euler) fluid. There have been many attempts at constructing such a theory, dating back to Onsager [1] (for a review see Ref. [2]). A motivation for this work has been to explain the generic occurrence of large isolated vortices in numerical simulations and observations on flows that may be approximated as two dimensional. Such "coherent structures" are of obvious importance in many phenomena. Also the results of a previous approximate statistical mechanics of the undamped undriven fluid were used to motivate the idea of an inverse cascade of energy to long wavelengths in the driven and damped (but high-Reynolds-number) flow, i.e., two-dimensional turbulence [3].

This subject has renewed interest with the recent modeling of large scale planetary atmospheric phenomena, such as the Great Red Spot of Jupiter, in terms of inviscid two-dimensional fluid dynamics [4,5], and subsequent numerical investigation of the time evolution of the Euler equations [6]. Large coherent vortices are indeed a typical final state for wide ranges of initial conditions and parameters. Another experimental application is to one-component plasmas in a strong magnetic field, which may under certain approximations be treated as an ideal two-dimensional fluid [7]. Recent experiments have investigated this analogy in some detail.

The statistical mechanics of the two-dimensional inviscid, incompressible fluid is an interesting field theory in its own right: it is the statistical mechanics of volume-preserving diffeomorphisms in the plane. However, it is convenient to discuss the approach in terms of the application to the experimental phenomena described above. In particular we will introduce the ideas by comparing

with the recent numerical evolution of the dynamics equations by Marcus [6].

We begin by describing the model [4,6]. The planet Jupiter for our purposes comprises a two-dimensional annulus, with rigid boundaries. The winds of Jupiter are realized by applying to the annulus an external potential, which is equivalent to some combination of a background shear, potential vorticity, and Coriolis force. The external potential possesses an orientation, which breaks the equivalence of positive and negative vorticity.

The initial conditions for the calculation are determined completely by the initial vorticity field. A typical initial vorticity field involves two spots of vorticity of opposite sign, placed on opposite sides of the annulus.

We may then follow the time evolution of the fluid [6]. The spot of vorticity with opposite orientation to the external potential breaks down and disperses, whereas the spot of vorticity with the same orientation as the external potential remains largely intact. At long times, only the vortex sharing the sense of the external potential persists.

Marcus [6] suggests that the persisting vortex possesses a number of characteristics that bear qualitative resemblance to Jupiter's Great Red Spot. In particular, the shape and vorticity profile of the persistent vortex are consistent with those of the spot. Of course, the model used in the simulation is idealized; one could, if one wished, include the effects of the free surface, bottom topography, planetary curvature, etc. Nevertheless, the simulation suggests that the basic physics of the phenomenon may well be encapsulated by the simplified model.

We now ask whether we can predict and explain the long-time evolution of flows such as the one introduced above without resorting to dynamics. There are two reasons for wishing to accomplish such a task. One is simply pragmatic: dynamical simulations demand exten-

sive computation and incur substantial error at long times. An independent method of calculating the asymptotic evolution of a flow provides a check on the dynamical calculation and might plausibly replace it. The other reason is that when dynamics are unimportant, as in conditions of statistical equilibrium, we may understand the basic physics through simpler, static considerations.

The exotic feature of the 2D Euler equations when a statistical mechanics is contemplated is that the flow preserves an *infinite number* of conserved quantities that are *dynamically relevant*. (These can be crudely represented as area integrals of all powers of the fluid vorticity.) Previous statistical-mechanical treatments have ignored all these extra conservation laws, or have explicitly maintained only one—the “enstrophy” (the quadratic power). Our treatment incorporates these conservation laws in the construction of the theory: Our results are therefore in general different than in previous formulations. We remark that, to our knowledge, no convincing justification for ignoring the higher-order conserved quantities in inviscid flow has been put forward. It has been suggested that the addition of viscosity will serve to disproportionately quench the higher-order quantities; however, we believe that the addition of viscosity to the problem is an extremely difficult theoretical question that cannot be addressed so glibly. We do not have any results on this question.

Our basic results are contained in Eqs. (5.6)–(5.8). Since the results depend in general on an infinite number of conserved quantities, they are necessarily somewhat complicated (although simple examples to be discussed below display the main content of the theory). Equation (5.7) is a field equation for the stream function  $\psi_0(\mathbf{r})$ , which gives the long-time vorticity-field distribution. The equation involves Lagrange multipliers  $\beta$  (the scaled inverse temperature) and the function  $\mu(\sigma)$ . These are determined implicitly by demanding that the energy and conserved quantities take on the specified values, for example, as set by the initial vorticity field in a numerical evolution. The extra conserved quantities are completely specified by the vorticity distribution function  $g(\sigma)$ , which specifies the amount of vorticity at the level  $\sigma$  (e.g., as set by the initial configuration).

We have described a microcanonical formulation, as would be used in predicting final long-time averages from a given initial state. The inverse temperature  $\beta$  simply plays the role of a Lagrange multiplier. It turns out that, for nontrivial solutions,  $\beta$  is often negative and always so in the case of vanishing net vorticity. (This latter result depends only on very weak assumptions about the regularity of the initial vorticity field.) Thus if we consider a canonical formulation we must consider equilibrium with a “heat bath” at *negative temperatures*. In particular, we cannot allow equilibrium with the molecular degrees of freedom, which are always at positive temperatures, as, for example, mediated by viscosity in a real fluid. There are certain results of this theory that will be discussed in more detail below, but which are at first sight counterintuitive and therefore worth a preliminary discussion.

There are no fluctuations at any physical length scale in the statistical equilibrium: Mean-field theory is exact.

The long-time solutions are steady (except for symmetry-related dynamics such as uniform rotation about the annulus) and in particular are nonchaotic. In addition, although the infinite number of conserved quantities play a crucial role in constructing the theory, integrals of the powers of vorticity measured on any physical scale [i.e., as determined by (5.6)–(5.8)] will *not* be the same as in the initial state. [We discuss this phenomenon in terms of a “bare” vorticity distribution  $g(\sigma)$  and a “dressed” vorticity distribution  $g_d(\sigma)$  in Sec. V D.] This surprising result appears entirely natural when the dynamic evolution from a smooth initial condition is considered. It is well known that the flow becomes highly filamented, with regions of high and low vorticity becoming intertwined on finer and finer length scales as time proceeds. On these fine scales the integrals of powers of the vorticity are indeed conserved; however if the fine scales are averaged over *before* the powers are taken, then the conservation laws will apparently be violated. In the asymptotic long-time limit that the statistical theory addresses, the mixing scales have become arbitrarily fine, and quantities defined on any physical length scale will show this violation. It should be remarked in particular that the conservation of enstrophy too is apparently violated: using this conservation as a diagnostic of good numerical convergence in a long-time evolution may not be appropriate.

There are two major assumptions in our work. The first is that we only consider initial vorticity distributions which are sufficiently regular: this leads to the prediction of asymptotic statistical states that are free of fluctuations. This assumption does not seem much of a restriction in the microcanonical formulation aiming at the long-time asymptotics from a smooth initial condition. We cannot, however, rule out more dynamic, and therefore perhaps more interesting, statistical states, perhaps corresponding to positive temperatures, if this regularity constraint is violated.

The second assumption, which is harder to justify in detail, is the assumption of ergodicity, i.e., we construct the averages assuming the fluid samples all configurations restricted only by the conserved quantities. Ergodicity is hard to justify in any interesting physical system, and is usually assumed without comment. The assumption may be riskier in the present case since the infinite number of conserved quantities may be imagined to place severe constraints on the passage between allowed regions of phase space. The justification of this assumption will have to rest on future comparisons with experiment and simulation. Certainly if the initial condition leads to a strongly dynamic state with the mixing of the vorticity as described above, one might hope that ergodicity would be well approximated. On the other hand, we would not be surprised if very special, highly symmetric initial conditions yield nonergodic dynamics even if weakly perturbed. The Kida solution of an oscillating elliptical vortex, if indeed stable, as has been suggested [18] may be just such an example.

The outline of the paper is as follows. After we remind the reader of some basic properties of Euler flow in Sec. II, we argue in Sec. III that earlier attempts at a comprehensive statistical mechanics for the two-

dimensional inviscid fluid have been unsuccessful. These failures originate primarily in the mistreatment of the infinite family of conserved quantities of Euler flow. Much of this section is of interest mainly to experts, and may be omitted on a first reading. Section IV lays out the computation that needs to be done, in a manner that highlights the role of the conserved quantities. In Sec. V we present two different methods for deriving the mean-field equations that equilibrium configurations of the fluid must satisfy, and we prove the “dressed-vorticity corollary.” In Sec. VI we solve some illustrative special cases of the mean-field equations, as well as rederiving the Lynden-Bell theory of star clusters which antedates our own statistical mechanics. We also point out the connection to the Debye-Hückel theory of electrolytes. We discuss the physical content of these theories in Sec. VII. In Appendix A we outline a recent development: the possibility of defining a series of *lattice* Euler flows with an arbitrary large, but finite, number of conserved quantities. Although these models do not have a simply analyzable statistical mechanics, they may be important for numerical simulations. In Appendix B we summarize our preliminary attempts at a numerical solution of the mean-field equations, with the eventual aim of detailed comparison with dynamical simulations, such as those of Marcus [6].

The major results of this work have been published in a Letter [9]. Here we present full derivations and some numerical comparisons.

## II. PRIMER ON THE IDEAL FLUID IN TWO DIMENSIONS

In this section we collect a few standard facts about Euler flow in two dimensions, to which we shall refer repeatedly in the ensuing text. A transparent derivation of the properties of Euler flow may be found in Ref. [1].

### A. Notation

The two-dimensional velocity field is denoted by  $\mathbf{u}(\mathbf{r})$ : the (scalar) vorticity field  $\omega(\mathbf{r})$  is defined as

$$\omega = \nabla \times \mathbf{u} = \partial_x u_y - \partial_y u_x . \quad (2.1)$$

We label by  $\Omega$  the region containing the fluid;  $p$  is the pressure,  $\rho$  the density.

### B. Euler equation

Following Landau and Lifshitz [11] we observe that in a coordinate system comoving with an (infinitesimal) fluid element  $v$ , the force on the fluid element is given by

$$-\oint_{\partial v} p \, d\mathbf{S} = - \int_v \nabla p \, dV , \quad (2.2)$$

where  $\partial v$  is the boundary of  $v$ , so that Newton’s equation reads

$$\rho \frac{D\mathbf{u}}{Dt} = - \nabla p , \quad (2.3)$$

where  $D/Dt$  denotes a covariant (convective, material) derivative. In a stationary coordinate system, the covariant derivative may be rewritten to obtain Euler’s equation

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = - \frac{1}{\rho} \nabla p . \quad (2.4)$$

When we require the density  $\rho$  to be constant, the equation for mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 \quad (2.5)$$

entails the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0 , \quad (2.6)$$

which will be taken for granted in the remainder of our work.

Taking the curl of Euler’s equation (2.4) and using the incompressibility condition, we readily obtain in two dimensions

$$\frac{D\omega}{Dt} = \frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla) \omega = 0 . \quad (2.7)$$

That is, the convective derivative of the vorticity vanishes. In two dimensions (2.7) is entirely equivalent to (2.4) and (2.6) (see below).

### C. Stream function

The vanishing of  $\nabla \cdot \mathbf{u}$  in two dimensions entails the existence of a scalar function (the stream function)  $\psi(\mathbf{r})$  such that

$$\mathbf{u} = \nabla \times \psi \equiv (\partial_y \psi, -\partial_x \psi) \quad (2.8)$$

and consequently  $\omega(\mathbf{r}) = -\nabla^2 \psi(\mathbf{r})$ .

### D. Conserved quantities

For any closed path  $\partial\eta(t)$  moving with the fluid in time  $t$ , and whose interior  $\eta$  is contained in the fluid, the circulation

$$\Gamma = \oint_{\partial\eta(t)} \mathbf{u} \cdot d\mathbf{l} = \int_{\eta(t)} \omega \, d^2\mathbf{r} \quad (2.9)$$

is conserved by Euler flow. In fact, in two dimensions, given *any* function  $f(\omega)$  of the vorticity field, the quantity  $\int_{\eta(t)} f(\omega(\mathbf{r})) \, d^2\mathbf{r}$  is conserved by the flow:

$$\begin{aligned} \frac{d}{dt} \int_{\eta(t)} d^2\mathbf{r} f(\omega(\mathbf{r})) \\ = \int_{\eta(t)} d^2\mathbf{r} f'(\omega) \left[ \frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega \right] = 0 . \end{aligned} \quad (2.10)$$

If we take  $f(\omega) = \omega^2$ , the corresponding integral invariant is called the “enstrophy,” a term apparently originating in Ref. [12].

### E. Boundary conditions

Ordinarily, we shall assume rigid boundaries: that is,  $\mathbf{u}$  is assumed to be tangent to the boundary  $\partial\Omega$  of a bounded planar region  $\Omega$  containing the fluid. When  $\Omega$  possesses  $n$  holes, we may define the quantities  $\Gamma_n$  to be the circulations around each of these holes. We observe [13] that together  $\omega(\mathbf{r})$  and  $\Gamma_1, \dots, \Gamma_n$  uniquely determine the velocity field  $\mathbf{u}$ .

### III. REVIEW OF STATISTICAL MECHANICS OF THE TWO-DIMENSIONAL FLUID

#### A. Introduction

Our discussion throughout will be confined to the two-dimensional Newtonian fluid. The Navier-Stokes equations for an incompressible fluid

$$\rho \left[ \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla p + \nu \nabla^2 \mathbf{u}, \quad (3.1)$$

$$\nabla \cdot \mathbf{u} = 0$$

are thought to be a valid description of a viscous fluid in a variety of physical regimes. We shall not discuss when and why the two-dimensional fluid is a good physical model for fluids in a three-dimensional world. We refer the reader to the text by Pedlosky [14] for a justification of the two-dimensional model. We remark only that a variety of laboratory and geophysical flows display essentially two-dimensional behavior, a nearly invariant prerequisite for which is rotation of the plane containing the fluid about the perpendicular axis. The well-known Taylor-Proudman column (see, e.g., Ref. [15]) graphically illustrates this kind of phenomenon in the laboratory; planetary atmospheres provide a geophysical setting. The elimination of the third dimension entails several consecutive approximations, and one must maintain careful control of the time and length scales that one wishes to describe.

Similar attention to time and length scales is needed to justify the additional approximation of discarding the viscous term in the Navier-Stokes equation to derive the Euler equations, which in two dimensions take the form (2.7)

$$\frac{\partial \omega}{\partial t} = -(\mathbf{u} \cdot \nabla) \omega, \quad (3.2)$$

$$\nabla \cdot \mathbf{u} = 0.$$

Later on, we shall give an informal argument that the physics of certain flows may permit us to neglect the effect of viscosity on length scales of interest. For the bulk of our work, we shall simply assume that the physics is described by the *inviscid* Euler equations in two dimensions, (3.2).

#### B. The Onsager theory of negative temperature states

Onsager [1] was the first to attempt construction of a statistical mechanics for the two-dimensional fluid. Since subsequent papers refer to this one with hardly an exception, we discuss the ideas contained in his paper in detail.

Onsager wants to explain the generic occurrence of large isolated vortices in unsteady flow. He begins by restricting himself to a subset of Euler flows: those that can be described by a system of point vortices. The point-vortex system consists of a linear superposition of point sources of vorticity,

$$\omega(\mathbf{r}) = \sum_i \omega_i \delta(\mathbf{r} - \mathbf{r}_i). \quad (3.3)$$

Writing  $\nabla^2 \psi(\mathbf{r}) = -\omega_0 \delta(\mathbf{r} - \mathbf{r}_0)$  for a single point vortex with vorticity  $\omega_0$  located at  $\mathbf{r}_0$ , we obtain  $\psi(\mathbf{r}) = \omega_0 \mathcal{G}(\mathbf{r}, \mathbf{r}_0)$ . Here  $\mathcal{G}(\mathbf{r}, \mathbf{r}_0)$  is the Green's function for the Laplacian on the region containing the fluid, and its form depends on the boundary conditions. For example, free boundary conditions yield a logarithmic potential

$$\mathcal{G}(\mathbf{r}, \mathbf{r}_0) = -\frac{1}{2\pi} \ln(|\mathbf{r} - \mathbf{r}_0|/R_0),$$

where  $R_0$  is an arbitrary constant with dimensions of length. Obviously,  $\mathcal{G}(\mathbf{r}, \mathbf{r}_0)$  corresponds to the potential at  $\mathbf{r}$  due to a vortex at  $\mathbf{r}_0$ .

The potential of a point vortex is finite at any finite distance from the source. There exists no *a priori* bound on how closely two point vortices may approach each other.

It had been observed much earlier that the equations of motion of a superposition of  $N$  point vortices can be derived from a Hamiltonian [16]. It is precisely this feature that leads Onsager to restrict himself to the special case of point vortices. Without a Hamiltonian, it is not obvious how to construct a statistical mechanics; once one has a Hamiltonian, no choice in that construction remains (see Sec. IV for details).

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \omega_i \omega_j \mathcal{G}(\mathbf{r}_i, \mathbf{r}_j) \quad (3.4)$$

and the equations of motion are derived by regarding the two components of the position vector of each vortex as conjugate variables,

$$\omega_i \frac{d\mathbf{r}_i}{dt} = \nabla_i \times \mathcal{H}. \quad (3.5)$$

Notice that phase space coincides with configuration space. Using Onsager's notation, we write the volume of phase space near some configuration

$$d\Omega = dx_1 dy_1 \cdots dx_N dy_N. \quad (3.6)$$

If we demand that the fluid be contained in some compact region of space, we see that  $\int d\Omega$ , the *total* volume of phase space, is finite.

The boundedness of phase space has a peculiar consequence. Ordinarily, we expect that, as a function of energy, the volume of phase space available to a system,  $\Omega(E)$ , increases exponentially [17]. But here, the finite phase volume precludes this alternative, and the possibility exists that the volume of phase space available to the system could *decrease* as we increase the energy. Put another way, if we take the entropy at some system energy to be given by  $S = \ln \Omega(E)$ , then, above some energy  $E_m$ , the quantity  $dS/dE$ , which is formally equal to the inverse of the temperature  $T$ , may be *negative*. In this regime the energy is so high that vortices, with the same sign, cluster together, and vortices with opposite signs repel. The equilibrium consists of two isolated clusters of vortices. For a finite number  $N$  of point vortices in a bounded volume this is indeed what happens.

Addition of the usual kinetic momentum term  $p^2/2m$

to the Hamiltonian (3.4) would eliminate the negative-temperature regime. The total phase volume would no longer be finite, and the momentum degrees of freedom could accommodate any energy. Microscopic degrees of freedom, for example, are usually (with the exception of spin systems) described by a Hamiltonian with a kinetic momentum, and cannot take on negative temperature. These remarks emphasize a basic feature of Onsager's theory for macroscopic vortices in fluids: the fluid system at negative temperature cannot be in equilibrium with molecular degrees of freedom. Negative temperatures are higher than any positive temperature and a negative-temperature system, on being placed in thermal contact with an ordinary positive-temperature heat bath, will dump energy into the bath.

While this analysis does encompass the essential physics of the problem, Onsager remarks that it cannot apply quantitatively to many flows of physical interest. Flows such as those in the model for the Red Spot discussed earlier involve *continuous* distributions of vorticity. But how to approximate a continuous distribution of vorticity by means of point vortices is not obvious, and in fact, generally not possible. One way to comprehend the difficulty is to look at quantities conserved by Euler flow. Consider, for example, the integral of any finite power of the vorticity. For a vorticity field consisting of point sources, these integrals involve powers of  $\delta$  functions and are therefore singular. Yet in typical physical models the vorticity field has perfectly well-defined moments, to which we ought to be able to assign any consistent set of values.

### C. Subsequent approaches

A direct approach to the statistical mechanics of a continuous vorticity field is taken by Lee [18]. In his statistical field theory, the only conserved quantity is the energy. Rewriting the Euler equations in Fourier space, he obtains a Liouville theorem

$$\frac{\delta \dot{\omega}_{\mathbf{k}}}{\delta \omega_{\mathbf{k}}} = 0, \quad (3.7)$$

implying incompressible flow in the spectrally decomposed phase space. Lee's derivation applies in any dimension; in Sec. IV we will give an argument specific to the 2D Euler fluid. The Liouville equation is crucial to the statistical treatment since it determines the measure on our phase space. Imposing only energy conservation at finite temperature, Lee finds an energy spectrum of the form  $E(k) \propto k$ , which is ultraviolet divergent. He imposes an ultraviolet momentum cutoff to obtain a finite energy. Lee remarks that, in two dimensions, conservation laws preclude simple (i.e., conservation of energy alone) ergodic behavior, and in consequence he restricts himself to three dimensions.

As an aside we remark that an analogous problem involving an infinite set of constants of motion was stated and solved contemporaneously by Lynden-Bell [19] in the context of stellar dynamics. We shall discuss this work in Secs. VI and VII. The connection to Euler's equation was apparently not realized prior to the present work.

Kraichnan [3] constructs a statistical mechanics by keeping as his conserved quantities not only the energy  $E$  but also the enstrophy  $\Omega$ . He truncates the Fourier representation of the Euler equations to obtain a *truncated* dynamics which explicitly conserves these quantities, and for which a Liouville theorem holds in the remaining finite number of spectral variables [2,20]. Since both the energy and the enstrophy are quadratic in the stream function, this field theory is Gaussian. By the standard procedure he obtains an equilibrium spectrum of the form

$$\begin{aligned} E(k) &= 1/(\beta k^2 + \alpha), \\ \Omega(k) &= k^2/(\beta k^2 + \alpha), \end{aligned} \quad (3.8)$$

where  $\alpha$  and  $\beta$  are inverse temperatures conjugate to the energy and the enstrophy, respectively. As in the theory that exclusively conserves energy, we need a large momentum cutoff at any finite temperature. We refer to this theory that conserves only energy and enstrophy as the energy-enstrophy theory.

Kraichnan justifies the elimination of constants of motion other than the energy and enstrophy on grounds that the cutoff dynamics conserve only quadratic constants of motion. Agreement with point-vortex theory is also deduced (we shall show below that this correspondence may only be justified in a very limited regime). The reader will find additional supporting arguments of an informal nature in an article by Kraichnan [20]. The truncated dynamics, when used properly, is claimed to be faithful to both inviscid and viscous fluids in statistical equilibrium.

Within the context of the 1967 theory, the energy-enstrophy theory contains three parameters: a momentum cutoff, a temperature conjugate to the energy, and a temperature conjugate to the enstrophy. Each of these quantities is to be finite. If we take the limit of infinite momentum cutoff with the remaining two parameters finite, we deduce easily from (3.8) that the energy diverges logarithmically, and the enstrophy quadratically with the cutoff. In order to treat the continuum limit, without artificial cutoffs, finiteness of  $\alpha$  and  $\beta$  must be relaxed. Kraichnan [20] and Basdevant and Sadoury [21] point out that by choosing the following scaling for the energy and enstrophy temperatures one can obtain finite values for the energy and enstrophy in the (continuum) limit of infinite momentum cutoff  $k_m$

$$\begin{aligned} \beta &\rightarrow \pi k_m^2 / 2\Omega, \\ \alpha / \beta &\rightarrow -k_0^2 + k_m^2 \exp(-k_m^2 E / \Omega), \end{aligned} \quad (3.9)$$

where  $k_0$  is an infrared momentum cutoff determined by the finite system size. Note that this scaling is only possible at negative values of the inverse energy temperature  $\alpha$ . We will obtain these sorts of scalings as a consequence of our general theory as well: see Secs. V and VI.

A dynamical simulation by Deem and Zabusky [22] elicited a spurt of publications in which the predictions of the energy-enstrophy theory are compared with results of dynamical simulation. The comparison is generally quite favorable. We mention two examples. Fox and Orszag [23] test by means of numerical simulation an explanation

of the Deem and Zabusky results in terms of the energy-entropy theory. They find general agreement with the theory, except for substantial deviations at low and high wave numbers, which they attribute either to insufficient relaxation time or to the effect of additional constants of the motion. The computational technology of fluid simulation has evolved in the last 20 years, and the extent to which numerical simulation by present standards would validate these results is unclear. Basdevant and Sadourny [21] confirm the energy-entropy theory for the *truncated* dynamics by means of an Arakawa code, which explicitly conserves energy and entropy. These latter authors emphasize that the energy-entropy theory ought only to apply to the truncated dynamics and not to the full inviscid dynamics.

#### D. Mean-field theory and random-phase approximation for the point-vortex limit

In the mid 1970s a large number of authors derived and studied mean-field equations for the equilibrium configuration of a system of point vortices [24–32]. Lundgren and Pointin [26,27], for example, derive two limits of the point-vortex system as  $N$ , the number of vortices, diverges. The two limits are distinguished by the scaling of the energy with  $N$ . In the “low-energy limit,” the energy scales as the number of vortices, whereas in the “high-energy limit,” the energy scales as the *square* of the number of vortices.

For ease of discussion in what follows, we consider only systems with vanishing total vorticity, except where otherwise noted.

The low-energy limit can apply only at positive temperatures, for reasons that emerge below, and entails an extensive thermodynamic scaling: for nontrivial results we require the density to be constant in the limit of large  $N$ , with energy proportional to  $N$ . This regime corresponds to the neutral two-dimensional Coulomb gas, to which the standard thermodynamic prescription may be applied (see also Sec. III E).

The high-energy limit obtains at negative temperatures for neutral systems (for non-neutral systems the high-energy limit is also correct at positive temperatures and the low-energy limit is trivial). For the high-energy limit to be valid, the area of the system is best thought of as remaining *finite* in the absence of additional constraints on the phase space, such as angular momentum conservation [26]. In the high-energy limit, the area of the system remains constant and the density of vortices diverges as  $N$ , with energy proportional to  $N^2$ . Furthermore, because the high-energy limit is not extensive, *the temperature must be scaled* as  $1/N$  to yield a nontrivial equilibrium. One can derive a mean-field equation for the spatial distribution of vorticity as  $N \rightarrow \infty$ ,

$$\begin{aligned} \omega(\mathbf{r}) &= -\nabla^2\psi \\ &= \exp(-\psi/\bar{T}) \left[ \int_{\Omega} d^2\mathbf{r} \exp(-\psi/\bar{T}) \right]^{-1}, \end{aligned} \quad (3.10)$$

where  $\bar{T}$  denotes the *scaled* temperature. This equation has been deduced by a number of authors, using a variety of approximations. Lundgren and Pointin [27] claim to derive it rigorously, but a truncation of the frequency

spectrum at the very beginning of their argument depends on an unproven assumption that two particular limits commute. On the other hand, we expect that (3.10) is nevertheless valid in some limits (see Sec. VI).

Using numerical methods, many authors went on to compare the long-time dynamics of point vortices with the predictions of this statistical-mechanical theory of point vortices. They obtain good agreement between the long-time spatial distribution of vortices in dynamical simulations and the equilibrium vortex-density profiles calculated from Eq. (3.10). We draw attention in particular to Refs. [24,26,31,33]. Aref and Siggia [33] study long-time dynamics of point vortices in a shear layer.

In calculating the vortex-density response function in the low-energy limit, Lundgren and Pointin [27] use an approximation introduced by Edwards [34] in a calculation of the properties of a neutral plasma at (finite) positive temperature. Edwards approximates the Jacobian in a change of variables from particle to collective coordinates, and is emulated by Edwards and Taylor [35], Seyler [36,37], Seyler *et al.* [38], Taylor [39], Taylor and McNamara [40],[41] and others in treatments of the 2D plasma. The approximation, which they call the random-phase approximation (RPA), is equivalent to that of Pines and Bohm [42]. The RPA is a well-understood *perturbative* tool which has been used in a variety of physical contexts [43]. In the particular case of particles with a Coulomb interaction, the RPA is only valid in a *screened* phase, where the number of particles within a Debye screening length,  $\lambda_D = (2\pi e^2 n / T)^{-1/2}$ , of one another is large ( $e$  is the charge of a particle;  $n$  is the particle density). Under these conditions, charge fluctuations around the mean charge density are small, justifying a perturbative treatment. A linear-response calculation of the density-density correlation yields [42]

$$\langle |\rho_k|^2 \rangle = nk^2 / (k^2 + \lambda_D^{-2}), \quad (3.11)$$

which is to be compared with the entropy spectrum of the Kraichnan theory, Eq. (3.8). The similarity of Eqs. (3.8) and (3.11) has led several authors to conclude that the 2D Coulomb gas possesses the same energy spectrum as the energy-entropy theory of the Euler fluid (see Refs. [44,45,20,37] and the low-energy limit of Ref. [31]); however, this identification can in fact be made only in the limited regime in which the RPA is applicable. In particular, whenever the fluid undergoes *macroscopic* charge separation in equilibrium, the RPA breaks down, and a nonperturbative calculation is required. In the limit of infinite charge density, the energy spectrum obtained from (3.8) applies only to a homogeneous vorticity field with no structure on any finite length scale.

#### E. Rigorous results

We next discuss two mathematical papers that describe rigorous results on the statistical mechanics of point vortices and Euler flow. Fröhlich and Ruelle [46] prove that extensive negative-temperature states do not exist in the point-vortex gas; Benfatto, Picco, and Pulvirente [44] claim that the statistical mechanics of an Euler fluid is given by the energy-entropy theory. We will argue that

each of these results, while correct, applies only in regimes with trivial structure at finite length scales. In both cases, their conclusions depend on using a temperature that is not appropriately scaled with the number of degrees of freedom. Their arguments and conclusions can be completely understood within the context of the earlier studies we have sketched above.

Fröhlich and Ruelle [46] treat a neutral collection of point vortices on a torus. They wish to prove that there can be no negative temperatures in the thermodynamic limit of Onsager's point-vortex gas. By thermodynamic limit, they mean the limit in which the volume of the fluid diverges, while the density and energy per vortex converge to finite values; the temperature, of course, remains finite. Whereas Onsager [1] argues that for a *finite* collection of vortices there exists an energy at which  $dS/dE$  changes sign from positive to negative, Fröhlich and Ruelle [46] show that this energy itself diverges as the number of point vortices goes to infinity. Consequently, a regime where  $S$  decreases as a function of  $E$  is unattainable in this particular thermodynamic limit.

In fact, this limit does not correspond to interesting solutions in the fluid. This outcome may be attributed in a simple way to screening. First, let us observe with Fröhlich and Ruelle [46] that the absence of a scale in the logarithmic potential permits us to rescale the volume of the fluid to a finite value as  $N \rightarrow \infty$ . So we may just as well talk about a limit in which the density diverges, but the energy per vortex remains finite. We understand the physics of this system very well at positive temperatures: vorticity fluctuations will be *screened*, and the long-range potential becomes effectively short ranged. At infinite vortex density, the screening length vanishes and the interaction becomes irrelevant. Consequently, this thermodynamic limit yields a vanishing mean vorticity (the system being overall neutral) and trivial correlations.

To see this more explicitly, think for a moment of  $N$  positively charged vortices contained in a domain  $\Omega$  and distributed uniformly in a blob of finite radius. Now, multiply the number of vortices by some factor  $\eta$ , and ask what would be the energy of a blob with precisely the same relative vortex charge distribution. It is not difficult to see that the *long range* of the logarithmic interaction entails that the energy scales by a factor  $\eta^2$ . If we were not to rescale the energy, then the  $\eta N$  vortices would distribute themselves more evenly over  $\Omega$ : the normalized amplitude of the blob would decrease. In any mean-field description, the normalized amplitude of a charge inhomogeneity will remain constant *only if the energy is scaled by the square of the number of charges*.

On the other hand, if we scale the energy in the manner suggested by the high-energy limit, that is as  $N^2$ , then a new feature emerges. While, at positive temperatures, screening dominates since opposite signs attract, at negative temperatures where opposite signs repel, the screening of the long-range potential is suppressed. Macroscopic charge inhomogeneities persist (and are, in fact, unavoidable) in the limit of large  $N$ . As discussed earlier, the scaling of the temperature as  $1/N$  is corequisite to the scaling of energy with  $N^2$ . Fröhlich and Ruelle [46] elim-

inate any possibility for negative temperatures and macroscopic charge inhomogeneities in the thermodynamic limit by their most basic assumption of the scaling that constitutes a thermodynamic limit.

Benfatto, Picco, and Pulvirente [44] also predicate their argument on an unnecessary restriction of the scalings that they permit in the thermodynamic limit. These authors once again treat neutral flow on a torus, represented by  $N$  blobs of vorticity with radius  $\epsilon$  and charge  $\pm\sqrt{\sigma}$ . Their Hamiltonian describes a gas of particles interacting with a logarithmic potential which is smoothed over a length of order  $\epsilon$ , so as to remove the short-distance logarithmic divergence. They construct a canonical partition function with inverse temperature  $\alpha$ , where  $\alpha$  is a *finite* quantity that is *not* scaled with the number of particles. As  $N \rightarrow \infty$  at constant  $V$ , the radius and charge squared per vortex scale as  $\epsilon = N^{-\delta}$  and  $\sigma = (2\pi)^2/\beta N$ , respectively. The quantity  $\delta$  must be chosen correctly from the interval  $(0, \beta/2\pi\alpha)$ . The principal result of Benfatto, Picco, and Pulvirente [44] is that for large  $N$ , this system may be described by the energy-entropy partition function, with energy temperature  $\alpha$  and entropy temperature  $\beta$ .

Benfatto, Picco, and Pulvirente [44] assume that both  $\alpha$  and  $\beta$  possess finite and positive values. For  $\alpha$  finite, we expect the vortices to be uncorrelated in the infinite-density limit. For example, in the case of periodic boundary conditions, it has been shown by Lundgren and Poin-tin [26,27] that the uncorrelated vortices will have an energy (contributed by the image charges) diverging as  $\ln N$ , where we have corrected for the scaling of the charge used by Benfatto, Picco, and Pulvirente [44]. As we have observed earlier, the energy-entropy partition function also yields a logarithmically diverging energy for this parameter range.

In summary, Benfatto, Picco, and Pulvirente [44] have shown that the low-energy limit coincides with the energy-entropy theory for a neutral vortex gas at positive temperature in the dense limit. This regime is homogeneous, and has no interesting structure at scales on the order of the system size. Its behavior on scales large compared with the screening length should be given simply by a theory that incorporates the energy and particle number as conserved quantities. For a gas of  $\pm 1$  charged vortices, the summed square vorticity is sufficient to determine the latter. Higher-order couplings are expected to be irrelevant on these scales. Neither Fröhlich and Ruelle [46] nor Benfatto, Picco, and Pulvirente [44] have anything to say about regimes in which the fluid has non-trivial structure on length scales larger than the screening length.

#### F. Appropriate scaling of the temperature

We now assume that the proper way to use these statistical methods is to take the limit of an infinite number of degrees of freedom, and scale the temperature(s) in such a way as to extract the desired energy (and other conserved quantities such as entropy). This scaling will fall out, in a natural way, of the full theory presented in Sec. V. We have seen a special case of it in our discussion of the point-vortex mean-field theory. We may further il-



lustrate what is involved using the energy-ensrophy theory of Kraichnan [3], where Eq. (3.9) constitutes another example of such a temperature rescaling.

As observed in Sec. III C, we can choose temperatures  $\alpha$  and  $\beta$  so that the energy-ensrophy theory yields any consistent *total* energy  $E$  and enstrophy  $\Omega$ . Values of  $\Omega$  are in fact constrained by a choice of value for  $E$ . An elementary variational argument reveals that the minimum value of  $\Omega$  corresponds to  $\Omega_{\min}(E) = k_0^2 E$ , where  $|k_0|$  is the system's smallest wave number.

A body of literature investigates the properties of flows possessing the minimum value of  $\Omega$  consistent with a given  $E$ . The rationale is a "minimum enstrophy principle" based on crude arguments about the action of viscosity [47]. We shall not discuss "minimum enstrophy vortices" [48] in any detail here since they do not generalize to the full theory; however, Carnevale and Frederiksen [49] point out that these same flows do emerge from the energy-ensrophy theory in the limit described by Eq. (3.9): All of the energy resides in the smallest wave number  $k_0$ ; the enstrophy in the smallest wave number is exactly  $\Omega_{\min}(E) = k_0^2 E$ ; whatever enstrophy is left over sits at *infinite* wave number. This partitioning of the enstrophy carries an interesting implication: The energy is sufficient to determine the macroscopic flow completely. The enstrophy exceeding  $\Omega_{\min}(E)$  disappears completely from the flow at infinite time, since it drains to infinitesimal spatial scales at infinite wave number. Consequently, all long-time flows possessing the same energy must be identical. The dependence of the energy-ensrophy theory on two parameters is illusory.

There is nothing unsound about a one-parameter theory; we shall see that the one-parameter theory, which we derive as a special case of our formalism at the end of Sec. VI, comprises a crude approximation to the full statistical mechanics of the two-dimensional inviscid fluid. The two main lessons to take away from the one-parameter theory, which are essential ingredients in the full theory, are the proper choice of the temperature variable that is conjugate to the energy and the fact that long-time flows may push part of a conserved variable to infinite wave number, leading to its apparent nonconservation in the macroscopic flow. The latter was alluded to in Sec. I and will also be discussed in detail in Sec. V.

We have not discussed the following relevant issues, for the details of which we refer the reader to the literature.

- (1) Dynamical properties of point vortices.
- (2) Applications of statistical mechanics to geophysical flows.
- (3) Turbulence. Equilibrium statistical mechanics is not relevant to turbulent cascades; however, following Onsager, we do hope that statistical mechanics may yield a description of turbulent flow on length scales of order of the system size.
- (4) Dynamical simulation of two-dimensional fluids. A thorough understanding of numerical computation of inviscid fluid flow, especially with regard to its faithfulness to the conservation laws, may be necessary in order to generate simulations that confirm the predictions of statistical mechanics.

#### IV. STATISTICAL MECHANICS

Our construction of a statistical-mechanical formalism for the two-dimensional inviscid fluid is a fairly straightforward application of completely standard ideas from statistical thermodynamics. Landau and Lifshitz [17] review these ideas.

In short, we calculate the expectation value of a quantity  $Q$  by the following rule: Average  $Q$  over all regions of phase space sharing given fixed values of the conserved quantities. Underlying this rule is the basic assumption of statistical mechanics, the ergodic hypothesis, which postulates that our averaging assigns to each element of phase space an identical weight.

We dissect this rule into a few discrete steps.

(1) Define phase-space variables so that the dynamics preserve arbitrary phase-space volumes: phase-space flow must be *incompressible*. Since the ergodic hypothesis ascribes weights in proportion to phase-space volume, the phase-space flow preserves relative probabilities (weights) in the chosen variables. Equivalently, we need to choose variables for which a Liouville theorem holds.

(2) Determine those functions  $\mathcal{C}$  of the phase variables which are conserved by the dynamics. For a typical particle Hamiltonian, the only such quantities are energy and total density. Anything we wish to calculate will be parametrized by these quantities, so we shall fix them to particular values,  $\hat{\mathcal{C}}$ .

(3) Given a quantity  $Q$  whose expectation in statistical equilibrium is required, we calculate its average over all of the phase space in which the conserved quantities take their chosen values. In this manner, we arrive at  $\langle Q \rangle_{\hat{\mathcal{C}}}$ , the expectation value of  $Q$  in statistical equilibrium.

The recipe we have given is a microcanonical formulation. We might just as well employ a canonical or grand canonical approach; in the case of interest here it is simplest to think within a microcanonical framework, but one obtains identical results no matter which of the three ensembles one chooses. For purposes of calculation, we shall find the grand canonical point of view most useful.

Statistical mechanics is most often applied to dynamics that can be derived, at least in some limit, from a Hamiltonian. The reason is that (canonical) Hamiltonian dynamics manifestly preserves phase-space volumes. The Liouville theorem follows trivially, almost tautologically, from the Hamiltonian equations of motion,

$$\frac{\partial \dot{x}_i}{\partial x_i} + \frac{\partial \dot{p}_i}{\partial p_i} = \frac{\partial}{\partial x_i} \frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i} \frac{\partial H}{\partial x_i} \equiv 0. \quad (4.1)$$

In a closed box, the usual many-particle Hamiltonian  $H(\mathbf{x}_1, \mathbf{p}_1, \dots, \mathbf{x}_N, \mathbf{p}_N)$ , with particle coordinates  $\mathbf{x}_i$  and momenta  $\mathbf{p}_i$ , possesses one conserved quantity, the energy, which coincides with the scalar value of the (manifestly) conserved quantity  $H$ . (We do not allow particles to enter or leave the box). Given a value of the energy  $E_0$ , we may calculate the expectation value of a quantity  $Q(\mathbf{x}_1, \mathbf{p}_1, \dots, \mathbf{x}_N, \mathbf{p}_N)$  by integrating over all values of  $\mathbf{x}_i, \mathbf{p}_i, i = 1, \dots, N$  on the manifold determined by the constraint  $E_0 = H(\mathbf{x}_1, \mathbf{p}_1, \dots, \mathbf{x}_N, \mathbf{p}_N)$ .

We follow an analogous recipe for the two-dimensional inviscid fluid.



(1) We choose the phase-space variables  $\omega(\mathbf{r})$  for which a Liouville theorem, Eq. (3.7), holds. Our phase space consists of scalar fields defined on the region  $\Omega$  containing the fluid.

(2) We set the values of quantities conserved by Euler flow. These quantities comprise the energy, and some infinite set of scalars which determines the vorticity distribution. We are now restricted to the scalar fields on  $\Omega$  which share the fixed values for these quantities.

(3) We integrate over all fields on the manifold determined by the constraints given in (2).

The main hurdle we face is in setting up the calculation of (3) in such a way that we can carry it out. Before we do so, we shall discuss steps (1) and (2) in greater detail.

#### A. Hamiltonian formulation of inviscid fluid flow

Just as in the discrete-particle case, a Hamiltonian formalism may be used to set up our approach to statistical mechanics. In fact, any explicit use of the Hamiltonian dynamics for our system seems arguably superfluous. Lynden-Bell [19] sets up the problem for the collisionless Boltzmann equation without any explicit recognition that his equations of motion could be recast in Hamiltonian form. The feature of the equations of motion, which in both cases leads immediately to statistical mechanics, is that they give rise to a phase-space flow that is incompressible. That is, one may derive directly from the equations of motion a Liouville theorem.

Although the Hamiltonian nature of the system is unnecessary for construction of the statistical mechanics, it deserves some emphasis for two reasons: first, it underlies the well-known, indeed classic, Hamiltonian dynamics of point vortices, in which Onsager's statistical mechanics originated; second, it falls under the rubric of the "noncanonical Hamiltonian formalism" into which a variety of physical systems have been collected (see, e.g., Ref. [13]), each of which is amenable to the statistical-mechanical program which we are constructing in this work. These systems, among which are examples of classical and relativistic dynamics in both two and three space dimensions, each possess at least one infinite family of conservation laws. On these grounds we have chosen merely to outline very briefly the basic features of the Hamiltonian formalism for ideal fluid mechanics in two dimensions. A substantial body of literature exists to which the reader may refer for details.

Underlying the Hamiltonian description is a symmetry of the physics: the equations of motion in the form (2.3) are invariant under volume-preserving coordinate reparameterizations. This invariance, a kind of Newtonian covariance principle known also as "particle-relabelling symmetry," can be viewed as a gauge symmetry, and through Noether's theorem gives rise to the infinite set of conservation laws of two-dimensional Euler flow [50]. These same coordinate transformations, which constitute the group of volume-preserving diffeomorphisms,  $\mathcal{D}_V(\Omega)$ , form the configuration space for fluid motions. Starting from an arbitrary reference coordinate frame, an element of  $\mathcal{D}_V(\Omega)$  carries particles in the

reference frame to a new configuration which may be labeled by this group element. Any motion of the fluid is described by the action of a group element on a configuration; Arnol'd [51] describes a variational method to derive the motion of the fluid, which of course must correspond to the Euler equations.

Rather than discuss Arnol'd's construction, we shall follow the (equivalent) route of Marsden and co-workers (see, e.g., Holm *et al.* [13]). (We ignore any technicalities involving boundary conditions, etc.) For arbitrary functions  $F$  and  $G$  of  $\omega$  we define a Poisson bracket (see also Appendix A):

$$\{F, G\}(\omega) = \int_{\Omega} d^2\mathbf{r} \omega(\mathbf{r}) \left\{ \frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega} \right\}_{xy}, \quad (4.2)$$

where  $\{f, g\}_{xy} = (\partial_x f)(\partial_y g) - (\partial_x g)(\partial_y f)$ , the usual (canonical) Poisson bracket in the variables  $x$  and  $y$ . The Hamiltonian is the kinetic energy of the fluid,

$$\mathcal{H} = \frac{1}{2} \int d^2\mathbf{r} \mathbf{u}^2 = \frac{1}{2} \int d^2\mathbf{r} \psi \omega,$$

the Poisson bracket of which we take with the field  $\omega$ , as usual

$$\frac{\partial \omega}{\partial t} = \{\omega, \mathcal{H}(\omega)\}, \quad (4.3)$$

which by way of an easy calculation yields Euler's equations.

In particular, for any integral of the form  $C_f = \int d^2\mathbf{r} f(\omega)$ , the Poisson bracket of  $C_f$  with any functional of  $\omega$  yields zero, implying that these integrals would be conserved quantities for any Hamiltonian system possessing this Poisson bracket. Such quantities are labeled "distinguished functionals," by Olver [52], who argues that because of their trivial character with respect to the Poisson bracket, they do not suffice to make the Hamiltonian system integrable.

Not surprisingly, the Hamiltonian description of point vortices can be viewed as a special case of this more general formulation, as discussed by Marsden and Weinstein [53].

#### B. Liouville theorem

Because  $\omega$  is not a canonical coordinate, we have no conjugate variable with which to write a Liouville equation in the form (4.1). Nevertheless, as alluded to in Eq. (3.7), a Liouville theorem may easily be written in terms of the Fourier components of  $\omega$ . The derivation for 2D Euler flow is straightforward. Using the stream function  $\psi$  in Fourier space we write

$$\begin{aligned} \mathbf{u}(\mathbf{r}) &= i \sum_l (l_y, -l_x) \psi_l e^{i\mathbf{l} \cdot \mathbf{r}}, \\ \omega(\mathbf{r}) &= \sum_{\mathbf{m}} \mathbf{m}^2 \psi_{\mathbf{m}} e^{i\mathbf{m} \cdot \mathbf{r}}. \end{aligned} \quad (4.4)$$

The equation of motion for  $\omega$  becomes

$$\dot{\omega}_{\mathbf{m}} = - \sum_l (\mathbf{l} \times \mathbf{m}) \omega_l \psi_{\mathbf{m}-\mathbf{l}}, \quad (4.5)$$

and differentiation with respect to  $\omega_n$  yields

$$\frac{\delta\dot{\omega}_m}{\delta\omega_n} = 2(\mathbf{n} \times \mathbf{m})\psi_{m-n}. \quad (4.6)$$

Setting  $\mathbf{n}=\mathbf{m}$  yields the desired result  $\delta\dot{\omega}_m/\delta\omega_m=0$ , which demonstrates precisely the incompressibility of phase flow in the space of  $\omega_m$ 's. Now we observe that the transformation to real space variables is trivial: it is merely a linear superposition of  $\omega_k$ 's with coefficients that do not depend on the vorticity field in any way. Consequently, we shall obtain a factor in the partition function, corresponding to the determinant of the transformation between real space and Fourier variables. Since the Fourier transform is an orthogonal transformation, we obtain from this determinant only a trivial constant which we may ignore.

The conserved quantities make no difference to the particular volume form we use in the phase space, provided that we restrict ourselves in the functional integration to the manifold determined by these same constraints. For example, a microcanonical partition function for a particle Hamiltonian would entail the integral

$$Z(E) = \int \prod_i dp_i dq_i \delta(H(\mathbf{x}_1, \mathbf{p}_1, \dots, \mathbf{x}_N, \mathbf{p}_N) - E) \quad (4.7)$$

with phase volume form  $\prod_i dp_i dq_i$ . This latter is the measure on phase space conserved by the Liouville flow, *not* the measure on the surface to which the single conserved quantity, the Hamiltonian, constrains the integration. This measure may be complicated, depending on the derivative of  $H$  perpendicular to the energy hypersurface. In a parallel manner, we expect a microcanonical partition function for the Euler equations to take the form (up to some overall factor accounting for the change of variables from real space to Fourier space)

$$Z(E) = \int \prod_i d\omega_i \delta(\mathcal{H}(\omega) - E) \prod_j \delta(C_j(\omega) - \hat{C}_j), \quad (4.8)$$

where  $\prod_j$  denotes a product over all independent conserved functions of the vorticity.

It is this partition function (4.8) whose properties we need to compute. We carry out the necessary computations, on the grand canonical form of (4.8), in Sec V.

## V. DERIVATION OF THE MEAN-FIELD EQUATIONS

In this section we present two derivations of the mean-field equations. Very similar to that of Lynden-Bell [19], the first derivation [9] is combinatorial and relies on the weakness of the divergence in the Coulomb-type interaction between vortices at small distances. On very short length scales the vortices may be treated as an ideal gas of hard-core particles, whereas on large length scales the interaction energetics are determined entirely by the averaged Coulomb potential of small patches over which the ideal-gas degrees of freedom have been integrated out. This separation of length scales is the key to the exactness of mean-field theory. We present the complete derivation here; a summary may be found in Ref. [9].

A second and more formal derivation may be comforting to those familiar with the standard mappings and manipulations used in critical phenomena. The formal de-

vice is the Kac-Hubbard-Stratanovitch transformation which allows one to convert the long-range Coulomb interaction into a purely local square-gradient interaction. The exactness of mean-field theory is a consequence of the standard wisdom in critical phenomena that interactions of sufficiently long range give rise to mean-field critical behavior. More precisely, if the interaction in  $d$  dimensions decays less rapidly than  $r^{-3d/2}$ , mean-field critical behavior will result. The Coulomb interaction  $r^{2-d}$  [ $\ln(r)$  in  $d=2$ ] clearly satisfies this condition, and decays so slowly (in fact growing in  $d=2$ ) that mean-field theory holds over the *entire phase diagram*.

Both derivations generalize to higher dimensional models, Lynden-Bell's [19] being an example. As will emerge later, Debye-Hückel theory of the three-dimensional plasma [54] is a special case of our theory, the limit of a dense point-charge distribution.

### A. Definitions, conservation laws, and results

To begin with, we write the kinetic energy of the Euler fluid in terms of the vorticity field

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} \int d^2\mathbf{r} |\mathbf{u}(\mathbf{r})|^2 \\ &= -\frac{1}{2} \int d^2\mathbf{r} \psi(\mathbf{r}) \nabla^2 \psi(\mathbf{r}) \\ &= \frac{1}{2} \int d^2\mathbf{r} \int d^2\mathbf{r}' \omega(\mathbf{r}) \mathcal{G}(\mathbf{r}, \mathbf{r}') \omega(\mathbf{r}'), \end{aligned} \quad (5.1)$$

where  $\psi(\mathbf{r})$  is the (scalar) stream function,  $\mathbf{u}(\mathbf{r}) = \nabla \times \psi(\mathbf{r})$  is the velocity field, and  $\omega(\mathbf{r}) = \nabla \times \mathbf{u}(\mathbf{r}) = -\nabla^2 \psi(\mathbf{r})$  is the vorticity field. The Green's function  $\mathcal{G}(\mathbf{r}, \mathbf{r}')$  satisfies  $-\nabla^2 \mathcal{G}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$ , is symmetric in its arguments, and in each variable satisfies the same boundary conditions that  $\psi(\mathbf{r})$  does, for example  $\psi = \text{const}$  at a rigid free-slip boundary. For *free* boundary conditions, as appropriate to an unbounded system, the Green's function can be written down explicitly:

$$\mathcal{G}(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi} \ln \left| \frac{\mathbf{r} - \mathbf{r}'}{R_0} \right|,$$

where  $R_0$  is a reference separation. For an arbitrary vorticity configuration, we have

$$\psi(\mathbf{r}) = \int d^2\mathbf{r}' \mathcal{G}(\mathbf{r}, \mathbf{r}') \omega(\mathbf{r}'), \quad (5.2)$$

the boundary conditions on  $\mathcal{G}(\mathbf{r}, \mathbf{r}')$  ensuring the corresponding ones on  $\psi(\mathbf{r})$ . For the particular case of periodic boundary conditions, we have to be more careful. We require a uniform, compensating background vorticity in order to define  $\mathcal{G}(\mathbf{r}, \mathbf{r}')$ . Equivalently, we demand that the system be neutral:  $\int d^2\mathbf{r} \omega(\mathbf{r}) = 0$ . The equation for  $\mathcal{G}(\mathbf{r}, \mathbf{r}')$  becomes  $-\nabla^2 \mathcal{G}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') - 1/V$ . The  $1/V$  term ensures that the left-hand side integrates to zero, and removes the zeroth-order Fourier component from the  $\delta$  function. Neutrality ensures that (5.2) still holds.

Next we discuss the conserved quantities of Euler flow. Incompressibility,  $\nabla \cdot \mathbf{u} = 0$ , has been incorporated through the existence of the stream function. The energy was already displayed in Eq. (5.1). We suppose that the fluid is confined to a bounded domain, so that the net momentum vanishes. When the region possesses azimu-

thal symmetry, the angular momentum will be conserved. Only the component perpendicular to the flow plane is nonvanishing, and is given by

$$L \equiv \int_{\Omega} d^2\mathbf{r} \mathbf{r} \times \mathbf{u}(\mathbf{r}) \\ = -\frac{1}{2} \int_{\Omega} d^2\mathbf{r} r^2 \omega(\mathbf{r}) + \frac{1}{2} \int_{\partial\Omega} r^2 \mathbf{u}(\mathbf{r}) \cdot d\mathbf{l} . \quad (5.3)$$

Angular momentum will be conserved only if the domain has cylindrical symmetry. In that case the last term has the value  $R^2\Gamma/2$  where  $R$  is the radius of the system, and  $\Gamma = \int_{r=R} \mathbf{u} \cdot d\mathbf{l} = \int d^2\mathbf{r} \omega(\mathbf{r})$  is the circulation, which is also a conserved quantity, as we show below. For an annulus, with inner radius  $r=R_i$  and outer radius  $r=R_o$ , things are a bit more complicated. The last term becomes  $R_o^2\Gamma_o - R_i^2\Gamma_i$ , and the difference  $\Delta\Gamma = \Gamma_o - \Gamma_i$  arises from a static source of vorticity located inside  $r=R_i$ , which enforces the boundary conditions at the inner radius. Irrespective of angular-momentum considerations, the circulation around any impenetrable boundary is a conserved quantity.

One may also consider a boundary that *comoves* with the fluid, locally adjusting its shape with the flow so that no fluid breaches it. Incompressibility entails that any such boundary encloses a constant area. It follows from the vanishing of the convective derivative of the vorticity,  $D\omega/Dt=0$ , that the circulation around the moving boundary is also constant in time, leading to an infinite set of conservation laws. The most convenient way to parametrize these conservation laws is, for each real number  $\sigma$ , to define the quantity  $\Omega_{\sigma}$ , the region of fluid on which  $\omega \leq \sigma$ . The area of this region is conserved by the flow, and the normalized area

$$G(\sigma) = V(\Omega_{\sigma})/V = \frac{1}{V} \int d^2\mathbf{r} \eta(\sigma - \omega(\mathbf{r})) \quad (5.4)$$

is a constant of motion. Here  $\eta(\sigma)$  is the step function:  $\eta(\sigma)=1$  for  $\sigma \geq 0$ , and  $\eta(\sigma)=0$  for  $\sigma < 0$ . Perhaps more convenient is the derivative

$$g(\sigma) = \frac{dG}{d\sigma} = \frac{1}{V} \int d^2\mathbf{r} \delta(\sigma - \omega(\mathbf{r})) . \quad (5.5)$$

We may interpret  $g(\sigma)d\sigma$  as the fractional area on which  $\sigma \leq \omega \leq \sigma + d\sigma$ .

Now the question arises: does the single variable  $\sigma$  suffice to parameterize all relevant conservation laws? At first glance it might seem that a single variable is inadequate. As is well known, a region  $\Omega_0$  which is connected at time 0 remains connected for all time (see, e.g., Arnold [51]). In principle, two distinct blobs of vorticity cannot evolve into a single blob, or vice versa, even though we can easily envision cases where both configurations possess the same function  $G(\sigma)$ . Loosely speaking, a full set of conservation laws ought to specify the number of distinct disconnected domains into which each set  $\{\omega(\mathbf{r})=\sigma\}$  is divided, the topologies of these domains (simply connected, annular, figure eight, etc.), and the fractional areas of the domains, only the net area of which are determined by  $g(\sigma)$ . The flow will preserve each of these properties.

It should be observed that conservation of connectivity

differs from conservation of area integrals  $g(\sigma)$  in a basic way. Up to sets of measure zero, the topological constraints do not preclude the vorticity field from taking any particular spatial configuration we wish to choose. That is, the set of configurations accessible to the topologically constrained fluid is dense in the set of configurations available to the *unconstrained* fluid. A ring of vorticity can evolve into two widely separated blobs, provided the topology is preserved by a filament of vorticity. Since we generally confine ourselves in this discussion to vorticity fields with a finite maximum vorticity, a fine thread of vorticity can contribute very little to the dynamics, especially if the dynamics are in fact dominated by structures on much larger scales. In contrast, the area integrals *do* affect the accessible vorticity configurations, essentially by rendering the vorticity field itself incompressible. When we alter the value of an area integral, we find that new vorticity fields arise which cannot necessarily be duplicated by the action of any volume-preserving diffeomorphism on the original flow. Of course, through this alteration, vorticity fields also arise which *can* be duplicated by an area-preserving transform. Our statistical average will range over all accessible vorticity fields, subject to the conservation laws. Consequently, we neglect the topological laws, which entail no restriction on the vorticity field, but we take into account explicitly the area integrals to the extent that they in fact constrain the vorticity field.

This basic assumption of our work can be justified by several lines of physical argument. (1) We are interested exclusively in long-time properties of the flow. In the infinite-time limit the fluid will be kneaded on all scales: infinitely folded and stretched. As stated above a single connected blob of vorticity can give birth to two (apparently) separated blobs, provided that the two blobs are in fact joined by a thread of vorticity of infinitesimal width. The finite maximum of the vorticity suggests that the thread cannot have a dynamical effect. (2) Topological conservation laws of this type exist in other dynamical systems without constraining ergodicity. For example, phase-space flow for any Hamiltonian system may be viewed as the flow of an incompressible fluid of probability density [55]. The volume of any closed hypersurface in phase space is preserved by the flow, although its shape may change. Nevertheless, phase-space flows can be *mixing*: the average of the probability density becomes *uniform* in phase space. The topology of the particular closed hypersurface does not matter. Mixing can be thought of as a source of macroscopic irreversibility [55] and is accompanied by a loss of information in the evolution from microscopic to coarse-grained vorticity distribution, which we discuss later. (3) Point-vortex systems are subject to these same topological constraints. For point-vortex systems the area integrals are singular, and their only effect is to set the relative numbers of vortices carrying given charges. The area integrals no longer prevent adjacent vortices from approaching each other arbitrarily closely, so that the point-vortex gas is compressible; nevertheless, point vortices yield a divergence-free velocity field, and any curves we draw in the fluid must maintain their connectivities. But the er-

godicity of a system of point vortices has been amply demonstrated by a number of authors [24,26,31,33]. These authors find that dynamical simulation of systems of point vortices confirms the predictions of statistical-equilibrium theory. Kraichnan [3] has also argued that the topological constraints have no effect on the flow at long times. In contrast to conservation of connectivity, the conservation of  $g(\sigma)$  affects the fluid motion in a fundamental way.

The result of the statistical mechanics, derived in Secs. V B and V C is that the statistical averages are completely dominated by a single macroscopic configuration (i.e., mean-field theory is exact). This configuration is conveniently represented in terms of a function  $n_0(\mathbf{r}, \sigma)$ , the local (but coarse-grained) density of vorticity with value  $\sigma$ . We find  $n_0$  is determined self-consistently as

$$n_0(\mathbf{r}, \sigma) = \frac{\exp(-\bar{\beta}\{\sigma[\psi_0(\mathbf{r}) - h(\mathbf{r})] - \mu(\sigma)\})}{\int_{-\infty}^{\infty} d\sigma' \exp(-\bar{\beta}\{\sigma'[\psi_0(\mathbf{r}) - h(\mathbf{r})] - \mu(\sigma')\})} \quad (5.6)$$

where the stream function  $\psi_0(\mathbf{r})$ , with accompanying vorticity  $\omega_0(\mathbf{r})$ , is determined by  $n_0$ ,

$$-\nabla^2 \psi_0(\mathbf{r}) = \omega_0(\mathbf{r}) = \int_{-\infty}^{\infty} d\sigma \sigma n_0(\mathbf{r}, \sigma). \quad (5.7)$$

In Eq. (5.6) the Lagrange multiplier function  $\mu(\sigma)$  enforces, and is determined by, the  $g(\sigma)$  constraint

$$g(\sigma) = \frac{1}{V} \int d^2\mathbf{r} n_0(\mathbf{r}, \sigma), \quad (5.8)$$

and  $\bar{\beta}$  is determined by fixing the energy

$$E = -\frac{1}{2} \int d^2\mathbf{r} \omega_0(\mathbf{r}) \psi_0(\mathbf{r}).$$

The field  $h(\mathbf{r})$  may include external potentials coupling to the vorticity field via

$$\Delta H = - \int d^2\mathbf{r} h_{\text{ext}}(\mathbf{r}) \omega(\mathbf{r}) \quad (5.9)$$

such as might come, for example, from a Coriolis force, and also additional Lagrange multiplier terms to enforce any additional constraints due to the symmetry of the system such as angular-momentum conservation. Note that to solve equations (5.6) and (5.7) we must solve a single spatial partial-integro-differential equation for  $\psi_0(\mathbf{r})$ , but depending parametrically on a function determined by functional constraints. These full equations are certainly quite complicated. Some simple special cases described in Sec. VI may make their content more apparent.

We now proceed to the derivation of these equations. Results and consequences are taken up again in Sec. VI.

### B. Combinatorial derivation

As discussed above we derive the equilibrium statistics under the assumption that, at most,  $H$ ,  $L$ , and  $g(\sigma)$  need to be explicitly accounted for. For counting purposes we discretize space into a lattice with spacing  $a$ , and perform calculations with fixed  $a$ . We shall eventually take the limit  $a \rightarrow 0$ . Convergence is more or less ensured by the softness of the Coulomb interaction at short distances.

More explicitly, we define the coarse-grained vorticity field

$$\omega_i = \frac{1}{a^2} \int_{\mathcal{B}_i} d^2\mathbf{r} \omega(\mathbf{r}) \quad (5.10)$$

where  $\mathcal{B}_i$  denotes the lattice square centered at  $\mathbf{r}_i$ . The lattice clearly breaks (local) Galilean invariance, so an associated discrete dynamics can probably not be realized. We are investigating discretizations that preserve Galilean invariance, but we expect them to yield the same equilibrium statistics (see Appendix A). The discrete static energy is perfectly well defined,

$$\mathcal{H}_a = \frac{1}{2} a^4 \sum_{i,j} \mathcal{G}_{ij} \omega_i \omega_j. \quad (5.11)$$

The matrix  $\mathcal{G}_{ij}$  is the inverse of the discrete Laplacian

$$-\sum_{\text{NN}\gamma} (\mathcal{G}_{i+\gamma,j} - \mathcal{G}_{ij}) = \delta_{ij}, \quad (5.12)$$

where NN stands for nearest neighbors. We need to choose a scale  $R_0$  which sets the value of  $\mathcal{G}_{0,0}$ . A standard choice is to take  $\mathcal{G}_{0,0} = 0$ , which corresponds to  $R_0/a$  equal to a particular fixed constant [56], but we shall later require that  $R_0$  be of the order of the system size, independent of  $a$  as  $a \rightarrow 0$  [see the discussion following (5.49)]. Appropriate boundary conditions should be imposed as well.

The canonical statistical configuration average is also well defined and consists of a sum over all permutations of the  $N = V/a^2$  boxes, in which we regard boxes possessing the same vorticity  $\omega$  as indistinguishable. The permutations are weighted by the Gibbs factor  $e^{-\beta_a \mathcal{H}_a}$ , where  $\beta_a = 1/T_a$  sets the average energy. We shall see later that  $T_a$  must scale with  $a$ . This prescription obviously respects conservation of the function  $g(\sigma) = N_\sigma/N$ , which merely counts  $N_\sigma$ , the number of squares with vorticity  $\sigma$ .

In order to carry out the configuration sum, we take advantage of the separation of scales alluded to earlier. We need to alter the discretization slightly: we discretize the field  $\omega(\mathbf{r})$  on a *finer* scale than the matrix  $\mathcal{G}(\mathbf{r}, \mathbf{r}')$ . In other words, we satisfy (5.12), but allow the indices  $i, j$  to run over a lattice with spacing  $l \gg a$ . We define

$$\mathcal{H}_{l,a} = \frac{1}{2} a^4 \sum_{\substack{i,j \\ a,\beta}} \mathcal{G}_{ij} \omega_{i\alpha} \omega_{j\beta}, \quad (5.13)$$

where the Greek indices run over the  $(l/a)^2 a$  cells within each  $l$  cell, which is labeled by Latin indices. In what follows we shall consider the limit  $a \rightarrow 0$  for fixed  $l$ , then take  $l \rightarrow 0$  in the end. Since  $\mathcal{G}_{ij}$  is well behaved for small  $|i-j|$ , this procedure should yield the correct continuum limit, so long as the bulk correlations are much larger than  $l$ .

Equation (5.13) is constructed in such a way that permutations of the  $\omega$ 's within a given  $l$  cell do not affect the energy. We may perform the sum over this restricted set of permutations explicitly. In order to control the limiting procedure properly, the allowed values of the vorticity must be discretized as well. A convenient way to do this is to partition the *range*  $[0,1]$  of  $G(\sigma)$  into  $p_l$  uniform

intervals of width  $\Delta_l = 1/p_l$ , which will vanish when  $l \rightarrow 0$ . For small  $\Delta_l$  this corresponds to a partitioning of the  $\sigma$  axis into intervals  $\sigma_k \leq \sigma < \sigma_k + \Delta\sigma_k$ , where  $G(\sigma_k) = k\Delta_l$ , and  $g(\sigma_k)\Delta\sigma_k \approx \Delta_l$  with  $k = 0, 1, \dots, p_l$ . Strictly speaking, we must assume here that  $g(\sigma)$  has compact support; otherwise  $\Delta\sigma_0$  and  $\Delta\sigma_{p_l}$  could be infinite [i.e.,  $G^{-1}(0) = -\infty$  and/or  $G^{-1}(1) = +\infty$ ]. If  $g(\sigma)$  decays to zero sufficiently rapidly, we could relax this condition, and allow the support of  $g(\sigma)$  to diverge at the same time that  $l \rightarrow 0$ . We shall have more to say about what happens when  $g(\sigma)$  does not decay sufficiently rapidly later—this condition obtains for *point vortices*.

We now introduce the following notation. Let  $N = V/a^2$  be the number of  $a$  cells and  $M = V/l^2 \equiv N/n$  be the number of  $l$  cells. Let  $v_i(\sigma_k)$  be the number of  $a$  cells in  $l$ -cell  $i$  with vorticity  $\sigma_k$ , and let  $v^T(\sigma_k) = \sum_i v_i(\sigma_k)$  be the total number of  $a$  cells with vorticity  $\sigma_k$ . Note that  $\sum_k v_i(\sigma_k) = n$ ,  $\sum_k v^T(\sigma_k) = N$ , and that

$$v^T(\sigma_k)/N = \Delta_l \approx g(\sigma_k)\Delta\sigma_k. \quad (5.14)$$

The last equality holds generally, even when the intervals  $\Delta_l$  are not independent of  $k$ . The average vorticity in  $l$ -cell  $i$  is

$$\bar{\omega}_i = \frac{1}{n} \sum_k \sigma_k v_i(\sigma_k) = \frac{1}{n} \sum_\alpha \omega_{i\alpha}. \quad (5.15)$$

The energy (5.13) remains constant so long as the  $\bar{\omega}_i$  are fixed throughout. We now examine the combinatorial factor that results from distributing vorticity in all possible ways for given values of  $\{v_i(\sigma_k)\}$ . This factor is given by

$$W_l\{v_i(\sigma_k)\} = \prod_i \frac{n!}{\prod_k v_i(\sigma_k)!}, \quad (5.16)$$

which represents the number of ways of distributing each  $v_i(\sigma_k)$  among the  $n$   $a$  cells in  $l$ -cell  $i$ . The total partition function is

$$Z[g] = \text{tr}^{\{v_i(\sigma_k)\}} [W_l\{v_i(\sigma_k)\} e^{-\beta_a \mathcal{H}_{l,a}\{v_i(\sigma_k)\}}], \quad (5.17)$$

where the trace is over all distinct ways of assigning the  $\{v_i(\sigma_k)\}$  with fixed  $\{v^T(\sigma_k)\}$  given by (5.14).

We must be careful to normalize the trace correctly. The Liouville theorem implies that the correct configuration sum should have uniform measure in the space of fields  $\omega(\mathbf{r})$ , or, on a lattice, in the space of the  $\omega_{i\alpha}$ . Before discretizing  $g(\sigma)$ , the trace is defined by

$$\text{tr}^{\{\omega_{i\alpha}\}} = \prod_{i,\alpha} \int_{-\infty}^{\infty} d\omega_{i\alpha} \frac{1}{q_0}, \quad (5.18)$$

where  $q_0$  is a reference vorticity needed to make  $Z$  dimensionless. Upon discretizing  $g(\sigma)$  this becomes

$$\begin{aligned} \text{tr}^{\{\omega_{i\alpha}\}} &= \prod_{i,\alpha,k=1}^p \left[ \frac{\Delta\sigma_k}{q_0} \right] \\ &= \prod_i \sum_{\alpha} \prod_k \left[ \frac{\Delta\sigma_k}{q_0} \right]^{v^T(\sigma_k)}, \end{aligned} \quad (5.19)$$

where the sum is now over the discrete values  $\omega_{i\alpha} \in \{\sigma_k\}_{k=1}^p$ . It is understood that appropriate  $\delta$  functions should be introduced in (5.18) and (5.19) to preserve the conservation laws. In (5.19) they take the form

$$\Delta(v, g) \equiv \prod_k \delta_{v^T(\sigma_k), \sum_i v_i(\sigma_k)} \prod_i \delta_{n, \sum_k v_i(\sigma_k)}, \quad (5.20)$$

where we remind you that  $v_i(\sigma_k) = \sum_\alpha \delta_{\omega_{i\alpha}, \sigma_k}$ . Our final expression for the partition function then becomes

$$\begin{aligned} Z(g) &= \text{tr}^{\{v_i(\sigma_k)\}} \left[ \prod_k \left[ \frac{\Delta\sigma_k}{q_0} \right]^{v^T(\sigma_k)} \right. \\ &\quad \left. \times W_l\{v_i(\sigma_k)\} \Delta(v, g) e^{-\beta \mathcal{H}} \right] \end{aligned} \quad (5.21)$$

in which the trace is now a *free* sum over all  $0 \leq v_i(\sigma_k) \leq n$ . It is easy to check that when  $\mathcal{H} \equiv 0$ , (5.21) yields the expected hard-core ideal-gas result

$$Z(g, \mathcal{H} \equiv 0) = \frac{N!}{\prod_k \left[ v_T(\sigma_k)! \left[ \frac{\Delta\sigma_k}{q_0} \right]^{-v_T(\sigma_k)} \right]}. \quad (5.22)$$

Using Stirling's formula  $m! \approx \sqrt{2\pi m} (m/e)^m$ , which is valid for large  $m$ , we may further evaluate (5.21). Define the  $l$ -scale entropy by

$$S_l = \ln \left[ \prod_k \left[ \frac{\Delta\sigma_k}{q_0} \right]^{v^T(\sigma_k)} W_l\{v_i(\sigma_k)\} \right]. \quad (5.23)$$

We obtain

$$\begin{aligned} S_l &= \sum_k v^T(\sigma_k) \ln \left[ \frac{\Delta\sigma_k}{q_0} \right] + N \ln(n) \\ &\quad - \sum_{i,k} v_i(\sigma_k) \ln[v_i(\sigma_k)] + O[\ln(N)]. \end{aligned} \quad (5.24)$$

On defining  $n_i(\sigma_k)\Delta\sigma_k = v_i(\sigma_k)/n$ , and dropping terms of  $O[\ln(N)]$ , (5.24) becomes

$$\begin{aligned} S_l &= -N \sum_{i,k} \frac{\Delta\sigma_k}{M} n_i(\sigma_k) \ln[q_0 n_i(\sigma_k)] \\ &\rightarrow -N \int d^2\mathbf{r} \int d\sigma n(\mathbf{r}, \sigma) \ln[q_0 n(\mathbf{r}, \sigma)] \text{ as } n \rightarrow \infty. \end{aligned} \quad (5.25)$$

Similarly, the logarithm of Eq. (5.22) has the continuum limit

$$S(\mathcal{H} \equiv 0) \rightarrow -N \int d\sigma g(\sigma) \ln[q_0 g(\sigma)], \quad (5.26)$$

where we have used Eq. (5.14). The field

$$n(\mathbf{r}, \sigma) = \int_{l \text{ cell at } \mathbf{r}} \frac{d^2\mathbf{r}'}{l^2} \delta(\sigma - \omega(\mathbf{r}')),$$

which represents the fine-grained number density of vorticity  $\sigma$  at  $\mathbf{r}$ , obeys the two constraints

$$\int d\sigma n(\mathbf{r}, \sigma) = 1, \quad (5.27)$$

$$\frac{1}{V} \int d^2\mathbf{r} n(\mathbf{r}, \sigma) = g(\sigma),$$

representing incompressibility and the conservation laws, respectively. The coarse-grained vorticity at  $\mathbf{r}$  is

$$\bar{\omega}(\mathbf{r}) = \int d\sigma \sigma n(\mathbf{r}, \sigma) \quad (5.28)$$

so that the Hamiltonian reads

$$\mathcal{H} \rightarrow \frac{1}{2} \int d^2\mathbf{r} \int d^2\mathbf{r}' \int d\sigma \int d\sigma' \sigma n(\mathbf{r}, \sigma) \sigma' n(\mathbf{r}', \sigma') \times \mathcal{G}(\mathbf{r}, \mathbf{r}'), \quad (5.29)$$

as  $n \rightarrow \infty$ . We now make the key observation which leads to the scaling of the temperature with  $a$ , and to mean-field theory: the coarse-grained entropy  $S_l$  diverges as  $VN = V^2/a^2$  and it is extensive relative to  $a$  in the sense that it is proportional to the number of  $a$  lattice

sites. Extensivity ordinarily refers to scaling with the volume  $V = V(\Omega)$ ; however, we are interested in the limit  $N \rightarrow \infty$ ,  $V = \text{const}$ . The energy (5.29) is finite as  $a \rightarrow 0$ , scaling only with  $V(\Omega)$ , but as we shall see shortly, *not* linearly. Clearly, if  $\beta_a$  is independent of  $a$  in (5.17), the entropy will completely dominate the energy, (5.26) will ensue for any  $\mathcal{H}$ , and vorticity will be distributed uniformly without structure over the entire system. In order to have competition between energy and entropy, leading to nontrivial flows, *both* terms must be of the same order. This condition entails the scaling

$$\beta_a = \bar{\beta}/a^2 \Rightarrow T_a = \bar{T}a^2 \quad (5.30)$$

where  $\bar{\beta}$  remains fixed as  $a \rightarrow 0$ . The combination  $\beta_a \mathcal{H}_{l,a}$  then scales as  $V^p/a^2$ , where the value of the exponent  $p$  will be addressed below.

Returning now to (5.25), (5.29), and (5.17), the partition function reads

$$Z(g) = \lim_{N \rightarrow \infty} \text{tr}^n \exp \left[ -N \left[ \frac{1}{V} \int d^2\mathbf{r} \int d\sigma n(\mathbf{r}, \sigma) \ln[q_0 n(\mathbf{r}, \sigma)] + (\bar{\beta}/2V) \int d^2\mathbf{r} \int d^2\mathbf{r}' \bar{\omega}(\mathbf{r}) \bar{\omega}(\mathbf{r}') \mathcal{G}(\mathbf{r}, \mathbf{r}') \right] \right], \quad (5.31)$$

where the trace is over all fields  $n(\mathbf{r}, \sigma)$  subject to (5.27). It is now clear why mean-field theory is valid: in the limit  $N \rightarrow \infty$  the trace will pick out the minimum of the quantity in large parentheses. The average of the field  $n(\mathbf{r}, \sigma)$ , which we denote by  $n_0(\mathbf{r}, \sigma)$ , is then determined by minimizing the functional

$$-\frac{1}{\beta_a V} \ln Z = \mathcal{F}[n] = \frac{1}{\beta V} \int d^2\mathbf{r} \int d\sigma n(\mathbf{r}, \sigma) \ln[q_0 n(\mathbf{r}, \sigma)] + (1/2V) \int d^2\mathbf{r} \int d^2\mathbf{r}' \bar{\omega}(\mathbf{r}) \bar{\omega}(\mathbf{r}') \mathcal{G}(\mathbf{r}, \mathbf{r}') \quad (5.32)$$

subject to (5.27). In order to derive a differential equation from (5.32) we use Lagrange multipliers to enforce the constraints. (We could also at this stage include, for example, conservation of angular momentum in an axisymmetric situation and external potentials.) Define

$$\mathcal{G}[n, \mu, \lambda] = \mathcal{F}[n] - \int d\sigma \mu(\sigma) \int d^2\mathbf{r} \frac{1}{V} n(\mathbf{r}, \sigma) - \frac{1}{V} \int d^2\mathbf{r} \lambda(\mathbf{r}) \int d\sigma n(\mathbf{r}, \sigma), \quad (5.33)$$

where the constraints (5.27) determine  $\mu(\sigma)$  and  $\lambda(\mathbf{r})$ . The required differential equation is obtained by differentiating (5.33) freely with respect to  $n(\mathbf{r}, \sigma)$  and setting it equal to zero

$$\frac{\delta \mathcal{F}}{\delta n(\mathbf{r}, \sigma)} = 0 = \ln[q_0 n_0(\mathbf{r}, \sigma)] + 1 + \bar{\beta} \int d^2\mathbf{r}' \sigma \omega_0(\mathbf{r}') \mathcal{G}(\mathbf{r}, \mathbf{r}') - \bar{\beta}[\mu(\sigma) + \lambda(\mathbf{r})], \quad (5.34)$$

where  $\omega_0(\mathbf{r}) = \langle \omega(\mathbf{r}) \rangle = \int d\sigma \sigma n_0(\mathbf{r}, \sigma)$  is the equilibrium vorticity field. Define the equilibrium *stream function*

$$\psi_0(\mathbf{r}) = \int d^2\mathbf{r}' \mathcal{G}(\mathbf{r}, \mathbf{r}') \omega_0(\mathbf{r}') \quad (5.35)$$

which, since (5.35) is a linear relationship, is just the average  $\langle \psi(\mathbf{r}) \rangle$ . We then have

$$q_0 n_0(\mathbf{r}, \sigma) = \exp[-1 + \bar{\beta} \mu(\sigma) + \bar{\beta} \lambda(\mathbf{r}) - \bar{\beta} \sigma \psi_0(\mathbf{r})]. \quad (5.36)$$

The function  $\lambda(\mathbf{r})$  may be immediately eliminated by in-

tegrating both sides over  $\sigma$ ,

$$\exp(-\lambda(\mathbf{r})\bar{\beta} + 1) = \frac{1}{q_0} \int d\sigma \exp\{-\bar{\beta}[\sigma \psi_0(\mathbf{r}) - \mu(\sigma)]\} \quad (5.37)$$

so that

$$n_0(\mathbf{r}, \sigma) = \frac{\exp\{-\bar{\beta}[\sigma \psi_0(\mathbf{r}) - \mu(\sigma)]\}}{\int d\sigma' \exp\{-\bar{\beta}[\sigma' \psi_0(\mathbf{r}) - \mu(\sigma')]\}}, \quad (5.38)$$

which corresponds to (5.6), although the one-body field  $h(\mathbf{r})$  has been left out for simplicity. A differential equation for  $\psi_0(\mathbf{r})$  may be obtained by multiplying both sides of (5.38) by  $\sigma$  and integrating, leading to Eq. (5.7), and the function  $\mu(\sigma)$  is determined by integrating (5.38) over all space, leading to Eq. (5.8).

Not only is the temperature appearing in Eq. (5.38) scaled with the cutoff, it also is often negative in physical situations. In fact for total vorticity zero (neutral system) the *only* nontrivial Euler states are necessarily described by negative temperatures. To see this consider the *neutral* lattice Coulomb gas at positive temperature  $T_{\text{Coul}}$  described by the Hamiltonian  $\mathcal{H}_a$  in (5.11), but *without* the  $a^4$  prefactor,

$$\mathcal{H}_{\text{Coul}} \equiv \frac{1}{2} \sum_{i,j} \mathcal{G}_{ij} \omega_i \omega_j, \quad (5.39)$$

$$\sum_i \omega_i = 0.$$

This system is well understood. Its average energy scales linearly with  $N$ , the number of lattice sites. The low-temperature phase consists of tightly bound, oppositely charged dipole pairs (we consider the two-species case  $\omega_i = \pm q$  or 0 for simplicity.) There is a transition at finite temperature  $T_c$  (the Kosterlitz-Thouless transition in two dimensions) to a neutral-plasma phase, in which the oppositely charged vortices are decoupled, which persists for arbitrarily high temperature. In both phases, correlations decay on the scale of the lattice spacing  $a$  (as power laws for  $T < T_c$  and exponentially for  $T > T_c$ ). The origin of the decay of correlations is screening: oppositely charged vortices attract and interfere with one another, yielding an effective interaction whose range is *finite* on the scale of the lattice spacing  $a$ . It is clear that any coarse grained  $l$  cell will be effectively *neutral* when  $l \gg a$ , and that vortices in different  $l$  cells will be *uncorrelated*. These features violate the assumptions upon which we based our argument that the lattice Hamiltonian (5.13) and the  $a \rightarrow 0$  limit will yield the correct continuum limit, and suggest that the positive-temperature neutral system is not properly described by a such a theory.

There is a simpler way to understand the failure of (5.13). Since (5.39) yields an energy which scales linearly with  $N$ , the equivalent hydrodynamic energy (5.13) will scale as  $Na^4 = Va^2$ , and *vanish* with  $a$ . Equivalently, we have the correspondence  $\beta_{\text{Coul}} = \beta_a a^4 = \bar{\beta} a^2$  so that  $\bar{T} = T_{\text{Coul}} a^2$  vanishes with  $a$  at fixed  $T_{\text{Coul}}$ . Intuitively, a positive hydrodynamic energy requires macroscopic flows, on the scale of the system size, while screening implies flows only on the microscale, invisible on any macroscopic scale. As is apparent from (5.28) and (5.29), hydrodynamic flows require macroscopic non-neutral regions of vorticity. In a neutral system, charge can segregate macroscopically only if, by some means, like charged vortices happen to *attract* rather than repel. Homophilic charges correspond to reversing the sign of the energy (5.13) or (5.39), or equivalently of the temperature (5.30), and naturally lead us to examine *negative-temperature* states of  $\mathcal{H}_{l,a}$ . As we remarked in the historical review, negative temperatures are inaccessible to systems consisting of real particles, since (the kinetic part of) the energy is not bounded from above. Crudely, we need to ensure that  $\int d\varepsilon \rho(\varepsilon) \exp(-\varepsilon/T)$  be finite, where  $\rho(\varepsilon)$  denotes the density of configurations with energy  $\varepsilon$ . This condition will be satisfied when the energy is bounded from above and below, or when the configuration space is sufficiently constrained to yield at least exponential decay in  $\rho(\varepsilon)$  for both  $\varepsilon \rightarrow +\infty$  and  $\varepsilon \rightarrow -\infty$ .

Since inviscid Euler dynamics is an approximation that neglects coupling between molecular and hydrodynamic degrees of freedom, it is perfectly consistent for the degrees of freedom at these respective length scales to be out of equilibrium with each other: the former at positive temperature; the latter at negative temperature. Viscosity, an agent of diffusive transport, would provide a route for energy to drain from the hydrodynamic degrees of freedom and excite molecular degrees of freedom, yielding in the long-time limit a warmer fluid, devoid of macroscopic motion. (More generally, since viscosity

preserves the total vorticity, depending on boundary and initial conditions, a rigidly rotating fluid might result.)

In order to clarify further the idea of negative temperature, which originated in this context with Onsager [1], we may view it simply as a mathematical device for setting the energy. In the microcanonical ensemble, thermodynamics is inferred from appropriate averages over the phase-space hypersurface at a given energy  $E$ . By the standard argument, we allow  $E$  to fluctuate, while we fix only the average energy using the Lagrange multiplier  $\beta$  to obtain the canonical ensemble, both descriptions coinciding in the thermodynamic limit. If the density of configurations  $\rho(\varepsilon)$  decays rapidly enough, there is no reason for restricting  $\beta$  to be positive. Only by recognizing that any real system will be in contact with positive- $\beta$  molecular degrees of freedom do we restrict our attention to positive- $\beta$  equilibria. If contacts between the system of interest and the positive- $\beta$  bath are weak, there may arise a time scale much longer than intrasystem equilibration times, but much shorter than the intersystem equilibration time through the contacts, over which negative- $\beta$  equilibria are a valid thermodynamic description. This separation of time scales occurs in certain paramagnetic systems of nuclear moments in a crystal, where negative (spin-) temperature states have been experimentally demonstrated (see Landau and Lifshitz, Chap. 6 [Ref. 17]).

The presence of a separation of time scales may not be easy to establish, especially as different parts of our system may equilibrate more rapidly through the contacts than others. It is clear, for example, that viscosity ought to equilibrate first regions with large second derivatives in the velocity field, so that small-scale structures such as filaments of vorticity will be rapidly smoothed out, whereas large-scale structures will have longer lifetimes. We might hope that high-Reynolds-number turbulence, in which the dimensionless parameter reflects the dominance of convective processes over the diffusive effects of viscosity, might permit such a separation of time scales, particularly when an additional length scale (such as system size) exists to set the scale for coherent structures.

The reader may find it useful to think about positive and negative temperatures in the context of nearest-neighbor ferromagnetic spin models. There, negative-temperature equilibria are the usual positive-temperature equilibria of the corresponding nearest-neighbor *antiferromagnet*. Statistics of the *high-energy* states of a ferromagnet are simply those of the low-energy states of the antiferromagnet. In general, a sure sign of the existence of negative temperature states is an energy that remains finite when  $T \rightarrow +\infty$ . This energy is the *unweighted* average of the energy over all states, and the system can never achieve energies larger than this particular energy unless  $T$  becomes negative. We observe that the thermodynamics is continuous through  $T = \pm\infty$ , ( $\beta = 0$ ), *not*  $T = 0$ . The latter corresponds to the ground state for  $T \rightarrow 0+$ , and to the most excited state for  $T \rightarrow 0-$ .

In contrast to the ferromagnet, where interactions are short-ranged, the negative-temperature states of the Coulomb gas do not have energies that scale linearly in



system size. This property returns us to the question of the value of the exponent  $p$  below Eq. (5.30). An easy calculation reveals that rescaling the number of lattice sites by a factor  $\alpha$  yields an energy-rescaling factor  $\alpha^2$  (modulo  $\ln\alpha$  corrections that are unimportant in this argument). In a general dimension  $d$ , the energy rescaling factor is  $\alpha^{2/d+1}$ . So long as decreasing the coarseness of the mesh in (5.13) does not affect the macroscopic vorticity distribution (an assumption basically equivalent to convergence as  $a \rightarrow 0$ ), the energy  $\mathcal{H}_{l,a}$  will scale as  $N^2 a^4 = V^2$ . The combination  $\beta_a \mathcal{H}_{l,a}$  scales as  $V^2/a^2$ , i.e.,  $p=2$ . This value corresponds to the scaling for  $S_i$ , reconfirming that we have made the correct choice for  $\beta_a$ .

### C. Mean-field theory: Kac-Hubbard-Stratonovich transformation

We now turn to the second derivation of the mean-field equations. This time we introduce appropriate Lagrange multipliers into the statistical mechanics from the outset. Unlike the multipliers introduced in (5.33), which constitute merely an exact method for solving constrained differential equations, statistical Lagrange multipliers allow us to remove constraints on the partition sum directly. In the mean-field limit the two constructions coincide because only a single state contributes to the partition sum.

In standard fashion, we introduce into the Hamiltonian a Lagrange multiplier  $\mu_i$  for each conserved quantity,  $Q_i[\omega]$ ,

$$\bar{\mathcal{H}}[\omega] = \beta \left[ \mathcal{H}[\omega] - \sum_i \mu_i Q_i[\omega] \right]. \quad (5.40)$$

The partition function consists of an unconstrained trace:  $Z = \text{tr}^\omega \exp \bar{\mathcal{H}}$ . The value of a conserved quantity may be obtained from the appropriate free-energy derivative

$$q_i \equiv \frac{1}{V} \langle Q_i[\omega] \rangle = - \frac{\partial F}{\partial \mu_i}, \quad (5.41)$$

where  $F = -(\beta V)^{-1} \ln(Z)$ . Conservation of  $g(\sigma)$  leads to a term

$$\begin{aligned} \mathcal{H}_\mu &= - \int d\sigma \mu(\sigma) \int \frac{1}{V} d^2\mathbf{r} \delta(\sigma - \omega(\mathbf{r})) \\ &= - \frac{1}{V} \int d^2\mathbf{r} \mu(\omega(\mathbf{r})) \end{aligned} \quad (5.42)$$

with the function  $\mu(\sigma)$  to be determined from the functional derivative

$$g(\sigma) = - \frac{\delta F}{\delta \mu(\sigma)}. \quad (5.43)$$

An alternative formulation of this term is obtained by associating, to each  $n=1,2,3,\dots$  a Lagrange multiplier  $\mu_n$  with the corresponding power of the vorticity  $\Omega_n = \int d^2\mathbf{r} \omega^n(\mathbf{r})$ ,

$$\mathcal{H}_\mu = - \sum_n \mu_n \Omega_n. \quad (5.44)$$

We observe that the  $\mu_n$  are the Taylor coefficients of  $\mu(\sigma)$ ,

$$\mu(\sigma) = \sum_{n=1}^{\infty} \mu_n \sigma^n, \quad (5.45)$$

and corresponding to (5.43) we have

$$\frac{1}{V} \langle \Omega_n \rangle = \int d\sigma \sigma^n g(\sigma) = - \frac{\partial F}{\partial \mu_n} \quad (5.46)$$

so that  $\langle \Omega_n \rangle$  are the moments of  $g(\sigma)$ .

Angular-momentum conservation is incorporated by means of a term

$$\mathcal{H}_h = - \int d^2\mathbf{r} h(\mathbf{r}) \omega(\mathbf{r}), \quad (5.47)$$

where  $h(\mathbf{r}) = -\frac{1}{2} \alpha r^2 + h_{\text{ext}}(\mathbf{r})$ , and  $h_{\text{ext}}(\mathbf{r})$  contains any external fields that might couple linearly to the vorticity field. For example, the Coriolis force in the  $\beta$ -plane approximation yields  $h_{\text{ext}}(\mathbf{r}) = \beta r^3$ . The derivative  $-\partial f / \partial \alpha$  yields the first term in (5.3). To obtain the contribution to  $L$  from the circulation, observe that  $R_o^2 \Gamma_o - R_i^2 \Gamma_i = R_o^2 \Omega_1 + (R_o^2 - R_i^2) \Gamma_i$ , where  $\Omega_1$  is calculated from (5.43) or (5.46) and  $\Gamma_i$  is enforced by an imaginary point vortex at the origin of strength  $\Gamma_i$ . The latter appears as an additional external field  $h_i(\mathbf{r}) = \Gamma_i \mathcal{G}(\mathbf{0}, \mathbf{r})$  in (5.47), which may be seen most simply by separating out a static contribution  $\Gamma_i \delta(\mathbf{r})$  to the vorticity field in (5.1) and dropping the self-energy term proportional to  $\Gamma_i^2$ .

The complete Hamiltonian now reads

$$\begin{aligned} \bar{\mathcal{H}} &= \frac{1}{2} \beta \int d^2\mathbf{r} \int d^2\mathbf{r}' \mathcal{G}(\mathbf{r}, \mathbf{r}') \omega(\mathbf{r}) \omega(\mathbf{r}') \\ &\quad - \frac{\beta}{V} \int d^2\mathbf{r} \mu(\omega(\mathbf{r})) - \beta \int d^2\mathbf{r} h(\mathbf{r}) \omega(\mathbf{r}) \end{aligned} \quad (5.48)$$

and  $Z = \text{tr}^\omega [\exp(-\bar{\mathcal{H}})]$  is now a *free* trace over all vorticity configurations  $\omega(\mathbf{r})$ . Equation (5.48) is the Hamiltonian for a continuum, continuous-spin Ising model, with exchange constants  $\mathcal{G}(\mathbf{r}, \mathbf{r}')$ , spin-weighting factor  $\mu(\sigma)$ , and external magnetic field  $h(\mathbf{r})$ . More typically, for applications to critical phenomena,  $\mathcal{G}(\mathbf{r}, \mathbf{r}')$  is short-ranged [e.g.,  $\mathcal{G}(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \nabla^2$ ], and  $\mu(\sigma)$  is a low-order polynomial,  $\beta \mu(\sigma) = \frac{1}{2} r \sigma^2 + u \sigma^4$  (the  $\phi^4$  model). We are not restricted to analytic forms for  $\mu(\sigma)$ . The usual Ising discrete measure arises from the choice  $\exp[\beta \mu(\sigma)] = \frac{1}{2} [\delta(\sigma - 1) + \delta(\sigma + 1)]$ ; whenever the function  $g(\sigma)$  consists of a series of  $\delta$  functions (a finite number of vortex charge species), so does  $\exp[\beta \mu(\sigma)]$ .

We now introduce the Kac-Hubbard-Stratonovich transformation. The idea is the following. All of the terms in  $\bar{\mathcal{H}}$ , except for the first, are *local*. We may convert the first term into one that is purely local at the cost of introducing a new field  $\psi(\mathbf{r})$ . The basic formula we use is

$$\begin{aligned} &\exp \left[ \frac{1}{2} \sum_{i,j} A_{ij} s_i s_j \right] \\ &= \frac{1}{\mathcal{N}} \prod_i \int_{-\infty}^{\infty} d\psi_i \exp \left[ -\frac{1}{2} \sum_{i,j} (A^{-1})_{ij} \psi_i \psi_j - \sum_i \psi_i s_i \right], \end{aligned} \quad (5.49)$$

where  $A$  is any positive definite matrix,  $\{s_i\}$  any set of real or complex numbers, and  $\mathcal{N} = \det(2\pi A)^{1/2}$  is a normalization factor. If  $A$  is *negative* definite the same formula holds with  $\psi_i$  replaced by  $i\tilde{\psi}_i$  in the exponent on the right-hand side. We shall apply (5.49) with  $A_{ij} = -\beta^{-1}\mathcal{G}_{ij}$  and  $s_i = \beta\omega_i a^2$  [see (5.11)].

Since  $\mathcal{G}_{ij}$  is the inverse of the negative Laplacian operator  $-\Delta_{ij}$ ,  $\mathcal{G}$  will be positive definite except, perhaps, for the eigenvalue corresponding to the *constant* eigenfunction. If conducting boundary conditions are used, the  $\delta$  function  $\delta_{ij}$  has no constant component, so  $-\Delta_{ij}$  is positive definite and invertible and this problem does not arise. For periodic boundary conditions we must be more careful. In that case, the constant component,  $\mathcal{G}_0$  of  $\mathcal{G}_{ij}$  is arbitrary, and we choose it to be positive. This choice is equivalent to demanding that  $\mathcal{H}[\omega]$  be positive for a *uniform* vorticity field  $\omega(\mathbf{r}) \equiv \text{const} \neq 0$ , which may be ensured by requiring the reference separation  $R_0$  to be larger than the system size so that  $-(1/2\pi)\ln|(\mathbf{r}-\mathbf{r}')/R_0|$  is always positive. The constant  $\mathcal{G}_0$  does not affect the physics, since it only sets a reference energy, which corresponds to an additive term  $\frac{1}{2}\mathcal{G}_0\Omega_1^2$  and vanishes for periodic boundary conditions where  $\Omega_1=0$  [see the discussion below (5.2)].

With proper choice of  $\mathcal{G}$ , we have the representation for the discretized Coulomb energy

$$\exp\left[-\frac{1}{2}\beta_a a^4 \sum_{i,j} \mathcal{G}_{ij} \omega_i \omega_j\right] = \text{tr}^\psi \left[ \exp\left[\frac{1}{2}\beta_a \sum_{i,j} (\mathcal{G}^{-1})_{ij} \psi_i \psi_j - \bar{\beta} \sum_i \psi_i \omega_i\right] \right], \quad (5.50)$$

where

$$\text{tr}^\psi \equiv \mathcal{N}^{-1} \prod_i \int_{-\infty}^{\infty} d\psi_i,$$

$$\mathcal{N} = \det(2\pi\beta_a^{-1}\mathcal{G})^{1/2},$$

and  $\bar{\beta} = \beta_a a^2$  as before. We have incorporated a rescaling of the temperature with lattice spacing. Equation (5.50) is valid for  $\beta_a < 0$ . For  $\beta_a > 0$ ,  $i\tilde{\psi}_i$  must be used in place of  $\psi_i$  in the exponent. By construction

$$\frac{\delta \mathcal{F}[\psi]}{\delta \psi(\mathbf{r})} = 0 = -\nabla^2 \psi_0(\mathbf{r}) + \frac{1}{\mathcal{G}_0 V} \int d^2 \mathbf{r}' \psi_0(\mathbf{r}') - \frac{\int_{-\infty}^{\infty} \sigma d\sigma \exp\{-\beta[\sigma(\psi_0(\mathbf{r})-h(\mathbf{r}))-\mu(\sigma)]\}}{\int_{-\infty}^{\infty} d\sigma \exp\{-\bar{\beta}[\sigma(\psi_0(\mathbf{r})-h(\mathbf{r}))-\mu(\sigma)]\}} \quad (5.56)$$

and the averaged vorticity field is then given by

$$\omega_0(\mathbf{r}) \equiv \langle \omega(\mathbf{r}) \rangle = -\frac{\delta \mathcal{F}[\psi]}{\delta h(\mathbf{r})} = -\nabla^2 \psi_0(\mathbf{r}) + \frac{1}{\mathcal{G}_0} V \int d^2 \mathbf{r}' \psi_0(\mathbf{r}'). \quad (5.57)$$

Equation (5.57) implies that  $\psi_0(\mathbf{r}) = \int d^2 \mathbf{r}' \mathcal{G}(\mathbf{r}, \mathbf{r}') \omega_0(\mathbf{r}')$ , which yields

$$\frac{1}{\mathcal{G}_0} V \int d^2 \mathbf{r}' \psi_0(\mathbf{r}') = \Omega_1 / V.$$

But  $\Omega_1 = \int d^2 \mathbf{r}' \omega_0(\mathbf{r}')$  vanishes whenever the  $1/\mathcal{G}_0$  term is present as we remarked following (5.49). Evidently, we may discard the  $1/\mathcal{G}_0$  term. Equations (5.56) and (5.57) reduce to the quoted result (5.7), complete with external field  $h(\mathbf{r})$ . Finally, the constraint equation is

$$(\mathcal{G}^{-1})_{ij} = -\sum_{NN\delta} (\delta_{i+\delta,j} - \delta_{ij}) + (\mathcal{G}_0 N)^{-1} \quad (5.51)$$

where the constant term is present for periodic boundary conditions (we shall see shortly that it is canceled).

Now define the Laplace transform

$$\exp[\bar{\beta} W(\tau)] = \int_{-\infty}^{\infty} d\sigma \exp\{-\bar{\beta}[\sigma\tau - \mu(\sigma)]\}, \quad (5.52)$$

in which the variable  $\tau$  may be complex. The partition function may be written

$$Z = \text{tr}^\psi \exp\left\{\frac{1}{2}\beta_a \sum_{i,j} (\mathcal{G}^{-1})_{ij} \psi_i \psi_j + \bar{\beta} \sum_i W[\psi_i - h_i]\right\}. \quad (5.53)$$

In the continuum limit we have

$$\frac{1}{a^2} (\mathcal{G}^{-1})_{ij} \rightarrow -\nabla^2 \delta(\mathbf{r}-\mathbf{r}') + (\mathcal{G}_0 V)^{-1},$$

yielding

$$Z = \text{tr}^\psi \exp(-\beta_a V \mathcal{F}[\psi]), \quad (5.54)$$

where

$$\mathcal{F}[\psi] = -\frac{1}{V} \int d^2 \mathbf{r} \left\{ \frac{1}{2} \psi(\mathbf{r}) (-\nabla^2) \psi(\mathbf{r}) + W[\psi(\mathbf{r}) - h(\mathbf{r})] \right\} - \frac{1}{2\mathcal{G}_0} \left[ \frac{1}{V} \int d^2 \mathbf{r} \psi(\mathbf{r}) \right]^2. \quad (5.55)$$

$\bar{\beta} = \beta_a a^2$  has again emerged via (5.52) as the natural temperature variable. Mean-field theory results as the coefficient  $\beta_a V = \bar{\beta} N$  in (5.54) diverges as  $a \rightarrow 0$ . The saddle point of (5.55) determines the free energy, here a maximum, since  $\bar{\beta} < 0$ . For  $\bar{\beta} > 0$  we replace  $\psi(\mathbf{r})$  by  $i\tilde{\psi}(\mathbf{r})$  and seek the minimum; however, as is common in steepest-descent calculations, we shall always find the saddle point to be at *real* values of  $i\tilde{\psi}(\mathbf{r})$ . We shall write the free energy in the form (5.55) and remember to seek the appropriate extremum depending on the sign of  $\bar{\beta}$ .

We proceed to derive the mean-field equations from (5.55). The extremum  $\psi_0(\mathbf{r})$  is given by the equation

$$g(\sigma) = -\frac{\delta \mathcal{F}[\psi]}{\delta \mu(\sigma)} = \frac{1}{V} \int d^2 \mathbf{r} \frac{\exp\{-\bar{\beta}[\sigma(\psi_0(\mathbf{r}) - h(\mathbf{r})) - \mu(\sigma)]\}}{\int_{-\infty}^{\infty} d\sigma' \exp\{-\bar{\beta}[\sigma'(\psi_0(\mathbf{r}) - h(\mathbf{r})) - \mu(\sigma')]\}}, \quad (5.58)$$

which is (5.8). An equation for angular-momentum conservation can be similarly obtained by differentiating the  $\alpha$  dependence of  $h(\mathbf{r})$  [see (5.3) and the discussion following (5.47)]

$$\langle L \rangle = -\frac{1}{2} \int d^2 \mathbf{r} r^2 \omega_0(\mathbf{r}) + R_0^2 \Omega_1 + (R_0^2 - R_i^2) \Gamma_i. \quad (5.59)$$

This derivation of the mean-field equations, although requiring the introduction of the  $a$  lattice, made no use of the  $l$  lattice. The results of both derivations coincide, verifying the irrelevance of interactions between vortices below the microscopic scale  $l$ .

It is amusing to see the averaged flow potential  $\psi_0(\mathbf{r})$  emerge as the Kac-Hubbard transformed vorticity variable. The origin of  $\psi_0(\mathbf{r})$  lies in the form of the energy, which may be written  $\frac{1}{2} \int d^2 \mathbf{r} \omega(\mathbf{r}) \psi(\mathbf{r})$ , similar in form to the coupling term in (5.50), and in the relation  $\omega = -\nabla^2 \psi$  which minimizes the exponent in (5.50) for given  $\psi$ : mean-field theory enforces this relation *on average*. It is also interesting to see, from the first derivation of the mean-field equations, the interpretation of the integrand of the spatial integral in (5.58) as the microscale vorticity distribution function  $n_0(\mathbf{r}, \sigma)$  [see Eqs. (5.38) and (5.6)].

#### D. Dressed-vorticity corollary

Equations (5.6) to (5.8) are a complete solution to the equilibrium states of a Euler flow *given* the distribution function  $g(\sigma)$ . A key observation is that, except for  $\Omega_1$ , the vorticity-conservation laws are all *violated* on the macroscopic scale. If we introduce the *dressed* distribution function from the mean-field solution,

$$g_d(\sigma) = \frac{1}{V} \int d^2 \mathbf{r} \delta(\sigma - \omega_0(\mathbf{r})), \quad (5.60)$$

then, except under very special circumstances, we will find that  $g_d(\sigma) \neq g(\sigma)$ . Since  $g_d(\sigma)$  will be the distribution function observed on *any finite* length scale, we see that it is (experimentally) *impossible* to infer  $g(\sigma)$  from the equilibrium state alone. At first glance, this loss of microscopic information would apparently make a theoretical prediction impossible in the absence of any knowledge of the initial conditions: information about initial conditions is absent for geophysical flows like the ones on Jupiter.

Fortunately, knowing only  $g_d(\sigma)$ , we may make some *partial* predictions based on the following *dressed-vorticity corollary* [9]: The averaged vorticity field  $\omega_0(\mathbf{r})$  is the *maximum energy solution* (corresponding to  $\bar{T} \rightarrow 0^-$  or  $\bar{\beta} \rightarrow -\infty$ ) of the mean-field equations with constraint function  $g_d(\sigma)$ . For the maximum energy solution,  $g_d(\sigma) = g(\sigma)$ : The constraint function is unrenormalized.

To verify this claim, we consider again Eqs. (5.6)–(5.8), now with the constraint function  $g_d(\sigma)$ . As  $\bar{\beta} \rightarrow -\infty$ , we see that for each  $\mathbf{r}$ ,  $n_0(\mathbf{r}, \sigma)$  becomes peaked in  $\sigma$  around the maximum of the exponent  $\sigma[\psi(\mathbf{r}) - h(\mathbf{r})] - \mu(\sigma)$ . That is, (5.6) becomes

$$\lim_{\bar{\beta} \rightarrow -\infty} n_0(\mathbf{r}, \sigma) = \delta(\sigma - \omega_\infty(\mathbf{r})) \equiv n_\infty(\mathbf{r}, \sigma), \quad (5.61)$$

where  $\omega_\infty(\mathbf{r})$  satisfies

$$\frac{d\mu}{d\sigma}[\omega_\infty(\mathbf{r})] = \psi_\infty(\mathbf{r}) - h(\mathbf{r}) \quad (5.62)$$

(for  $\bar{\beta} \rightarrow +\infty$  we would look for the *minimum* rather than the maximum) which must be solved with the usual consistency equations. Equation (5.61) shows the equality of the bare and dressed distribution functions for  $\bar{\beta} \rightarrow -\infty$ . Since  $g_d$  is the bare distribution function for  $n_\infty$  we have

$$g_d(\sigma) \equiv \frac{1}{V} \int d^2 \mathbf{r} n_\infty(\mathbf{r}, \sigma). \quad (5.63)$$

Thus from (5.60), (5.61), and (5.63) we have

$$\frac{1}{V} \int d^2 \mathbf{r} \delta(\sigma - \omega_\infty(\mathbf{r})) = \frac{1}{V} \int d^2 \mathbf{r} \delta(\sigma - \omega_0(\mathbf{r})), \quad (5.64)$$

which establishes that  $\omega_\infty(\mathbf{r})$  has the same dressed distribution function as  $\omega_0(\mathbf{r})$ .

To prove that  $\omega_\infty(\mathbf{r}) = \omega_0(\mathbf{r})$  we argue as follows. It is easy to see that  $\omega_\infty(\mathbf{r})$  is the result of maximizing the energy (5.48) alone, subject to the constraint  $g_d(\sigma)$ , while  $\omega_0(\mathbf{r})$  is the result of maximizing the sum of the entropy and energy (5.32) or (5.33), with constraint  $g(\sigma)$ . We claim that  $\omega_0(\mathbf{r})$  yields the same energy as  $\omega_\infty(\mathbf{r})$ ; that is, the maximum possible energy subject to the coarse-grained constraint  $g_d(\sigma)$ . Suppose  $\omega_0(\mathbf{r})$  were to yield a *smaller* energy. Since  $\omega_0(\mathbf{r})$  and  $\omega_\infty(\mathbf{r})$  have the same dressed distribution function  $g_d(\sigma)$ , they are related by an area-preserving diffeomorphism: there exists some function  $\mathbf{M}: V \rightarrow V$  with unit Jacobian such that  $\omega_\infty(\mathbf{r}) = \omega_0(\mathbf{M}(\mathbf{r}))$ . We define  $\bar{n}_0(\mathbf{r}, \sigma) = n_0(\mathbf{M}(\mathbf{r}), \sigma)$ , which yields the same entropy and satisfies the same constraints (5.27) as  $n_0(\mathbf{r}, \sigma)$ , but has first moment  $\int d\sigma \sigma \bar{n}_0(\mathbf{r}, \sigma) = \omega_\infty(\mathbf{r})$ . Intuitively, we view  $\mathbf{M}(\mathbf{r})$  as a reshuffling of the  $l$  cells in the argument leading to (5.25), in which we keep the  $a$ -cell microstructure within each  $l$ -cell fixed. But since  $\omega_\infty(\mathbf{r})$  has a higher energy than  $\omega_0(\mathbf{r})$ , it is clear that  $\bar{n}_0(\mathbf{r}, \sigma)$  has a higher *free* energy (5.32) than  $n_0(\mathbf{r}, \sigma)$ . This deduction contradicts our assumption that  $n_0(\mathbf{r}, \sigma)$  gave the free-energy maximum. Provided  $\omega_\infty(\mathbf{r})$  is a unique energy maximum, we may conclude that  $\omega_0(\mathbf{r}) = \omega_\infty(\mathbf{r})$ , which establishes the corollary.

## VI. SIMPLE EXAMPLES AND COMPARISONS WITH PREVIOUS WORK

### A. Single charge-vortex fluid and point vortices

To illustrate the formalism we consider the case of two vorticity levels, which we take to be  $\sigma = q$  and  $\sigma = 0$  on

fractional areas  $\alpha$  and  $1-\alpha$ , respectively:

$$g(\sigma) = \alpha\delta(\sigma - q) + (1-\alpha)\delta(\sigma), \quad 0 \leq \alpha \leq 1. \quad (6.1)$$

The statistical mechanics is easily understood in terms of  $\rho_0(\mathbf{r})$ , the density of the charged species  $q$  at  $\mathbf{r}$ . Then

$$\omega_0(\mathbf{r}) = q\rho_0(\mathbf{r}) \quad \text{and} \quad \frac{1}{V} \int d^2\mathbf{r} \rho_0(\mathbf{r}) = \alpha. \quad (6.2)$$

For lattice discretization size  $a$  we must simply calculate the distribution of elementary squares with (circulation) charge  $qa^2$  in an effective potential  $\psi_0(\mathbf{r})$  determined by the coupling to the stream function [see the Hamiltonian (5.1)]. Since, as shown above, the energy is dominated by the long-length scales, whereas the entropy is dominated by the constant energy redistribution over the short length scales, this distribution reduces to the distribution of a hard-core ideal-gas in an external potential

$$\frac{\omega_0(\mathbf{r})}{q} = \rho_0(\mathbf{r}) = \frac{1}{1 + e^{\bar{\beta}[q\psi_0(\mathbf{r}) - \mu_q]}}, \quad (6.3)$$

with  $\bar{\beta} = \beta/a^2$  as before and  $\mu_q$  the relative chemical potential related to the general  $\mu(\sigma)$  via

$$\exp[\bar{\beta}\mu(\sigma)] = \exp(\bar{\beta}\mu_q)\delta(\sigma - q) + \delta(\sigma). \quad (6.4)$$

$\bar{\beta}$  and  $\mu_q$  are fixed by the total energy and the normalization condition (6.2). Thus in this simple two-level situation we solve the self-consistency condition

$$\nabla^2\psi_0 = \frac{-q}{1 + e^{\bar{\beta}[q\psi_0(\mathbf{r}) - \mu_q]}} \quad (6.5)$$

and the vorticity source density is given by the Fermi function. (We are again ignoring the conservation of angular momentum and external potentials for simplicity.)

We may simplify further to the special case of the *point-vortex limit* which we *define* as the limit in which the fractional area  $\alpha \rightarrow 0$  but  $q \rightarrow \infty$  so that the total charge (circulation)  $\alpha q$  remains fixed. In this limit where the density  $\rho$  of vortices becomes small everywhere, the exclusion effect of the hard core becomes unimportant and the 1 in the denominator of the right-hand side of (6.5) becomes negligible. This equation becomes

$$\omega_0(\mathbf{r}) = -\nabla^2\psi_0 = Ae^{-\bar{\beta}\psi_0(\mathbf{r})}, \quad (6.6)$$

with  $A$  a constant (into which we have absorbed the  $\mu_q$  term) to be determined by the integral condition

$$\frac{1}{V} \int d^2\mathbf{r} \omega_0(\mathbf{r}) = Q = \alpha q, \quad (6.7)$$

hence

$$A = Q \int d^2\mathbf{r} \frac{1}{V} \exp[-\bar{\beta}\psi_0(\mathbf{r})],$$

and where  $\hat{T} = \bar{\beta}q$  is the inverse of new rescaled temperature  $\hat{T} = \bar{T}/q$  that should remain fixed as  $q \rightarrow \infty$  in order that a well-defined limiting profile  $\omega_0(\mathbf{r})$  and energy exist. This requires that  $\bar{T} = \hat{T}Q\alpha^{-1}$  diverge as  $\alpha \rightarrow 0$ . The reason is that point vortices tend to collapse unless the temperature is very high: a hard core no longer keeps

them apart. Note that we have taken the limit  $a \rightarrow 0$  before the limit  $\alpha \rightarrow 0$ , so that the mean-field equations are valid at every step. Equation (6.6) has been written down many times as the mean-field equation for point vortices. Our derivation justifies the result as a particular limiting procedure. We reemphasize that as a description of the continuum fluid the point-vortex equation is only good in the restricted regime of small enough total circulation *and* large enough  $|T|$  (low enough energy) that  $\rho_0(\mathbf{r})$  is everywhere small compared with unity.

It is interesting to consider the phase diagram for this single-charge-species case in the  $\bar{\beta}-q$  plane for a *fixed* total vorticity  $\pi Q$ . In Fig. 1 we show results for  $Q=0.1$  for a disc geometry of radius unity. (For simplicity we do

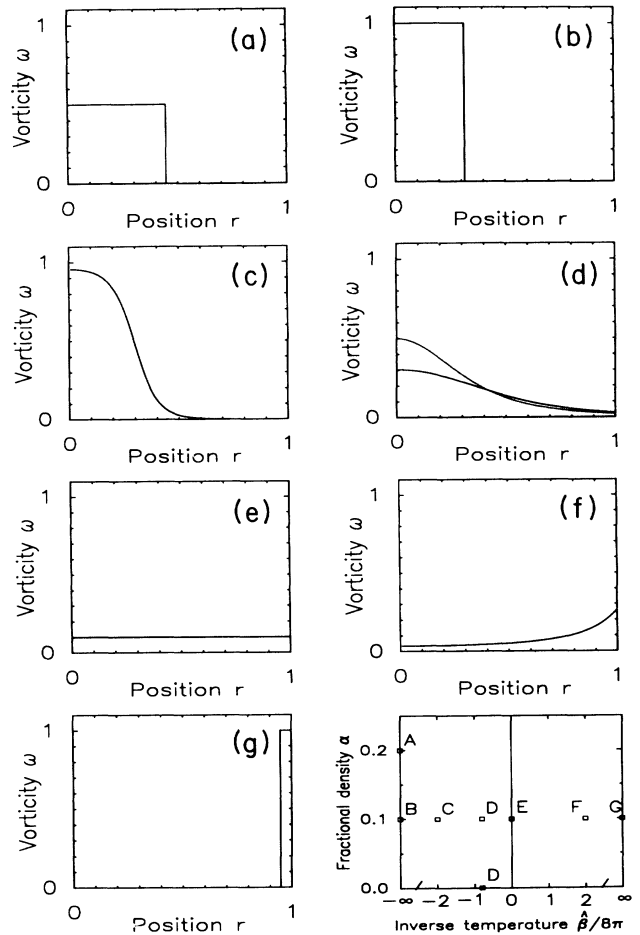


FIG. 1. Radial profiles in the single-charge case on the unit disk:  $g(\sigma) = (1-\alpha)\delta(\sigma) + \alpha\delta(\sigma - 1)$ , obtained numerically with various  $\alpha$  and scaled inverse temperature  $\bar{\beta}$ , as shown. The overall charge  $Q = \pi/10$  is fixed. (a) and (b) show the maximum-energy solution ( $\hat{T} \rightarrow 0^-$ ) for two different  $\alpha$ ; (c) shows an intermediate negative-temperature solution; (d) compares point vortices, Eq. (6.8) (dotted curve), and bounded vorticity at a negative temperature slightly above point-vortex collapse ( $\hat{\beta}_c/8\pi = -1$ ); (e) shows the structureless infinite-temperature solution; (f) shows an intermediate positive temperature solution; (g) shows the ground-state (minimum-energy) solution ( $\hat{T} \rightarrow 0^+$ ).

not impose an angular-momentum constraint, and have studied only axisymmetric solutions. The problem with fixed angular momentum may well involve nonaxisymmetric solutions [7], and it would be interesting to compare point-vortex and hard-core results in this case, too.) Axisymmetric solutions of (6.6) for point vortices may be obtained analytically (Ref. [57])

$$\omega_0(\mathbf{r}) = q\rho_0(\mathbf{r}) = Q \frac{(1+B)}{(1+Br^2)^2} \quad (6.8)$$

with

$$B = -\frac{\hat{\beta}}{8\pi + \hat{\beta}}.$$

Notice that the solution is defined only for  $\hat{\beta} > -8\pi$ : At  $\hat{\beta} = -8\pi$  there is a collapse to a point and for smaller  $\hat{\beta}$  the solution is not defined for point vortices. Clearly for  $\hat{\beta} \leq -8\pi$  and  $\alpha \neq 0$  the hard cores will be very important in determining the solution and the point-vortex model will not give a good approximation to Euler's equation.

Other "point-vortex" limits have been discussed in the literature. We might, for example, consider a limit with both  $a$  and  $\alpha$  vanishing simultaneously, in which case we have the appropriately rescaled temperature

$$\hat{T} = T/a^2q. \quad (6.9)$$

We might envision maintaining  $a^2q$ , the total vorticity per  $a$  cell, fixed as  $a \rightarrow 0$ , yielding a finite number  $N = Q/a^2q$  of point vortices in this limit, with  $\hat{T} \propto T$ . This scaling was treated rigorously by Fröhlich and Ruelle [46], in the limit  $V \rightarrow \infty$ , with  $n = N/V$  fixed. They obtain a fluctuating thermodynamics with positive-temperature equilibria only. It thus emerges that in order for mean-field theory to hold, we must allow  $a^2q \rightarrow 0$  and  $N \rightarrow \infty$  at fixed  $\hat{T}$ , yielding an infinite density of vortices, each with infinitesimal charge. We then obtain the "point-charge" limit described by (6.6).

### B. Doubly charged fluid

A slightly more complicated situation is with two finite charges (i.e., three vorticity levels  $\sigma = 0, q_1, q_2$ ). Then Eqs. (5.6)–(5.8) become

$$-\nabla^2\psi_0(\mathbf{r}) = q_1\rho_1(\mathbf{r}) + q_2\rho_2(\mathbf{r}) \quad (6.10)$$

with

$$\rho_i(\mathbf{r}) = \frac{\exp\{-\bar{\beta}[q_i\psi_0(\mathbf{r}) - \mu_i]\}}{1 + \sum_{i=1}^2 \exp\{-\bar{\beta}[q_i\psi_0(\mathbf{r}) - \mu_i]\}} \quad (6.11)$$

(the generalization to an arbitrary number  $n$  of charge species is now obvious). Here there are Lagrange multipliers  $\bar{\beta}, \mu_1, \mu_2$  fixed by demanding the correct energy and charge fractions

$$\frac{1}{V} \int d^2r \rho_i(\mathbf{r}) = \alpha_i. \quad (6.12)$$

This three-vorticity-level equation can be used as a crude model for more interesting physical situations such as shown in Fig. 2. Initial conditions as in Fig. 2(a) have

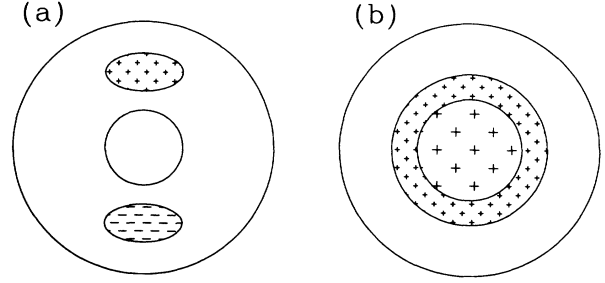


FIG. 2. Schematic plots of initial conditions using only a finite number of vorticity levels: (a) a typical three-level ( $\sigma = -1, 0$ , and  $+1$ ) initial condition used in Red Spot simulations, and (b) a three-level ( $\sigma = 0$  and two different positive values) initial condition that might be used to model the diocotron instability in cylindrically bound guiding-center plasmas.

been used in dynamical simulations to demonstrate the formation of coherent spots in planetary atmospheres. The configuration shown in Fig. 2(b) with a nonmonotonic radial vorticity profile can be used to study the diocotron instability in one-component plasmas in strong magnetic fields [58], which, for suitable parameters, can be mapped onto the Euler equation.

### C. Models of Jupiter's Red Spot

We turn now to the Red Spot, which for our purposes coincides with the persistent spot of cyclonic vorticity in Marcus's dynamical simulations [6]. Marcus carries out his calculation on a flat annulus with rigid boundaries. The effect of planetary curvature and rotation is incorporated by the  $\beta$ -plane approximation [14]. On the  $\beta$  plane we may define a *potential vorticity*  $\omega_p$ ,

$$\omega_p(\mathbf{r}) = \omega(\mathbf{r}) + \beta r.$$

The Euler dynamics on the  $\beta$  plane advectively conserves the potential vorticity  $\omega_p$ , and not  $\omega$ . For our purposes, we may account for the  $\beta$  plane by adding to the Hamiltonian an external potential  $-\beta r^3$ , and by replacing  $\omega$  with  $\omega_p$  in our statistical mechanics. A very thorough discussion of the various terms contributing to the  $\beta$ -plane Hamiltonian may be found in Ref. [6], and we feel no need to reproduce them here (see also Ref. [59]). We merely remark that the rotation contributes a term with no dependence on the spatial configuration of the potential vorticity, and the numerical values for the energy given below omit this contribution.

All parameters (i.e., values for all conserved quantities) for the statistical-equilibrium calculations described below are determined by discretizing the given initial conditions on the lattice [for details see Ref. [9(b)]]. The runs described below consisted of  $3.5 \times 10^2$  Monte Carlo sweeps (see Appendix B) with an acceptance ratio of  $1/18$ , and the maximum standard deviation was  $0.026$ . Due to the broken azimuthal symmetry, it was necessary to rotate each configuration to the center-of-vorticity frame before including it in the configuration average. The center of vorticity for these parameter values turns

out to correspond to the center of the single large spot appearing in the stationary state. In the figures, the vorticity profile is drawn in gray scale. Figure 3(a) shows initial conditions corresponding to a ring of potential vorticity. In Fig. 3(b) we show the results of the statistical-equilibrium calculation and in Fig. 3(c) we show the corresponding long-time dynamical result of Marcus [6]. In Fig. 4(a) the initial conditions are made up of two spots of

potential vorticity, each with opposite sign. Figure 4(b) demonstrates that in statistical equilibrium, only the cyclonic vortex (the vortex whose potential vorticity has the same orientation as the rotation determined by the  $\beta r^3$  potential) persists as a single condensed object; the anti-cyclonic vorticity is pushed to the boundaries. Figure 4(c) shows the corresponding result of Marcus [6].

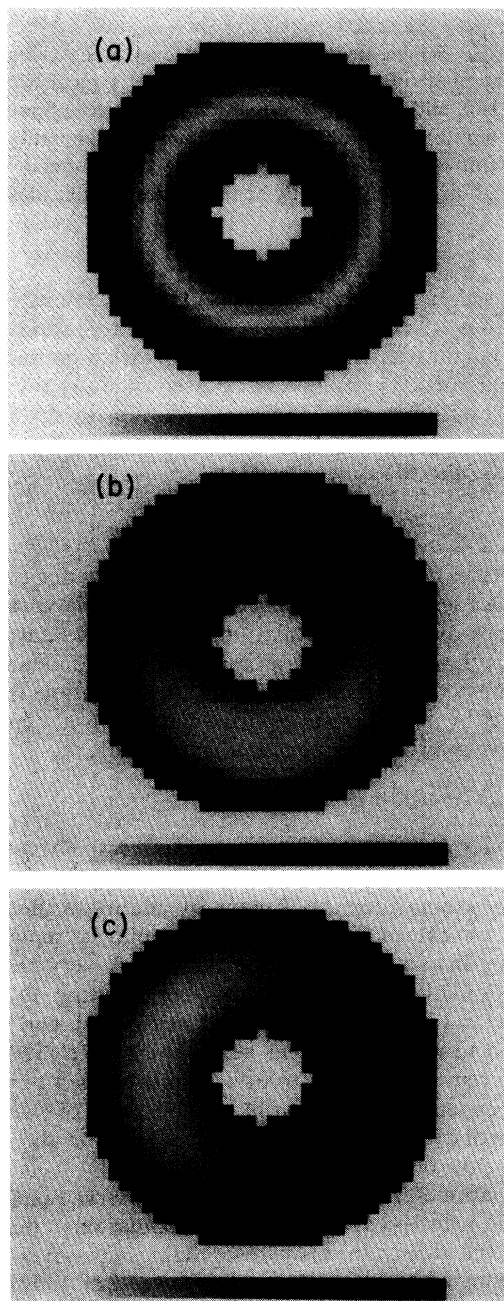


FIG. 3. Comparison of our Monte Carlo simulations with the long-time dynamics of Marcus (Ref. [6]): (a) the initial condition consisting of a ring of vorticity; (b) the Monte Carlo equilibrium state arising from (a); (c) the long-time dynamical state arising from (a). The zero of vorticity is at the midpoint of the gray scale.

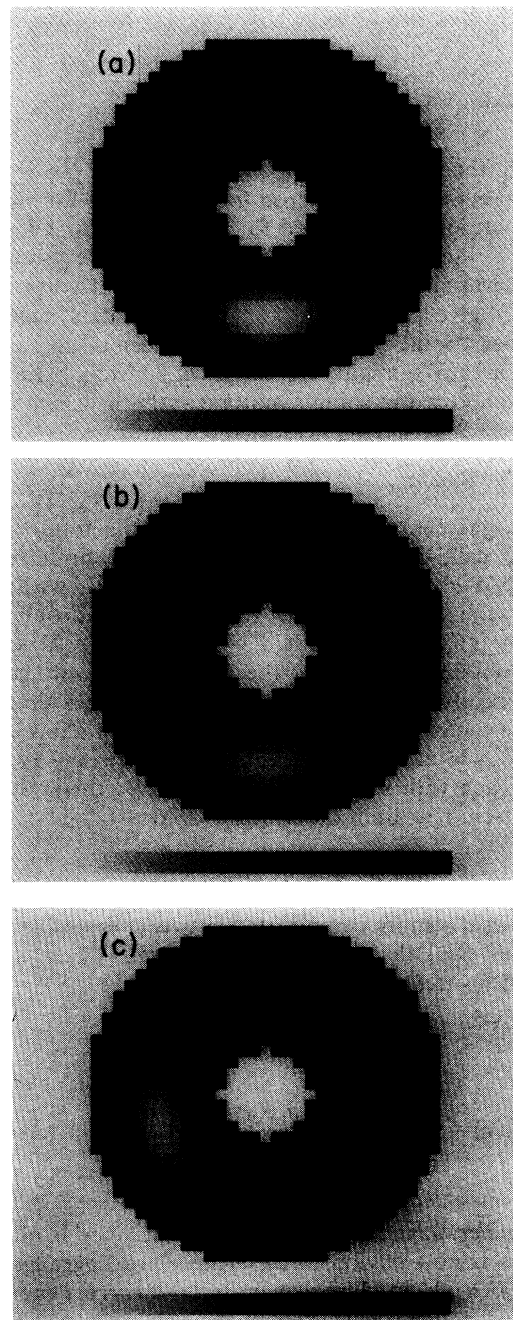


FIG. 4. Comparison of our Monte Carlo simulations with the long-time dynamics of Marcus (Ref. [6]): (a) the initial condition consisting of two oppositely charged symmetrically placed blobs; (b) the Monte Carlo equilibrium state arising from (a); (c) the long-time dynamical state arising from (a). The zero of vorticity is at the midpoint of the gray scale.

Clearly the comparisons in Figs. 3 and 4 yield strong qualitative similarities between the results of the two methods. A quantitative comparison remains to be done.

We would like to comment on the relation of the work of Smith and O'Neill [7] to our own simulations. Particularly noteworthy in their calculations was the emergence of a negative heat capacity, and differences between the results of canonical and microcanonical Monte Carlo calculations.

In contrast to the simulations described here, Smith and O'Neill confined themselves to the point-vortex limit. The situations shown above are far from that limit, and indeed in them we have encountered neither negative heat capacities nor differences between canonical and microcanonical ensembles. However, when we have attempted calculations in the point-vortex limit, we have found anomalous properties very similar to those seen by Smith and O'Neill. We have not yet studied them in sufficient detail to understand their origin, but we make the following two remarks: First, in higher dimensions, pertinent to gravitational systems, these phenomena have been observed both numerically and analytically (for a review, see Ref. [66]), and indeed play a central role in some theories of stellar systems. Whereas this kind of behavior is ruled out in most thermodynamic systems, the long-range attractive interaction makes it possible here. On the other hand, it has been argued (at least in the absence of the angular-momentum constraint) that this behavior ought to be absent in two dimensions [25]. Second, in the point-vortex limit it would seem to us uncertain whether one would expect to be able to sample the configuration space adequately by a Monte Carlo calculation without very careful attention to the phase space occupied by arbitrarily closely spaced vortices. We feel that a consistent explanation of the results of Smith and O'Neill [7] could be that they have obtained metastable states, and we believe that additional work would be necessary to rule out metastability.

#### D. Kraichnan energy-entropy theory

Another example worth discussing, because it arises so often in the literature, is the energy-entropy theory [20]. To make contact with these results we make the arbitrary choice

$$\mu(\sigma) = \frac{1}{2}\mu_2\sigma^2. \quad (6.13)$$

The integrals over  $\sigma$  in Eq. (5.52) may be done explicitly: we find (for  $\bar{\beta}\mu_2 < 0$ ),

$$W(\tau) = -\frac{\tau^2}{2\mu_2} + \frac{1}{2\bar{\beta}} \ln(-2\pi/\bar{\beta}\mu_2) \quad (6.14)$$

and hence

$$\mathcal{F}[\psi] = -\frac{1}{2V} \int d^2\mathbf{r} \left[ |\nabla\psi(\mathbf{r})|^2 - \frac{1}{\mu_2} [\psi(\mathbf{r}) - h(\mathbf{r})]^2 \right], \quad (6.15)$$

where we had dropped the trivial constant in (6.14). Let us restrict ourselves to  $h \equiv 0$  for simplicity. For  $\bar{\beta} > 0$  and

$\mu_2 < 0$ , we find  $\psi \equiv 0$  as the only consistent minimum (recall that in this case  $\psi$  ought to be replaced by  $i\psi$ ). For  $\bar{\beta} < 0$  and  $\mu_2 > 0$ , we seek the maximum, and the problem is well defined only if  $1/\mu_2 \leq \lambda_0$ , where  $\lambda_0 \approx 1/L^2$  ( $L$  being the linear system size) is the smallest eigenvalue of the negative Laplacian consistent with the boundary conditions. The mean-field equation for  $\psi_0$ ,

$$\nabla^2\psi_0(\mathbf{r}) + \frac{1}{\mu_2}\psi_0(\mathbf{r}) = 0, \quad (6.16)$$

is basically an eigenvalue problem for  $\mu_2$ , which is evidently not a free variable: It can take on only the degenerate value  $\mu_2 = 1/\lambda_0$ , and consequently  $\psi_0(\mathbf{r}) = A\psi_{\lambda_0}(\mathbf{r})$ , the normalized eigenfunction associated with  $\lambda_0$ , multiplied by an arbitrary amplitude. The order parameter is then  $\omega_0(\mathbf{r}) = (\lambda_0 A)\psi_0(\mathbf{r})$ , and the energy is  $E_0 = \lambda_0 A^2$  which is determined by the single free parameter  $A$ ,

$$\begin{aligned} \psi_0(\mathbf{r}) &= \sqrt{E_0/\lambda_0}\psi_{\lambda_0}(\mathbf{r}), \\ \omega_0(\mathbf{r}) &= \sqrt{E_0\lambda_0}\psi_{\lambda_0}(\mathbf{r}). \end{aligned} \quad (6.17)$$

For a square box of side  $L$ , with  $\psi_0$  vanishing on the boundary, we have

$$\psi_{\lambda_0} = (2/L) \sin\left[\frac{\pi x}{L}\right] \sin\left[\frac{\pi y}{L}\right], \quad 0 \leq x, y \leq L \quad (6.18)$$

with  $\lambda_0 = 2\pi^2/L^2$ . This stream function yields a bloblike shape centered on  $(L/2, L/2)$ . The constraint function  $g(\sigma)$  is given by

$$g(\sigma) = \frac{1}{V} \int d^2\mathbf{r} \frac{\exp\{\frac{1}{2}\bar{\beta}\mu_2[\sigma - \omega_0(\mathbf{r})]^2\}}{\sqrt{2\pi/(-\bar{\beta}\mu_2)}} \quad (6.19)$$

a nontrivial distribution of vorticity (observe that  $\int d\sigma g(\sigma) = 1$ , as we expect). By introducing the dressed distribution  $g_d(\sigma)$ , this expression can be rewritten in the form

$$g(\sigma) = \int_{-\infty}^{\infty} d\sigma' g_d(\sigma') \frac{\exp[\frac{1}{2}\bar{\beta}\mu_2(\sigma - \sigma')^2]}{\sqrt{2\pi/(-\bar{\beta}\mu_2)}}, \quad (6.20)$$

which may be inverted to yield

$$\begin{aligned} g_d(\sigma) &\equiv \frac{1}{V} \int d^2\mathbf{r} \delta(\sigma - \omega_0(\mathbf{r})) \\ &= \int_{-\infty}^{\infty} d\sigma' g(\sigma') \frac{\exp[-\frac{1}{2}\bar{\beta}\mu_2(\sigma - \sigma')^2]}{\sqrt{2\pi/(-\bar{\beta}\mu_2)}}. \end{aligned} \quad (6.21)$$

Note that  $g_d(\sigma)$  has compact support while  $g(\sigma)$  does not.

#### E. Lynden-Bell and Debye-Hückel theories

To conclude this section, we connect our work to that of Lynden-Bell [19] and to the Debye-Hückel theory [54] of electron systems. We must first agree to give up our interpretation of the theory as describing Euler's equa-



tion, which does not simply reduce to Coulomb form in higher dimensions  $d$ . The derivation of the mean-field equations is valid in any dimension, so long as  $\mathcal{G}(\mathbf{r}, \mathbf{r}')$  is replaced by the appropriate higher-dimensional Coulomb interaction between point vortices, and the temperature is scaled as  $\bar{\beta} = \beta_a a^d$  [see (5.30)]. The mean-field equations derived here may then be used to describe equilibria of three-dimensional Coulomb systems with continuous charge density, characterized by  $g(\sigma)$ . In particular, the point-vortex limit, described by (6.6), turns out to correspond to the Debye-Hückel theory for electrolytes [54]. In its quantum generalization, Thomas-Fermi theory, Boltzmann factors are replaced by free-electron Fermi functions [60].

The derivation is very simple. The local charge density  $n(\mathbf{r})$  is assumed to be governed by the local electric potential  $\phi(\mathbf{r})$  via the Boltzmann distribution

$$n(\mathbf{r}) = \exp\{-\beta[e\phi(\mathbf{r}) - \mu]\} \quad (6.22)$$

with the self-consistent relation

$$n(\mathbf{r}) = -\nabla^2 \phi(\mathbf{r}) \quad (6.23)$$

( $e$  denotes the electron charge). Here  $\phi(\mathbf{r})$  includes both externally applied fields, such as charged impurities, and those fields originating from the induced electron density. The chemical potential  $\mu$  is eliminated in favor of the average density

$$n_0 = \frac{1}{V} \int d^2\mathbf{r} n(\mathbf{r})$$

which yields

$$\exp(\beta\mu) = n_0 \int \frac{1}{V} \int d^2\mathbf{r} \exp[-\beta e\phi(\mathbf{r})], \quad (6.24)$$

and, upon substitution, the analog of (6.6). The validity of this self-consistent theory requires that  $\phi(\mathbf{r})$  vary slowly on the scale of the average separation between charges, and becomes exact in the limit of infinite density [compare with the discussion below (6.9)].

The theory developed by Lynden-Bell [19] is based on the collisionless Boltzmann equation for a gravitating mass distribution. He examines the distribution function  $f(\mathbf{r}, \mathbf{p}, t)$ , which denotes the probability density at time  $t$  for finding a particle at  $\mathbf{r}$  with momentum  $\mathbf{p}$ . The time evolution of  $f$  is assumed to be given by the Boltzmann equation

$$\frac{Df}{Dt} \equiv \dot{\mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{r}} + \dot{\mathbf{p}} \cdot \frac{\partial f}{\partial \mathbf{p}} + \frac{\partial f}{\partial t} = 0, \quad (6.25)$$

$$n_0(\mathbf{r}, \mathbf{p}, \sigma) = \frac{\exp(-\bar{\beta}\{\sigma[\psi(\mathbf{r}, \mathbf{p}) - h(\mathbf{r}, \mathbf{p})] - \mu(\sigma)\})}{\int_{-\infty}^{\infty} d\sigma' \exp(-\bar{\beta}\{\sigma'[\psi(\mathbf{r}, \mathbf{p}) - h(\mathbf{r}, \mathbf{p})] - \mu(\sigma')\})}, \quad (6.33)$$

which we interpret as the coarse-grained equilibrium distribution function for charge species  $\sigma$  at the coarse-grained phase space point  $(\mathbf{r}, \mathbf{p})$ . The resulting mean-field equations can be written

$$-\nabla^2 \psi_0(\mathbf{r}) = - \int d^d\mathbf{p} \int d\sigma \sigma n_0(\mathbf{r}, \mathbf{p}, \sigma) \quad (6.34)$$

where now the convective derivative is with respect to a  $2d$ -dimensional space. We have the relations

$$\begin{aligned} \dot{\mathbf{r}} &= \mathbf{p}, \\ \dot{\mathbf{p}} &= -\nabla\phi(\mathbf{r}, t), \\ \phi(\mathbf{r}, t) &= - \int d^{2d}\mathbf{r}' \mathcal{G}(\mathbf{r}, \mathbf{r}') f(\mathbf{r}', \mathbf{p}', t), \end{aligned} \quad (6.26)$$

where  $d^{2d}\mathbf{r}' \equiv d^d\mathbf{r}' d^d\mathbf{p}'$ ,  $\phi(\mathbf{r}, t)$  is the gravitational potential at  $\mathbf{r}$  due to the mass distribution  $f$ , and  $\mathcal{G}(\mathbf{r}, \mathbf{r}')$  is the appropriate  $d$ -dimensional Coulomb interaction (like charges attract at positive temperatures in gravitational systems). The total energy (kinetic plus potential) is given by

$$E = \int d^{2d}\mathbf{r} \frac{1}{2} p^2 f(\mathbf{r}, \mathbf{p}) - \frac{1}{2} \int d^{2d}\mathbf{r} d^{2d}\mathbf{r}' f(\mathbf{r}, \mathbf{p}) \mathcal{G}(\mathbf{r}, \mathbf{r}') f(\mathbf{r}', \mathbf{p}') \quad (6.27)$$

with the constraint function

$$g(\sigma) = \int d^{2d}\mathbf{r} \delta(\sigma - f(\mathbf{r}, \mathbf{p})). \quad (6.28)$$

Since  $\mathbf{p}$  is an unbounded variable  $\int g(\sigma) d\sigma$  is unbounded (the divergence is at small  $\sigma$ ); however, we do have the normalization

$$\int d\sigma \sigma g(\sigma) = \int d^{2d}\mathbf{r} f(\mathbf{r}, \mathbf{p}) = 1. \quad (6.29)$$

We are therefore led to consider the Boltzmann factor  $\exp(-\beta\mathcal{H})$  with, in an obvious shorthand,

$$\mathcal{H} = -\frac{1}{2} \int d^{2d}\mathbf{r} d^{2d}\mathbf{r}' f \mathcal{G} f - \int d^{2d}\mathbf{r} h f - \int d^{2d}\mathbf{r} \mu[f], \quad (6.30)$$

where  $h(\mathbf{p}, \mathbf{r})$  contains the  $\frac{1}{2}p^2$  term in (6.27), and any other "external fields" we might wish to add. The Kac-Hubbard-Stratanovitch transformation to the new field  $\psi(\mathbf{r}, \mathbf{p})$  is now straightforward. Since the Coulomb interaction  $\mathcal{G}$  does not depend on momentum, we use the identity (5.49) with  $\omega(\mathbf{r}) \equiv \int d^d\mathbf{p} f(\mathbf{r}, \mathbf{p})$ . With the definitions  $\bar{\beta} = \beta a^{2d}$  and

$$\exp[\bar{\beta}W(\tau)] = \int_{-\infty}^{\infty} d\sigma \exp\{-\bar{\beta}[\sigma\tau - \mu(\sigma)]\} \quad (6.31)$$

as before, the functional to be minimized becomes

$$\begin{aligned} \mathcal{F}[\psi(\mathbf{r})] &= \frac{1}{2} \int d^d\mathbf{r} |\nabla_r \psi(\mathbf{r})|^2 \\ &\quad - \frac{1}{2} \int d^{2d}\mathbf{r} W[\psi(\mathbf{r}) - h(\mathbf{r}, \mathbf{p})]. \end{aligned} \quad (6.32)$$

We may define

and the full equilibrium distribution is given by

$$f_0(\mathbf{r}, \mathbf{p}) = \int d\sigma \sigma n_0(\mathbf{r}, \mathbf{p}, \sigma) \quad (6.35)$$

with constraint function

$$g(\sigma) = \int d^{2d}\mathbf{r} n_0(\mathbf{r}, \mathbf{p}, \sigma). \quad (6.36)$$

These expressions correspond to the mean-field equations of Lynden-Bell [19].

## VII. SUMMARY AND CONCLUSIONS

We now turn to a discussion of the physical content of the theories we have derived in the last few sections. It is instructive to start with an account of the physics of the Lynden-Bell model [19]. As we have remarked earlier, the origin of the connection between Lynden-Bell's statistical mechanics and our own is the fundamental equation of motion (6.25). Obviously, we recover the Euler equations if we throw out the explicit dependence on the momentum in (6.25) and set  $d=2$ . An additional feature common to both the Euler fluid and stellar clusters is the existence of a scalar field, the evolution of which is described by the equation of motion, and which interacts through a long-range potential. The Coulomb potential governs the interaction of the vorticity in the fluid, and the mass density in the gravitational system. In both cases, the equation of the motion says that the *covariant* derivative of the scalar field vanishes, giving rise to the infinite family of conserved quantities, or Casimirs.

The collisionless Boltzmann equation describes the motion of point masses interacting by a gravitational potential, provided we do not permit the masses to collide. This motion is Hamiltonian:  $\mathbf{r}$  and  $\mathbf{p}$  constitute canonical coordinates; the introduction of collisions destroys the Hamiltonian character unless details of the interaction of nearby particles are added. Collisions, by permitting the merger of particles and inelastic scattering, further introduce source and sink terms into the Boltzmann equation, giving rise to a covariant derivative that no longer vanishes: the evolution of the mass density becomes *dissipative*. The collisionless Boltzmann equation represents an idealization of the motion of stars in which we view the discrete collection as a *continuous medium*, described by a continuous mass density  $f(\mathbf{r}, \mathbf{p}, t)$ . This idealization is only consistent to the extent that we may neglect collisions for the time scales in which we are interested.

In fact, the collisionless regime is of considerable interest to astrophysicists [61,62]. In studying the formation of, say, an elliptical galaxy from some ancient distribution of stars, they estimate the "collision time," which sets the time scales for which we may view the dynamics as described by a collisionless Boltzmann equation. Lynden-Bell asked the question: can statistical equilibrium theory for the *collisionless* dynamics yield the observed structure of stellar clusters? That is, (1) is it possible that the dynamics determining the mass and velocity distribution in a galaxy occur within the time for which collisions between stars can be neglected, and (2) if so, then do the collisionless degrees of freedom of the system equilibrate in this process. Antonov [63] had already argued that, in a physically significant regime, an isothermal system of gravitating *point* masses within an isolating spherical shell has a negative specific heat, and collapses catastrophically. If statistical mechanics was to have any bearing on the problem, equilibration of all degrees of freedom had to be excluded.

Lynden-Bell [19] tried to answer his question by con-

structing a statistical mechanics for the collisionless Boltzmann equation, obtaining generalizations of our mean-field equations; however, a lengthy and convoluted chain of results has led many workers, including Lynden-Bell, to conclude that the simple answer is *no*. (Less simple answers also exist: See Ref. [64]). It emerges that in an infinite volume, which most people seem to believe constitutes an appropriate physical boundary condition for galactic evolution, the mean-field equations have no solution. The star cluster can always achieve greater entropy by expelling a negligible mass to large distances, and increasing its density in a hot central core. Conservation of the mass distribution  $g(\sigma)$  cannot prevent this "violent relaxation," since the density can *increase* in the spatial dimensions, accompanied by a compensating *decrease* in the velocity dimensions. Astrophysicists must then explain how features of purely dynamical origin prevent complete equilibration, since observed galaxies evolve well into the collisional regime, and have evidently not collapsed.

Within a *finite* volume, it turns out that, at least for points, one encounters negative heat capacities and collapse under what are considered appropriate physical boundary conditions. Lynden-Bell and Wood [65] have claimed that finite cores rescue the system from these pathologies. We know of no effort to determine the extent to which equilibrium statistical mechanics is useful under these circumstances, which, in any event, may not be physically significant. For a partial review, see Ref. [66].

Our reason for discussing the Lynden-Bell theory in such detail is that we wish to contrast the Euler fluid with stellar mechanics. As Katz and Lynden-Bell [25] have demonstrated (without reference to the inviscid fluid), in a finite volume, the two-dimensional, attractive, single-charge-species point-vortex gas does not display the negative specific heat that is characteristic of the stellar thermodynamics. Nor is the single-species finite-core gas subject to collapse in infinite volume, since here we have only spatial, and not velocity, degrees of freedom. In fact, in two dimensions, we see no reason why solutions to the mean-field equations should not exist for any consistent set of values of the conserved quantities.

But the differences run even deeper. In fluid mechanics, it has long been observed that *in certain physical regimes* the Euler equations, equations of nondissipative motion for a continuous medium, yield a (perhaps surprisingly) correct description of the long-time behavior of real, viscous fluids. Marcus [6], for example, has successfully modeled a variety of nontrivial long-time laboratory flows [67,68] by means of *inviscid* dynamics. Onsager [1] proposed that statistical mechanics of inviscid flow could explain *turbulent* phenomena. Aref and Siggia [33] show that *inviscid* statistical mechanics models the turbulent shear layer.

In the case of fluids, it seems that the collisionless approximation can be extraordinarily effective, for reasons that we do not entirely understand. However, we should like to propose informally a mechanism that explains how inviscid statistical mechanics could usefully describe the long-time behavior of certain stationary *viscous* flows.

Our notions rely on a *separation of time scales*. Without wishing to assign any blame, we attribute some part of our thinking to Aref and Siggia [33].

We emphasize first that our statistical mechanics is meant only to be a *macroscopic* description of fluid properties. That is, we have nothing to say about correlations on scales small compared with the *finite* system size. We do not concede, for example, any but the most qualitative connection between the equilibrium properties that we hope to predict, and *turbulence*. The turbulent cascade, which we expect to dominate the small scales, is a stationary state which spans a set of equilibrium range wave numbers, forced at the low end, dissipated at the high end, and there is simply no reason to expect the turbulent scales to be in detailed balance. In an infinite volume, we would expect any forcing and dissipation to lead to the Kolomogorov cascade to large wave number and the inverse cascade to small wave number, as suggested by Kraichnan [3], or to the trivial long-wavelength properties suggested by Forster, Nelson, and Stephen [69].

We maintain that the finite size of the fluid container introduces a new scale into the problem. It allows us to envision a situation in which wave numbers of the order of inverse system size are *in equilibrium* with each other, while much larger wave numbers are out of equilibrium, their behavior being dominated by the energy cascades. Whereas small-scale correlations would be determined by properties of the forcing, correlations on larger scales might not be affected by small-scale forcing. Central to this picture is the notion that the time scale for equilibration of long-wavelength modes be small compared with the characteristic time scale for the forcing and dissipation of these modes by communication with the short wavelengths. The self-consistency of our picture at long wavelengths is supported by the dressed-vorticity corollary, which shows that, in equilibrium, we may ignore short-wavelength properties and focus on averages of the vorticity field over some scale determined by the system size. Any localized turbulence or viscosity-mediated diffusion that acts to smear the vorticity on this scale has no effect on the macroscopic flow.

A small-scale forcing of the kind we envision here may be a feature of some geophysical flows, where small-scale atmospheric storms drive the large-scale dynamics. In addition, it may be appropriate for certain laboratory flows, such as Swinney's spots [67,70].

As the above remarks no doubt convey, we do not have an adequate understanding of the applicability of a statistical-mechanical theory such as ours to fluid flows in general. In particular, the entire theory is based upon a presumption of ergodicity. We expect that there are many situations, such as steady laminar flow, for which this presumption is most certainly wrong. On the other hand, even small amounts of turbulence or externally induced noise could serve to open up all significant parts of phase space, so that flow becomes, effectively, ergodic. For example, Marcus [6] finds that an axisymmetric ring of vorticity is a steady flow in his dynamical simulation; however, when subject to an asymmetric perturbation that is ten orders of magnitude lower in relative vorticity, the flow destabilizes and evolves to a single blob. In oth-

er simulations, he finds that steady flows consisting of a number of separated blobs can be further mixed if he throws in filaments of vorticity, causing the blobs to merge and shed vorticity. Presumably, noise introduced by small-scale turbulence or other processes could play an analogous role in paring down the class of stable flows.

We have emphasized earlier that a number of authors have numerically demonstrated ergodic behavior for point vortices. An intrinsic property of point-vortex methods is that they preserve their (singular) values for the area integrals exactly. Their behavior may suggest that for a continuous vorticity field, a simulation faithful to the ideal Euler fluid would also be ergodic. On the other hand, it could be that the smoothing effects of dissipation stabilize flows that are not true long-time stationary states of the Euler equations. We have seen via the dressed-vorticity corollary that when we take the coarse-grained average of a finite-temperature equilibrium flow, we obtain a zero-temperature flow. Thus, viscosity might act locally to reduce the temperature, creating by its local smoothing effect stationary flows that are not in global equilibrium. Finally, Marcus [6] draws a distinction between "filamentous" and "nonfilamentous" flows, defined by the presence or absence of small-scale filaments of vorticity. He finds that nonfilamentous flows remember their initial conditions, in contrast to filamentous flows, in which only conserved quantities seem to matter. Perhaps any truly dissipationless flow ought to be filamentous and consequently ergodic, since viscosity would not be present to remove the small-scale filamentation. We would hope that our remarks apply also to high-Reynolds-number flows, where filamentation could occur on many more length scales than can be accommodated in any present-day numerical simulation. (These small-scale filaments should be distinguished from the microscopic filamentation discussed below.)

The objection might be raised that, even in the truly inviscid limit, the fluid never achieves a stationary state. For any finite time, the fluid is still evolving at some nonzero length scale that vanishes only at infinite time. In essence the turbulent equilibrium range steadily shifts to higher and higher wave numbers. Consequently, the fluid motion cannot be ergodic. We believe that this objection is of a formal nature. We would characterize the nonstationary behavior in a different manner: at any finite time, the fluid is still relaxing to an asymptotic equilibrium that will have structure at all length scales, i.e., the asymptotic state that maximizes the entropy and represents the long-time vorticity configuration consistent with ergodicity. Thus we would hope that as the equilibrium range moves to higher wave numbers it leaves behind an equilibrium state at smaller wave numbers which closely approximates the predictions of statistical mechanics.

The consistency of this characterization depends upon the irrelevance of short length-scale structures to the macroscopic vorticity configuration at long times. We would argue that Marcus [6] has observed just this irrelevance in his dynamical simulations. By careful control of the short-wavelength cutoff, he demonstrates that the long-time macroscopic vorticity configuration is

achieved *before* the enstrophy has reached the smallest scales of his numerics. If he increases his numerical resolution, Marcus obtains the same long-wavelength vorticity configuration, even though the simulation may now be continued to longer times in which structure at still smaller scales has emerged.

The long-range potential governing the interaction of vorticity is an additional factor suggesting that small-scale structures ought not to affect the macroscopic dynamics of a fluid near macroscopic equilibrium. Because of the long range, we may calculate the dynamics contributed by a small-scale structure to other structures at long distances by using the *average* of the vorticity over some small area. A similar feature emerges in the statistical mechanics: in equilibrium it is only the average of the vorticity field over small length scales that affects the macroscopic structures. Indeed, this is precisely what leads to the dressed-vorticity corollary.

These remarks also pertain to the divergence of integrals of the gradient of the vorticity in our statistical equilibria. In any statistical-mechanical system, one can define quantities that are finite for some class of initial conditions, but that diverge in statistical equilibrium. The defect lies not in statistical mechanics, but rather in the choice of quantities at which to look. For the Euler fluid, we would argue that the diverging integral vorticity gradients, which may be viewed as reflecting the width of the boundary between regions on which the microscopic vorticity field takes different values, are completely irrelevant to the macroscopic long-time flow. Our statistical formulation is consistent with many possible microscopic shapes for the vorticity field. It provides no useful information about them, and in turn, they do not matter to the macroscopic equilibrium at all.

Further difficulties arise in trying to test the theory by current numerical methods. Dynamical simulation of inviscid fluids is by no means a well-understood subject. In fact, appreciation of the constraining properties of the conservation laws has been sorely missing in this field. Even the very best studies seem to assume that the *macroscopic* enstrophy ought to be conserved at long times, whereas we have argued that it should *not* be conserved. Nor has anyone checked his or her simulation to ensure that he or she has controlled the vorticity distribution at the *microscopic* level. For example, let us suppose our initial conditions involve both positive and negative vorticity, and that the predicted equilibrium based on the microscopic vorticity distribution consists of spatially separated regions of positive and negative vorticity. We expect that, in evolving from the initial to long-time state, regions of positive and negative vorticity might be at times well mixed, with some of the vorticity again separating out asymptotically. Yet because of finite resolution, the numerical method could easily *cancel* the positive with negative vorticity in the well-mixed intermediate stages, yielding a long-time vorticity distribution *consistent* with the initial conditions, but incorrect nevertheless. We would be unable to distinguish the failure of statistical mechanics from the failure of the simulation. (Notice that point-vortex methods are not subject to this objection, since the methods explicitly conserve particle

number and charge). In spite of these realities, our recent preliminary comparison of predicted equilibrium flows with Marcus' dynamical simulations yields remarkable agreement (see Appendix B).

Certainly, possibilities exist for a numerical method that correctly incorporates the conservation laws. String theorists have uncovered a truncated dynamics that seems to converge, as the number of Fourier modes increases, to Euler dynamics [71,72] (see Appendix A). As the number of modes increases, so does the number of conservation laws. (This literature was related to us by A. Rouhi, who independently discovered the truncated dynamics.) Efforts are underway to design a computational method based on the truncated dynamics [73]. Perhaps this method will eventually confirm our equilibrium predictions.

We close by pointing out that there exist a variety of dynamical systems sharing the essential features of the Euler fluid that allowed us to construct a statistical mechanics. We refer the reader to Ref. [13] for a partial listing of systems with an infinite family of Casimir invariants. One particularly interesting example might be the "meteorological primitive equations," describing three-dimensional rotating, stratified, compressible flow of an ideal gas [74].

*Note added.* After this work was completed, we learned of parallel results using maximum entropy reasoning by R. Robert and J. Sommeria, *J. Fluid Mech.* **229**, 291 (1991). See also J. Sommeria, C. Staquet, and R. Robert (unpublished).

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#### APPENDIX A: $SU(n \rightarrow \infty)$ AND EULER'S EQUATION

A natural question to ask is whether or not there exists a *finite* dimensional (i.e., lattice) approximation to Euler's equation with a finite number of conserved quantities. That is, can one define a set of lattice degrees of freedom whose dynamics conserve some large, but finite, number of quantities, and which, in some sense, converge to the continuum Euler equation as the lattice size vanishes?

The answer to the question is yes [75], and the derivation is based on the Poisson-bracket formulation of Euler's equation. As usual the vorticity field plays a fundamental role. We begin with the continuum formulation. As usual the Hamiltonian is given by

$$H = \frac{1}{2} \int d^2r \int d^2r' \omega(\mathbf{r}) \omega(\mathbf{r}') \mathcal{G}(\mathbf{r}, \mathbf{r}'), \quad (\text{A1})$$

where  $\mathcal{G}$  is the appropriate Green function for the two-dimensional Laplacian. For any functionals  $F[\omega], G[\omega]$  we define the Poisson bracket [reproduced from Eq. (4.2)]

$$\{F[\omega], G[\omega]\} \equiv \int d^2r \omega(\mathbf{r}) \left\{ \frac{\delta F}{\delta \omega(\mathbf{r})}, \frac{\delta G}{\delta \omega(\mathbf{r})} \right\}_{xy}, \quad (\text{A2})$$

where  $\{, \}_{xy}$  is the usual Poisson bracket

$$\{f(\mathbf{r}), g(\mathbf{r})\}_{xy} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}, \quad \text{where } \mathbf{r} \equiv (x, y). \quad (\text{A3})$$

The time development of any quantity  $F$  is then given by

$$\frac{\partial F}{\partial t} = \{F, H\}, \quad (\text{A4})$$

and one indeed finds

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= -(\mathbf{v} \cdot \nabla) \omega, \\ \omega &= \nabla \times \mathbf{v}. \end{aligned} \quad (\text{A5})$$

Crucial properties of any Poisson bracket are (i) Distributive law

$$\{FG, H\} = F\{G, H\} + \{F, H\}G, \quad (\text{A6a})$$

(ii) Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0, \quad (\text{A6b})$$

as well as the obvious properties of linearity,  $\{aF + bG, H\} = a\{F, H\} + b\{G, H\}$ , where  $a$  and  $b$  are constants; and antisymmetry,  $\{F, G\} = -\{G, F\}$ . All of these are satisfied by (A2).

Let us now assume periodic boundary conditions, and to simplify notation, we take the fluid region to be the unit square. We may then define the Fourier coefficients  $\Omega(\mathbf{n})$  via

$$\begin{aligned} \omega(\mathbf{r}) &= \sum_{\mathbf{n}} \Omega(\mathbf{n}) e^{2\pi i \mathbf{n} \cdot \mathbf{r}}, \\ \Omega(\mathbf{n}) &= \int d^2r e^{-i2\pi \mathbf{n} \cdot \mathbf{r}} \omega(\mathbf{r}), \end{aligned} \quad (\text{A7})$$

where  $\mathbf{n} = (n_x, n_y)$  is an integer vector. For periodic boundary conditions  $\mathcal{G}(\mathbf{r}, \mathbf{r}')$  is defined only for *neutral* systems: Thus  $\Omega(\mathbf{0}) \equiv 0$ . In terms of the  $\Omega(\mathbf{n})$  we have

$$H = \frac{1}{2} \sum_{\mathbf{n} (\neq \mathbf{0})} \frac{1}{|\mathbf{n}|^2} \Omega(\mathbf{n}) \Omega(-\mathbf{n}) \quad (\text{A8})$$

and

$$\begin{aligned} \frac{\partial \Omega(\mathbf{n})}{\partial t} &= \{\Omega(\mathbf{n}), H\} \\ &= 4\pi^2 \sum_{\mathbf{m} (\neq \mathbf{0})} \frac{\mathbf{n} \times \mathbf{m}}{|\mathbf{m}|^2} \Omega(\mathbf{m}) \Omega(\mathbf{n} - \mathbf{m}), \end{aligned} \quad (\text{A9})$$

where we have used (here  $\mathbf{n} \times \mathbf{m} = n_x m_y - n_y m_x$ )

$$\{\Omega(\mathbf{m}), \Omega(\mathbf{n})\} = 4\pi^2 (\mathbf{n} \times \mathbf{m}) \Omega(\mathbf{m} + \mathbf{n}). \quad (\text{A10})$$

The structure of (A10) is that of a Lie algebra. The  $\Omega(\mathbf{m})$  may be considered the generators of the full algebra via multiplication and addition. From (A10) one may then derive the Poisson bracket for any two elements generated in this manner.

We next consider the conserved quantities

$$\begin{aligned} C_l &= \int d^2r [\omega(\mathbf{r})]^l \\ &= \sum_{\mathbf{m}_1, \dots, \mathbf{m}_l} \delta_{\mathbf{m}_1 + \dots + \mathbf{m}_l, \mathbf{0}} \Omega(\mathbf{m}_1) \cdots \Omega(\mathbf{m}_l), \quad l \geq 2. \end{aligned} \quad (\text{A11})$$

One may check that

$$\{C_l, \Omega(\mathbf{n})\} = 0 \quad \forall l, \mathbf{n}, \quad (\text{A12})$$

so that the  $C_l$  commute with all elements of the algebra, in particular with  $H$ . Such quantities are known as *Casimir operators* or *symplectic invariants*. Note that the  $C_l$  will be conserved for any Hamiltonian, not just the one we use.

We now view (A10) as the fundamental set of relations, and ask whether or not there exists a finite-dimensional approximation to them, i.e., can we truncate them, keeping only a finite set of  $\mathbf{n}$ ? One cannot do this arbitrarily since (A6) must be satisfied. For example, simply keeping only those  $\Omega(\mathbf{n})$  for which  $0 \leq |n_x|, |n_y| \leq M$  does not work since (A10) will generate  $\Omega$ 's outside this range. The algebra must therefore be truncated as well. The following prescription works: In order to close the algebra we define addition modulo  $N \equiv 2M + 1$  via

$$(\mathbf{m} + \mathbf{n}) \bmod N = ((m_x + n_x) \bmod N, (m_y + n_y) \bmod N) \quad (\text{A13})$$

for  $0 \leq |m_x|, |n_x| \leq M, 0 \leq |m_y|, |n_y| \leq M$ . We may assume that the results of the mod operation lie in this same domain by subtracting  $N$  if necessary, though this is only a convenience to fix the notation. We now periodize (A10) by defining

$$\{\Omega(\mathbf{m}), \Omega(\mathbf{n})\} = 2\pi N \sin \left[ \frac{2\pi}{N} \mathbf{n} \times \mathbf{m} \right] \Omega((\mathbf{m} + \mathbf{n}) \bmod N). \quad (\text{A14})$$

Henceforth we will drop the explicit mod  $N$  in sums, the periodicity being implicitly understood. One may check that (A14) satisfies the conditions (A6). More obvious candidates for the right-hand side of (A14), such as

$$\begin{aligned} N^2 \left[ \sin \left[ \frac{2\pi n_x}{N} \right] \sin \left[ \frac{2\pi m_y}{N} \right] \right. \\ \left. - \sin \left[ \frac{2\pi n_y}{N} \right] \sin \left[ \frac{2\pi m_x}{N} \right] \right] \end{aligned}$$

fail to satisfy at least one of the two conditions (A6). Clearly for fixed  $\mathbf{n}, \mathbf{m}$  (A14) converges to (A10) as  $N \rightarrow \infty$ . Since all Lie groups were classified long ago, it is not surprising that (A14) corresponds to something well known, namely the group  $SU(N)$  (note that here  $N$  must be odd). If one represents  $SU(N)$  by unitary  $N \times N$  matrices, the Poisson bracket is precisely the usual commutator. The dimension of  $SU(N)$  is  $N^2 - 1$  and the basis  $\Omega(\mathbf{n})$  spans the group. Explicit matrix representations

for the  $\Omega(\mathbf{n})$  may be written down [76] [these are different from the usual Cartan-Weyl basis which generalize the Pauli matrices for SU(2) and the Gell-Mann matrices for SU(3)].

Now what about Casimir operators? It turns out that the following operators commute with all others [77]

$$C_l = \sum_{\mathbf{m}_1, \dots, \mathbf{m}_l} \Omega(\mathbf{m}_1) \Omega(\mathbf{m}_2) \cdots \Omega(\mathbf{m}_l) \tilde{\delta}(\mathbf{m}_1 + \cdots + \mathbf{m}_l) \times \exp \left[ \frac{2\pi i}{N} \sum_{\substack{\alpha, \beta \\ \alpha < \beta}} \mathbf{m}_\alpha \times \mathbf{m}_\beta \right], \quad l=2, \dots, N \quad (\text{A15})$$

where  $\tilde{\delta}(\mathbf{n})$  is 1 if  $\mathbf{n} \bmod N = 0$ , and is zero otherwise. The main difference between (A15) and (A11) is the presence of the exponential factor. For fixed  $\{\mathbf{m}_\alpha\}$  this factor converges to unity. However, for fixed  $N$  only for a small subset of the  $\{\mathbf{m}_\alpha\}$  values  $(1/N)\mathbf{m}_\alpha \times \mathbf{m}_\beta$  will be small (roughly speaking, only for  $|\mathbf{m}_\alpha| \lesssim \sqrt{N}$ ). The same is true in (A14). Thus the convergence of (A14) and (A15) to (A10) and (A11) is highly nonuniform, and to what extent they truly approximate Euler's equation is not clear. As mentioned above, more rapidly convergent obvious choices do not yield a consistent group structure, so one may not be able to do any better. In real space we may define

$$\omega(\mathbf{r}_j) = \sum_{\mathbf{n}} e^{i2\pi \mathbf{n} \cdot \mathbf{r}_j} \Omega(\mathbf{n}), \quad (\text{A16})$$

$$\mathbf{r}_j \equiv (j_x/N, j_y/N),$$

where  $\mathbf{j} = (j_x, j_y)$  is an integer vector with  $0 \leq |j_\alpha| \leq M$ , and correspondingly

$$\Omega(\mathbf{n}) = \frac{1}{N^2} \sum_j e^{-i2\pi \mathbf{n} \cdot \mathbf{r}_j} \omega(\mathbf{r}_j). \quad (\text{A17})$$

Equation (A15) may be Fourier transformed to yield

$$C_l = \frac{1}{N^{2l}} \sum_{\mathbf{r}_1, \dots, \mathbf{r}_l} \omega(\mathbf{r}_1) \cdots \omega(\mathbf{r}_l) f_l(\mathbf{r}_1, \dots, \mathbf{r}_l), \quad (\text{A18})$$

where, as above,  $N\mathbf{r}_\alpha$  are integer vectors, and

$$f_l(\mathbf{r}_1, \dots, \mathbf{r}_l) = N^l \exp \left[ 2\pi i N \sum_{1 \leq \alpha < \beta \leq l} (-1)^{\alpha+\beta} \mathbf{r}_\alpha \times \mathbf{r}_\beta \right] \times \begin{cases} N^{-1}, & l \text{ odd} \\ \tilde{\delta} \left[ N \sum_{\beta=1}^l (-1)^\beta \mathbf{r}_\beta \right], & l \text{ even} \end{cases} \quad (\text{A19})$$

One may check that  $f_l$  is indeed translation invariant:  $f_l(\mathbf{r}_1 + \mathbf{r}, \dots, \mathbf{r}_l + \mathbf{r}) = f_l(\mathbf{r}_1, \dots, \mathbf{r}_l)$ . The  $\delta$  function for even  $l$  is essential. The obvious question is whether or not  $f_l(\mathbf{r}_1, \dots, \mathbf{r}_l)$  converges to the continuum result  $\delta(\mathbf{r}_2 - \mathbf{r}_1) \times \cdots \times \delta(\mathbf{r}_l - \mathbf{r}_1)$ , and, if so, how. To gain intuition consider

$$f_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = N^2 e^{2\pi i N (\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1)}. \quad (\text{A20a})$$

We would like to know if the left-hand side of (A20) converges to

$$\delta(\mathbf{r}_2 - \mathbf{r}_1) \delta(\mathbf{r}_3 - \mathbf{r}_1) \quad \text{as } N \rightarrow \infty. \quad (\text{A20b})$$

Equivalently, we consider convergence of  $N^2 e^{2\pi i N \mathbf{r} \cdot \mathbf{r}'}$  to  $\delta(\mathbf{r}) \delta(\mathbf{r}')$ . Let  $\varphi(\mathbf{r}, \mathbf{r}')$  be a continuous Fourier transformable function, and look at

$$\lim_{N \rightarrow \infty} \int d^2 r \int d^2 r' \varphi(\mathbf{r}, \mathbf{r}') N^2 e^{2\pi i N \mathbf{r} \cdot \mathbf{r}'} = \lim_{N \rightarrow \infty} N^2 \int d^2 r \tilde{\varphi}(\mathbf{r}, 2\pi N \mathbf{r}), \quad (\text{A21})$$

where the tilde denotes Fourier transform in the second argument. Since  $\tilde{\varphi}(\mathbf{r}, \mathbf{q})$  is continuous in  $\mathbf{r}$ , and is assumed to decay to zero with large  $|\mathbf{q}|$ , (A21) converges to

$$\frac{1}{(2\pi)^2} \int d^2 q \tilde{\varphi}(0, q) = \varphi(0, 0) \quad (\text{A22})$$

so that, at least in this integral sense, we have the requisite  $\delta$  function. Extending this intuition to higher  $l$  is straightforward. For sufficiently smooth  $\omega(\mathbf{r})$ , then, it is clear that (A18) converges to the continuum results (A11). If  $\omega(\mathbf{r})$  loses smoothness (as we certainly see that it must, if thermodynamics holds at long times) the  $C_l$  will clearly differ greatly from their continuum counterparts. However, this is very reasonable since we know that for any *fixed-length* scale resolution, the conservation laws are indeed violated for the continuum problem.

Finally, we may write down the discrete lattice *dynamics*. One finds the equation of motion for the  $\Omega(\mathbf{n})$

$$\frac{\partial \Omega(\mathbf{n})}{\partial t} = \sum_{\mathbf{m} (\neq 0)} \frac{2\pi N}{\Delta(\mathbf{m})} \sin \left[ \frac{2\pi}{N} \mathbf{n} \times \mathbf{m} \right] \Omega(\mathbf{m}) \Omega(\mathbf{n} - \mathbf{m}), \quad (\text{A23})$$

where

$$4\pi^2 \Delta(\mathbf{m}) = 4 - 2 \cos \left[ \frac{2\pi m_x}{N} \right] - 2 \cos \left[ \frac{2\pi m_y}{N} \right]$$

is the discrete Laplacian in Fourier space, so that

$$H_N = \frac{1}{2} \sum_{\mathbf{m} (\neq 0)} \frac{1}{\Delta(\mathbf{m})} \Omega(\mathbf{m}) \Omega(-\mathbf{m}). \quad (\text{A24})$$

In real space this yields

$$\frac{\partial \omega(\mathbf{r})}{\partial t} = \frac{1}{N^4} \sum_{\mathbf{r}_1, \mathbf{r}_2} g_N(\mathbf{r}_1 - \mathbf{r}, \mathbf{r}_2 - \mathbf{r}) \omega(\mathbf{r}_1) \omega(\mathbf{r}_2), \quad (\text{A25})$$

where

$$g_N(\mathbf{r}, \mathbf{r}') = -\frac{1}{N^2} \sum_{\mathbf{r}_3} \mathcal{G}(\mathbf{r}_3, 0) 2\pi N^3 \sin[2\pi N \mathbf{r}' \times (\mathbf{r} + \mathbf{r}_3)] = 2\pi N^3 \sin(2\pi N \mathbf{r} \times \mathbf{r}') / \Delta(N \mathbf{r}'). \quad (\text{A26})$$

The continuum result is

$$g_\infty(\mathbf{r}, \mathbf{r}') = [\nabla \times \mathcal{G}(\mathbf{r}, 0)] \cdot \nabla \delta(\mathbf{r}'). \quad (\text{A27})$$

Using the result

$$-2\pi N^3 \sin(2\pi N \mathbf{r} \cdot \mathbf{r}') \xrightarrow{N \rightarrow \infty} \nabla \delta(\mathbf{r}') \cdot \nabla \delta(\mathbf{r}), \quad (\text{A28})$$

which may be derived by differentiating the obvious relation

$$N^2 \cos(2\pi N \mathbf{r} \cdot \mathbf{r}') \xrightarrow{N \rightarrow \infty} \delta(\mathbf{r}) \delta(\mathbf{r}') \quad (\text{A29})$$

with respect to  $\mathbf{r} \cdot \mathbf{r}'$ , it is easy to see that (A26) indeed converges to (A27) as  $N \rightarrow \infty$ . Once again we see that convergence is rather slow and requires that  $\omega(\mathbf{r})$  be reasonably smooth.

It would obviously be very nice if one could derive our statistical mechanics from this method of discretizing Euler's equation. Unfortunately we see no obvious way of doing this. Both methods in Sec. V relied heavily on the single-site character of the conserved quantities (i.e., products of  $\omega$ 's on the same site *only*). The combinatoric argument took advantage of the obvious implementation of the conservation laws as a hard-core lattice gas—no longer the case for the  $SU(N)$  realization. The Kac-Hubbard-Stratanovitch transformation method used the single-site character in the definition of the function  $W(\tau)$ , Eq. (5.52), which would otherwise require far more than a single argument, and be far more difficult to calculate.

## APPENDIX B: NUMERICAL METHODS

The mean-field equations that we derived in Sec. V depend in a complicated way upon a large number of parameters. We do not know of a practical way to solve them directly, except in very simple cases. For a single species of point vortex in a disc with Dirichlet boundary conditions, an analytic solution is well known [57]. For a small number of nonzero charge species, a relaxational algorithm seems straightforward, and we have implemented this for axisymmetric equilibria (effectively a one-dimensional problem). In the single-species point-vortex limit, Smith [7] has employed a Newton continuation method that applies to (asymmetric) two-dimensional vorticity distributions. When required to deal with a large or infinite number of charge species, which is ordinarily the case under physical conditions, we have resorted to Monte Carlo methods.

We have used a simple relaxational numerical scheme to investigate the equilibria of the one- or two-charge case [Eq. (6.5) or (6.10)]. We will describe the procedure for the one-charge case: the generalization to more charges is clear.

We relax to the solutions of Eq. (6.5) by introducing a fictitious time variable  $\tau$  and solving

$$\frac{\partial \psi_0}{\partial \tau} = + \frac{\delta \mathcal{F}}{\delta \psi_0} = \nabla^2 \psi_0 + \frac{q}{1 + e^{\bar{\beta}(q\psi_0 - \mu_q)}}, \quad (\text{B1})$$

where  $\mathcal{F}[\psi_0]$  is the free energy introduced in Eq. (5.55),

$$\mathcal{F}[\psi_0] = - \frac{1}{V} \int d^2 r \left[ \frac{1}{2} (\nabla \psi_0)^2 + \bar{T} \ln(1 + e^{-\bar{\beta}(q\psi_0 - \mu_q)}) \right], \quad (\text{B2})$$

and the boundary conditions for no fluid flow through the

boundary at  $\mathbf{r}_b$  are

$$\psi(\mathbf{r}_b) = \psi_b. \quad (\text{B3})$$

For fixed  $\psi_b$  Eq. (B1) is relaxational, approaching the maximum of  $\mathcal{F}$ . For the axially symmetric case we have implemented the dynamics using a discrete mesh in the radial coordinate, and a semi-implicit, second-order accurate time evolution. In practice, during the relaxation, we have found it useful to take advantage of the fact that a constant added to  $\psi_0(\mathbf{r})$  can be absorbed into a redefinition of the chemical potential  $\mu_q$ , and to relax the value of  $\psi_b$  so that the field at the origin remains fixed at zero:  $\psi(\mathbf{r}=0)=0$ . We believe this will give a smoother dependence of the solutions on  $\mu_q$ .

The scheme may be readily generalized to include a finite number of vorticity levels, and also the conservation of angular momentum, at the cost simply of introducing more chemical potentials  $\mu_{q_i}$  and  $\mu_L$ . The nonaxisymmetric case, although more complicated numerically, is also an obvious extension.

For the special case of axisymmetric solutions of the one charge case we may eliminate the chemical potential by working directly with the density of charge  $\rho_1(\mathbf{r})$  and relaxing according to the equation

$$\frac{\partial \rho_1}{\partial \tau} = - \left[ \frac{\bar{T}}{q} \right] \nabla^2 \ln \left[ \frac{\rho_1}{1 - \rho_1} \right] + q \rho_1 \quad (\text{B4})$$

with the boundary condition

$$\frac{\partial}{\partial r} \ln \left[ \frac{\rho_1}{1 - \rho_1} \right] \Big|_{r=1} = \bar{\beta} q^2 \frac{\alpha}{2}, \quad (\text{B5})$$

which imposes a fixed total vorticity Eq. (6.7). This scheme is very useful to investigate the behavior in the  $(\alpha, \bar{\beta})$  plane as in Fig. 1, without having to evaluate the dependence  $\mu_q(\alpha, \bar{\beta})$ .

For the multilevel calculation we have relied on the Monte Carlo computation method, which we review here since readers with a fluid-mechanics background might not be familiar with this standard technique. By Monte Carlo, we refer to a class of rejection sampling methods related to the Metropolis algorithm [78]. These methods are based upon a stochastic dynamics that evolves the system toward a statistically stationary equilibrium state. The artificial dynamics is designed to sample the Boltzmann distribution  $\exp(-\beta \mathcal{H})$ , with statistics improving in time (or number of steps for a discrete dynamics). The stochastic dynamics needs to be chosen so that (1) it is ergodic; and (2) it satisfies detailed balance.

$$\mathcal{W}(X, Y) \exp[-\beta \mathcal{H}(X)] = \mathcal{W}(Y, X) \exp[-\beta \mathcal{H}(Y)], \quad (\text{B6})$$

where  $\mathcal{W}(X, Y)$  is the probability that the system moves to state  $Y$  at the next step, given that it is currently in state  $X$  [79]. We may divide  $\mathcal{W}(X, Y)$  into two independent parts: the conditional probability  $\mathcal{P}(X, Y)$  that a transition (or “move”) from  $X$  to  $Y$  is proposed, given that the system is in state  $X$ , and the conditional probability that a proposed transition is accepted,  $\mathcal{A}(X, Y)$ . If, as is customary, we choose  $\mathcal{P}(X, Y)$  constant for a set of  $Y$



related to  $X$  by an elementary move, and zero otherwise, then  $\mathcal{W}(X, Y) = \mathcal{A}(X, Y)$ . A variety of choices for  $\mathcal{W}(X, Y)$  are possible; the canonical Monte Carlo (Metropolis) method stipulates

$$\mathcal{W}(X, Y) = \max\{1, \exp[-\beta\mathcal{H}(Y)]/\exp[-\beta\mathcal{H}(X)]\} . \quad (\text{B7})$$

Once our Monte Carlo dynamics has relaxed the system to equilibrium, we may calculate sought-after quantities by averaging over the configurations given by the dynamics, provided we are careful to average only over configurations separated by sufficiently many time steps that the correlations between them are small.

A closely related “microcanonical” Monte Carlo method has been suggested by Creutz [80]. He partitions the energy between the system of interest and an additional degree of freedom, the “demon.” Moves are restricted to the phase space determined by conservation of the sum of the energies of system and demon.

We have made use of both canonical and microcanonical methods. We treat the energy and angular momentum separately; we have found it convenient to calculate with either (1) microcanonical moves separately in both energy and angular momentum; or (2) canonical moves in the energy but microcanonical moves in the angular momentum. For the regimes in which we are interested, both methods yield identical results.

Our configuration space is made up of the square lattice of sites enclosed by a disc or annulus. A move consists of the exchange of the value of the vorticity field on a specified site, with the value of the vorticity on a distinct site, chosen randomly on the lattice. We may view a move as the exchange of a pair of lattice sites. In practice, we achieve more rapid relaxation by the exchange of two independently chosen pairs of lattice sites on each move [7]. We find that this two-pair exchange yields the same equilibrium average as one-pair exchange. Note that a pair exchange can involve any two lattice sites, not merely nearby ones.

We have used both Green's functions and relaxational methods to calculate the energy difference between a given state and a proposed move. The Green's function of the Laplacian is convenient to compute for a disc, where the Dirichlet boundary conditions can be satisfied by means of image charges. In this case, we can confirm that both ways of calculating the energy yield identical results. On the annulus, we have calculated energies by using a relaxational method to solve the Poisson equation on the lattice, by means of a standard five-point first-order discretization.

Our calculations have been done with a  $32 \times 32$  square lattice. The outer boundary of the disc or annulus is always a circle of radius 16 lattice units; only squares within the boundaries play a role in the computation. A single Monte-Carlo sweep (MCS) is defined as  $n/2$  *accepted* moves, where  $n$  is the number of squares on which the vorticity achieves an absolute value greater than  $0.0001$  times the maximum absolute value of the vorticity,  $|\omega|_{\max}$ . Before collecting statistics, we allowed our lat-

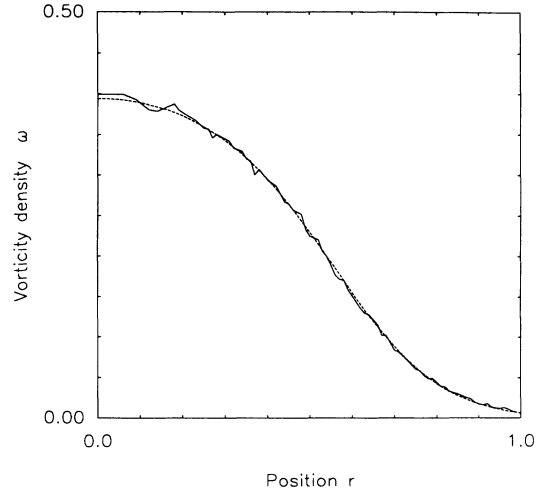


FIG. 5. Comparison of the numerical solution of the mean-field equations with a two-level initial condition,  $g(\sigma) = 0.7358(\sigma) + 0.2658(\sigma - 1)$  (dashed curve), with the corresponding Monte Carlo equilibrium state (solid line).

tices to equilibrate for several hundred MCS, by which time a stationary state is achieved, and further equilibration appears to be unnecessary. In the parameter ranges chosen below, we obtain identical results whether we anneal our lattices from higher temperatures, or quench them directly from the prescribed vorticity configuration. In parameter ranges similar to those used below, we have found that equilibration is rapid and longer runs yield quantitatively indistinguishable long-time configurations.

In Fig. 5 we show the equilibrium vorticity profile on a disc of radius 1 for a simple set of parameter values. We have chosen energy  $E = 0.034$  and angular momentum [redefined for convenience by removing the part with no configuration dependence—see Eq. (5.3)]  $L = \int d^2\mathbf{r} \omega r^2 = 0.212$ . The (bare) vorticity distribution  $g_b$  was chosen to be  $0.2658(\sigma - 1) + 0.7358(\sigma)$ . The smooth curve shows the equilibrium vorticity profile as calculated by numerically solving the radial ordinary differential equation for a particular choice of  $\bar{\beta}$  and  $\mu$ . The jagged curve shows the equilibrium vorticity profile as calculated by the Monte Carlo computation described above, with the relaxational Poisson algorithm. For the Monte Carlo calculation, the energy and angular momentum were determined by discretizing on the square lattice the equilibrium vorticity profile calculated from the ordinary differential equation and were found to be  $E = 0.036$  and  $L = 0.204$ . The bare vorticity distribution was, of course, the  $g_b$  specified above. We ran a microcanonical Monte Carlo calculation for  $5 \times 10^3$  MCS, with an acceptance ratio of  $1/6$ . The vorticity was averaged in the azimuthal direction, and has a maximum standard deviation of  $0.026$ . Evidently, the Monte Carlo calculation can yield an excellent approximation to the exact solution of the mean-field equations.

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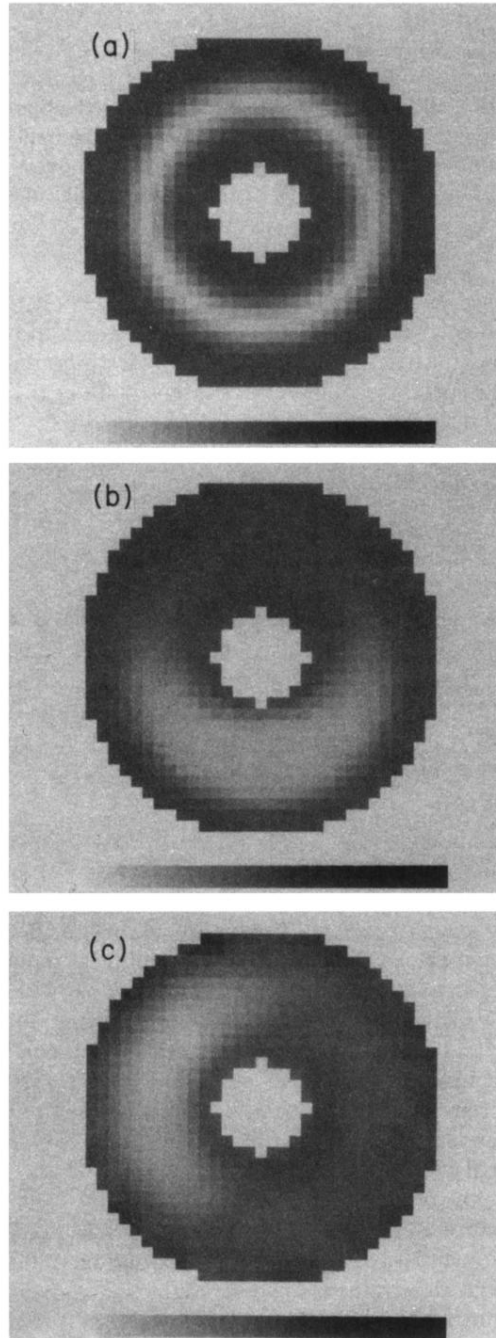


FIG. 3. Comparison of our Monte Carlo simulations with the long-time dynamics of Marcus (Ref. [6]): (a) the initial condition consisting of a ring of vorticity; (b) the Monte Carlo equilibrium state arising from (a); (c) the long-time dynamical state arising from (a). The zero of vorticity is at the midpoint of the gray scale.

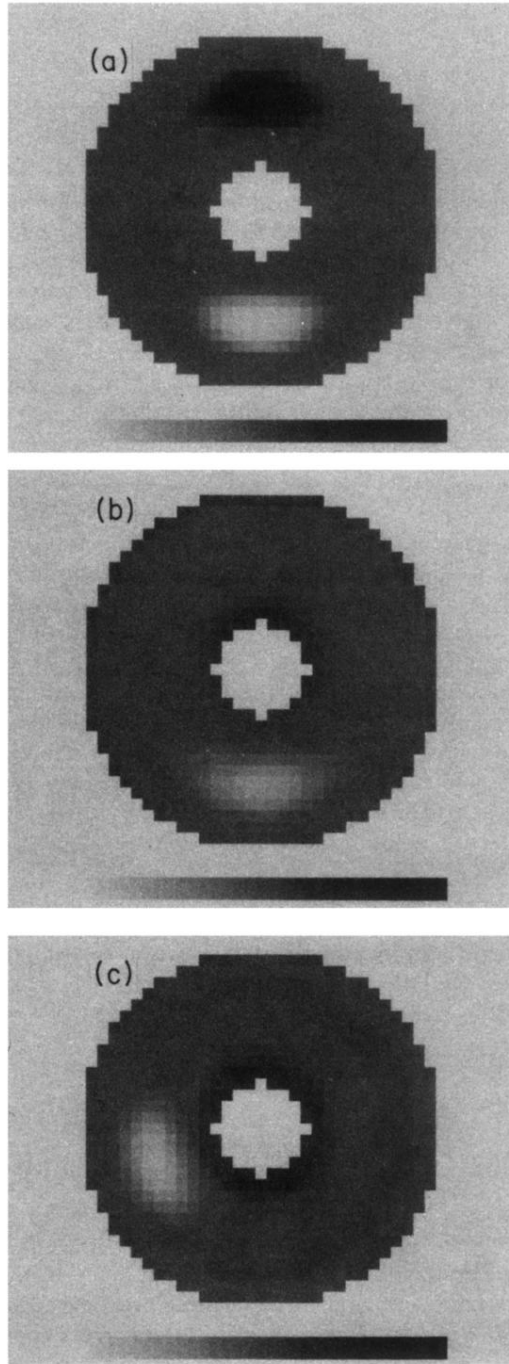


FIG. 4. Comparison of our Monte Carlo simulations with the long-time dynamics of Marcus (Ref. [6]): (a) the initial condition consisting of two oppositely charged symmetrically placed blobs; (b) the Monte Carlo equilibrium state arising from (a); (c) the long-time dynamical state arising from (a). The zero of vorticity is at the midpoint of the gray scale.